SOME MATCHING THEOREMS

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A graph G is said to be *even* if its vertices can be put into two distinct classes A and B so that no two vertices of the same class are joined by an edge. If in addition each class contains exactly n representatives, G is said to be of *type* (n, n). In what follows all our graphs will be of this form. Let every vertex in A be joined to every vertex in B. Then we say that G is *saturated*.

A *matching* of G is a set of edges covering every vertex just once. It was shown recently [1] that if G has more than $(1/2+c)n^2$ edges, $c > 0$, then it cannot have a unique matching. The method of proof depended upon a result of Žnam. This result allowed one to find disjoint saturated subgraphs G_i of G which were of type (r, r) with $r > c' \log n$ and such that every matching of ΣG_i could be extended to a matching of G. In the present note we show that it suffices to find a subgraph of G whose edges are distributed with some regularity, further we obtain a better estimate for the number of matchings.

THEOREM 1. *Let G be an even graph of type (n,n) and suppose that G has at least $(1/2+c)n^2$ edges, and has at least one matching. Then G has at least*

(1) $$2^{\mu}\,\mu\,!$$

distinct matchings, where

(2) $$\mu = [1/2\, m], \qquad m \geqslant \alpha n, \qquad \alpha = 1-(1-2c)^{1/2},$$

and m is an integer.

In particular, if c is fixed and n large, the number of distinct matchings exceeds

$$(n\,!)^{c_1}$$

where $c_1 > 0$ depends only upon c.

PROOF. Let the vertices of G be $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, and let the given matching be that which associates a_i with b_i for $i = 1, \ldots, n$.

Let $\rho(a_i)$ denote the number of b's joined to a_i and let $\sigma(b_i)$ denote the number of a's joined to b_i. Plainly

$$\sum_{i=1}^{n} \{\rho(a_i) + \sigma(b_i)\} \geqslant (1+2c)\, n^2,$$

since the sum on the left is twice the number of edges of G.

Let N denote the number of values of i for which $\rho(a_i) + \sigma(b_i) \geqslant (1+\alpha)n$. Then since $\rho(a_i) \leqslant n$ and $\sigma(b_i) \leqslant n$ the sum is at most

$$2nN + (1+\alpha)n\,(n-N).$$

It follows that

$$N \geqslant n(2c-\alpha)\,/\,(1-\alpha) = \alpha n,$$

the last equation being a consequence of the definition of α in (2).

We can therefore suppose that

$$\rho(a_i) + \sigma(b_i) \geqslant (1+\alpha)n,$$

for $i = 1, \ldots, m$ where m is the least integer $\geqslant \alpha\, n$.

For given i, let N_0 denote the number of j $(1 \leqslant j \leqslant n)$, distinct from i, for which there is no edge $a_i b_j$ or $b_i a_j$; let N_1 denote the number for which there is one such edge; and let N_2 denote the number for which there are two such edges. Then

$$N_0 + N_1 + N_2 = n-1$$

$$N_1 + 2N_2 \geqslant (1+\alpha)\, n - 1.$$

It follows on subtraction that $N_2 \geqslant \alpha\, n$, whence $N_2 \geqslant m$.

Thus, for each $i = 1, \ldots, m$ there exist

$$r_1^{(i)}, \ldots, r_m^{(i)},$$

such that a_i is joined to each b_r and b_i is joined to each a_r for the above values of r. This enables us to construct a variety of distinct matchings of G, as follows.

We replace the edges a_1b_1 and a_rb_r by the edges a_1b_r and a_rb_1 for $r = r_1^{(1)}, \ldots, r_m^{(1)}$. This is possible in m ways. When any such choice has been made, we consider the least i different from 1 and r (thus $i = 2$ or 3), and we replace the edges a_ib_i and a_sb_s by a_ib_s and a_sb_i for $s = r_1^{(i)}, \ldots, r_m^{(i)}$ provided $s \neq 1$ or r. This is possible in at least $m-2$ ways. Next we consider the least j which is different from $1, r, i, s$ (thus $j \leqslant 5$) and make a similar replacement, which is possible in at least $m-4$ ways, and so on.

The number of distinct matchings so obtained is at least

$$\prod_{0 \leqslant r < \frac{1}{2}m} (m-2r).$$

Since $m \geqslant 2\mu$, this is at least $2^{\mu}\,\mu\,!$, as stated.

The final clause is an immediate deduction, for if c is fixed, then so is α and $\mu > (n\,!)^{c_1}$.

Whilst giving a non-trivial result for any $c > 0$, as c approaches $1/2$ c_1, does not approach 1 as one would expect. For larger values of c the following result is perhaps therefore of interest.

THEOREM 2. *Let G satisfy the hypotheses of Theorem 1 with $2c > \sqrt{3}-1$. Then G has at least $m!$ distinct matchings where m is an integer satisfying*

$$m + 1 \geqslant n(2c-(2-4c)^{1/2}).$$

PROOF. We use the notation of the previous theorem. It is plain that we can find a value of i so that

$$\rho\,(a_i) + \sigma\,(b_i) \geqslant (1+2c)\,n.$$

Without loss of generality we can take $i = 1$. Let k be the least integer satisfying $k \geqslant 2cn$. Then arguing as in the proof of the previous theorem we see that we may assume that

$$a_ib_1 \text{ and } a_1b_i \qquad (i = 1, \ldots, k)$$

are all edges of G.

For any ϑ satisfying $0 < \vartheta < 1$, let N_ϑ denote the number of values of i so that $\rho(a_i) \geqslant \vartheta n$. We obtain the estimates

$$nN_\vartheta + \sigma\vartheta_n (n-N_\vartheta) \geqslant \Sigma\, \rho(a_i) \geqslant (1/2+c)\, n^2,$$

so that

$$N_\vartheta \geqslant n\,(1/2 + c - \vartheta)\,/\,(1-\vartheta).$$

Hence choosing $\vartheta = 1 - (1/2-c)^{1/2}$ and putting V for the least integer not less than ϑ_n we see that $N_\vartheta \geqslant V$. Of these values of i at least $k + V - n$ satisfy $i \leqslant k$ and by relabelling, if necessary, we can therefore assume that

$$\rho(a_i) \geqslant V \qquad (i = 1, \ldots, k + V - n).$$

Moreover for any such i the number of edges a_ib_j with $j \leqslant k + V - n$ is at least

$$(k+V-n) + V - n = k + 2V - 2n.$$

Consider now the subgraph G' with vertices a_i, b_j; $i,j = 1, \ldots,$ $\ldots, k + V - n$. By addition of edges of the type a_sb_s, $k + V - n < s \leqslant n$ we can clearly extend any matching of G' to one of G. We now derive matchings of G' by constructing distinct cycles all of which have an edge in common.

Defining $\rho'(a_i)$ and $\sigma'(b_i)$ in analogy with the definitions in Theorem 1 we see that

$$\rho'(a_i) \geqslant k + 2V - 2n, \qquad (i = 1, \ldots, k + V - n)$$

and

$$\rho'(a_1) = \sigma'(b_1) = k + V - n.$$

We construct cycles all containing the edge a_1b_1. First choose a value of j satisfying $1 < j \leqslant k + V - n$ so that a_1b_j is an edge of G'. This is clearly possible in $\rho'(a_1) - 1$ ways. Let this value be j_1. Now $a_{j_1} b_{j_1}$ is an edge of G' and we choose $j_2 \neq 1, j_1$ so that $a_{j_1} b_{j_2}$ is an edge of

G'. This is possible in $\rho'(a_{j_1}) - 2$ ways. Then $a_{j_2} b_{j_2}$ is an edge of G' and so on until after $k + 2V - 2n - 1$ choices we reach the edge $a_s b_s$ where $s \neq 1$. We now complete our cycle with the edge $a_s b_1$ since $a_i b_1$ is an edge for *any* a_i in G'.

In this manner the number of cycles and therefore the number of matchings of G' is at least

$$\prod_{i=1}^{k+2V-2n} (\rho'(a_{j_1}) - i) \geqslant (k+2V-2n-1)\,!$$

Noting that the restriction $2c > \sqrt{3} - 1$ guarantees $k + 2V - 2n$ exceeds a positive multiple of n we see that the proof is complete.

By a simple modification of the argument in Theorem 1 it is easily seen that G cannot have a unique matching if it has more than $1/2\, n(n+1)$ edges. In a certain sense this result is best possible as can be seen on considering the graph with edges $a_i b_j$ for $1 \leqslant i \leqslant j \leqslant n$. This clearly has $1/2 n\,(n+1)$ edges and just one matching.

Finally we noted that the value of c_1 in theorem 1 cannot exceed $(2c)^{1/2}$. Consider the graph G with edges $a_i b_j$ for i, j satisfying $1 \leqslant i \leqslant j \leqslant n$; $n - [n\sqrt{2c}] < i \leqslant n$ and $i \leqslant j \leqslant n$. Then G has more than $(1/2+c)\, n^2$ edges (taking of course $0 < c < 1/2$), but only $\exp\,(1+o\,(1)\,)\,\sqrt{2c}\, n \log n)$ matchings. Indeed it seems likely that this upper bound is more nearly the value of c_1 to be expected.

REFERENCE

1. ELLIOTT, P.D. *Even Graphs* (To appear in Mathematika).

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