ON THE SUM $\sum\limits_{n=1}^{x} d[d(n)]$

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Denote by $d(n)$ the number of divisors of n. It is well-known that

$$\sum_{n=1}^{x} d(n)=x\log x+(2c-1)x+0(x^{\eta}),\ \eta<\tfrac{1}{3}.$$

Ramanujan [**1**] investigated the function $d[d(n)]$, but I believe the following simple result is new :

Theorem.
$$\lim_{x=\infty}\frac{1}{\log\log x}\sum_{n=1}^{x} d\{d(n)\}=d_2 \qquad \ldots(1)$$

where $0<d_2<\infty$ is a constant.

Proof. The proof of (1) is simple. Denote by $s_1=1<s_2<\ldots$ the sequence of integers all whose prime factors occur with an exponent greater than 1. Clearly every integer can be uniquely written in the form $s_i\, q_i$ where q_i is square free and $(s_i, q_i)=1$. Thus we evidently have $\Big[$ $V(q)$ denotes the number of prime factors of q and in $\Sigma' q<\dfrac{x}{s_i}$, $(q, s_i)=1$ $\Big]$

$$\sum_{n=1}^{x} d\{d(n)\}=\sum_i{\sum}' d\{d(s_i, q)\}=\sum_i{\sum}' d\{2^{v(q)}\cdot d(s_i)\} \qquad \ldots(2)$$

Put $d(s_i)=2^{d_i}\beta_i$, β_i odd. Then from (2) we have

$$\sum_{n=1}^{x} d\{d(n)\}=\sum_i{\sum}' d(2^{v(\theta)+d_i}\cdot\beta_i)=\sum_i d(\beta_i){\sum}'\{v(q)+d_i+1) \qquad \ldots(3)$$

For fixed i we evidently have by interchanging the order of summation (p, q, r are primes)

$$\sum{}' \{v(q)+d_i+1) = \sum{}' v(q)+0(x) = \frac{x}{s_i}\sum_{\substack{p\leqslant x\\ p+s_i}} \frac{f(s_i)\left(1-\frac{1}{p}\right)}{p}+0(x)$$

$$=\frac{x\, f(s_i)}{s_i}\sum_{p\leqslant x}\frac{1}{p}+0(x)=x \log\log x\, \frac{f(s_i)}{s_i}+0(x), \qquad \text{...(4)}$$

where

$$f(s_i)=\prod_{q+s_i}\left(1-\frac{1}{q^2}\right)\prod_{s+s_i}\left(1-\frac{1}{s}\right). \qquad \text{...(5)}$$

The $0(x)$ in (4) is not uniform in i. Put now

$$\sum_{i=1}^{\infty}\frac{d(\beta_i)\, f(s_i)}{s_i}=d_2. \qquad \text{...(6)}$$

It is easy to see that the series (6) converges. To see this observe that

$$\sum_{i=1}^{\infty}\frac{d(\beta_i)f(s_i)}{s_i}<\sum_{i=1}^{\infty}\frac{d(\beta_i)}{s_i}\leqslant\sum_{i=1}^{\infty}\frac{d(s_i)}{s_i}$$

$$=\prod_{p}\left(1+\sum_{k=2}^{\infty}\frac{k+1}{p^k}\right)<\infty. \qquad \text{...(7)}$$

(3) and (4) clearly implies that for every fixed i_0

$$\sum_{n=1}^{x}d\{d(n)\}=x\log\log x\sum_{i\leqslant i_0}\frac{d(\beta_i)\, f(s_i)}{s_i}+\sum_{i>i_0}d(\beta_i)\sum{}'\{v(q)+d_i+1\}+0(x)$$

...(8)

Thus by (6) and (7), (8) implies (1) if we can show that for every $\epsilon>0$ there is an i_0 so that

$$\sum_{i>i_0}d(\beta_i)\sum{}'\{v(q)+d_i+1)<\epsilon x\log\log x. \qquad \text{...(9)}$$

By $d(s_i)=2^{d_i}\beta_i$ we clearly have $d(\beta_i)\{v(q)+d_i+1\}\leqslant d(s_i)v(q)$. Thus instead of (9) it will suffice to show

$$\sum_{i>i_0}d(s_i)\sum{}' v(q)<\epsilon\, x\log\log x_i] \qquad \text{...(10)}$$

Clearly we have for $x>x_0$

$$\sum{}' v(q)\leqslant\sum_{n\leqslant x/s_i}v(n)\leqslant\frac{x}{s_i}\sum_{p\leqslant x}\frac{1}{p}<\frac{2x\log\log x}{s_i}. \qquad \text{...(11)}$$

From (10) and (11) we have for sufficiently large i_0,

$$\left(\text{by (7)} \sum_i \frac{d(s_i)}{s_i} < \infty\right)$$

$$\sum_{i>i_0} d(s_i) \sum{}' v(q) < 2x \log\log x \sum_{i>i_0} \frac{d(s_i)}{s_i} < \epsilon x \log\log x$$

which proves (9) and hence the proof of our theorem is complete.

Put $d(n)=d_1(n), d_k(x)=d\{d_{k-1}(n)\}$ and denote by $\log_k n$ the k-fold interated logarithm. It seems likely that $(0<d_k<\infty)$

$$\lim_{x=\infty} \frac{1}{\log_k x} \sum_{n=1}^{x} d_k(n) = d_k.$$

I have verified this for $k=3$, the proof is similar but much more complicated than for $k=2$ and probably could be made to work in the general case but I have not carried out the details.

Denote by $l(n)$ the smallest integer k for which $d_k(n)=2$. It seems to be very difficult to get good limitations for the growth of $l(n)$, no doubt the problem is somewhat artificial.

REFERENCE

1. Ramanujan, S. : *On highly composite Numbers*, Collected Papers (Cambridge), 1927.

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