ON THE IRRATIONALITY OF CERTAIN SERIES

By

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In a previous paper [**1**] I proved that

$$\sum_{n=1}^{\infty} \frac{d(n)}{t^n} = \sum_{n=1}^{\infty} \frac{1}{t^n - 1} \qquad \ldots(1)$$

is irrational for every integer $t \geqslant 2$. Denote by $V(n)$ the number of distinct prime factors of n. I conjectured in [**1**] that

$$\sum_{n=1}^{\infty} \frac{V(n)}{t^n} = \sum_{p} \frac{1}{t^p - 1}$$

is also irrational. I have not yet been able to prove this conjecture. In fact I know no example of an infinite sequence $n_1 < n_2 < \ldots$ and $t \geqslant 2$ for which

$$\sum_{t=1}^{\infty} \frac{1}{t^n - 1} \qquad \ldots(2)$$

is rational, though it seems likely that this can happen. I am going to prove the following

THEOREM. *Let* $(n_i, n_j) = 1$, $\sum_{i=1}^{\infty} 1/n_i \leqslant \infty$. Then

$$\sum_{i=1}^{\infty} \frac{1}{t^{n_i} - 1}$$

is irrational for every $t \geqslant 2$.

By more complicated arguments one can show that the condition $(n_i, n_j) = 1$ is superfluous. We do not give the details since I do not think that the condition $\sum_{i=1}^{\infty} 1/n_i < \infty$ is very relevant. In fact it could be replaced by a weaker but more complicated condition. I would expect that the series (2) is always irrational if $n_{k+1} - n_k \to \infty$ (perhaps even $n_k/k \to \infty$ suffices).

The terms of the series (1) can be defined by the following recussion : $u_1=t-1$, $u_{n+1}=tu_n+t-1$. One would expect that if a series is defined by this recursion, then $\sum_{n=1}^{\infty} 1/u_n$ is irrational for any positive integral value of u_1. This I have not been able to prove, not even if $t=2$. For $t=2$ one would have to prove that $\sum_{n=1}^{\infty}\frac{1}{(2^n-1)l^n}$ is irrational for every positive integer l, but this I have not been able to do. Incidentally, I cannot show that $\sum_{n=1}^{\infty}\frac{1}{n!-1}$ is irrational.

Now we have to prove our theorem. Denote by $V^*(m)$ the number of divisors of m amongst the n_i. We evidently have

$$\sum_{k=1}^{\infty}\frac{1}{t^{n_k}-1}=\sum_{m=1}^{\infty}\frac{V^*(m)}{t^m}=\alpha. \qquad ...(3)$$

As in [1] we show that irrationality of α by showing that the t-ary development of α is infinite but that it contains arbitrarily many 0's. To show this let k be sufficiently large. We first of all try to find integers y for which

$$V^*(y+i)=t^i,\ i=1,\ldots,k. \qquad ...(4)$$

We now give k congruences for y.

$$y+1\equiv 0 \pmod{\prod_{i=1}^{t} n_i},$$

$$y+2\equiv 0 \pmod{\prod_{i=t+1}^{t^2+t} n_i}.$$

Now if $n_1=2y+3\equiv 0 \pmod{n_1}$. Thus if $n_1=2$ our third congruence is $y+i\equiv 0 (\mathrm{mod}\ \Pi n_i)$, $t^2+t+1<i<t^3+t^2+t-1$, in other words i runs through t^3-1 values. If $n_1>2$ then $t^2+t+1<i<t^3+t^2+t$(*i.e.*, i runs through t^3 values). In the jth congruence $1\leqslant j\leqslant k$, we demand that $y+j$ should be a multiple of the first r_j n's which have not yet been used in the first $j-1$ congruences where r_j is determined so that the first j congruences assure that $y+j$ is divisible by precisely t of the first $r_1+\ldots+r_j$ n's. It is easy to see that r_j is uniquely determined and its value depends only on the sequence $n_1<n_2<\ldots$(here we strongly use that the n's are relatively prime in pairs). Put $\sum_{j=1}^{k} r_j=l$, $A_l=\prod_{i=1}^{l} n_i$. y is uniquely determined mod A_l by these k congruences. We clearly have

$$t^k<l\leqslant\sum_{i=1}^{k} t^i$$

and $y+j$ is divisible by precisely t^j of the n_i not exceeding n_l (since the

n's of index greater than $r_1+\ldots+r_j$ but not exceeding $r_1+\ldots+r_k=l$ can never divide $n+j$. To see this let

$$\sum_{i=1}^{j} r_i < u \leqslant \sum_{i=1}^{l} r_i$$

and let $j, j' < j' \leqslant k$ the least integer for which $n+j' \equiv 0 \pmod{n_u}$. By definition of our congruences $n_u \geqslant u > t^{j'} > j'$, hence $n+j$ is not congruent to 0 (mod n_u) as stated).

Let y_0 be the smallest positive solution of our congruences. We evidently have $0<y_0<A_l$. Let x be sufficiently large and put

$$y=y_0+sA_l,\ 0\leqslant s<\frac{x}{A_l}. \qquad \ldots(4)$$

We shall now show that there is an s satisfying (4) for which

$$V^*(y+i)=t^i,\ 1\leqslant i\leqslant k \qquad \ldots(5)$$

and

$$0 < \sum_{j>k} \frac{V^*(y+j)}{t^{y+j}} < \frac{1}{t^{y+k/2}} \qquad \ldots(6)$$

(5) and (6) imply that there are at least $\frac{k}{2}$ 0's following the y's t-ary digit of α and since this holds for every k and since (6) also implies that not all digits following the y's are 0, we have proved that α is irrational.

Thus to complete our proof we only have to show (5) and (6) hold for a y satisfying (4). In view of our k, congruences (5) are satisfied if

$y+i=y_0+sA_l+i$ is not congruent to 0 (mod n_j), $1\leqslant i\leqslant k,\ n_l<n_j\leqslant X$. ...(7)

We estimate from above the number of values of s for which (7) is not satisfied for some i or n_j. For fixed i and j the number of solutions of (7) is at most $\left[\frac{X}{A_l n_j}\right]+1$. Put $N(x)=\sum_{n_j\leqslant x} 1$, since $\sum_{j=1}^{\infty} \frac{1}{n_j} < \infty$ we have $N(x)=0(x)$. Thus finally the number of values of s for which (7) is not satisfied for all relevant values of i and j is for sufficiently large k and x at most $\left(\sum_{j>k}\frac{1}{n_j}<\epsilon \text{ for } k>k_0\right)$

$$\frac{X}{A_l}\sum_{j>k}\frac{1}{n_j}+kN(X)<\frac{X}{2A_l} \qquad \ldots(8)$$

Now we deal with (6). Put for $j>k$

$$V^*(y+j)=V_1^*(y+j)+V_2^*(y+j) \qquad \ldots(9)$$

where

$$V_1^*(y+j)=\sum_{\substack{n_i/(y+j)\\ i\leqslant l}} 1,\ V_2^*(y+j)=\sum_{\substack{n_i/(y+j)\\ i>l}} 1$$

For $j \leqslant 2k$ we have

$$V_1^*(y+j) < k, \qquad \ldots(10)$$

since from our congruences it follows that if $u \leqslant l$ then $n_u \mid (y+i)$ for some $0 < i \leqslant k$ hence if for $j \leqslant 2k$, $n_u \mid (y+j)$ we have $n_u < 2k$, hence by $N(x)=o(x)$, $u<k$ (for $k>k_0$) and hence (10) follows.

For $j > 2k$ we evidently have

$$V_1^*(y+j) \leqslant l < t^{k+1}. \qquad \ldots(11)$$

From (10) and (11) we have for $k > k_0$

$$\sum_{j>k} \frac{V_1^*(y+j)}{t^{y+j}} < k \sum_{j>k} \frac{1}{t^{y+j}} + t^{k+1} \sum_{j>2k} \frac{1}{t^{y+j}} < \frac{1}{2t^{k/2}}. \qquad \ldots(12)$$

Now we prove the following

Lemma. *For all but* $\frac{X}{4A_l}$ *values of s we have for every* $j > k$,

$$V^*(y+j) < j^2. \qquad \ldots(13)$$

To prove our lemma we first of all observe that (13) is trivially satisfied for $j > x$, since if $j < x$ then

$$V^*(y+j) < y + j < 2j < j^2.$$

We evidently have for a fixed $j < x$, $y < x$ and $k > k_0$

$$\left(\text{in } \Sigma_1\ y = y_0 + sA_l,\ 0 \leqslant s < \frac{x}{A_l}\right)$$

$$\sum\nolimits_1 V_2^*(y+j) \leqslant \sum_{k<n_i \leqslant 2X} \left[\left(\frac{X}{A_l n_i}\right) + 1\right] < N(2X) + \frac{X}{A_l} \sum_{n_i > k} \frac{1}{n_i} < \frac{X}{A_l}.$$

Thus for any fixed j, the number of values of which (13) does not hold is less than $\frac{x}{j^2 A_l}$. Hence the total number of values of s for which (13) does not hold for some $j < k$ is less than

$$\frac{X}{A_l} \sum_{j>k} \frac{1}{j^2} < \frac{X}{4A_l}$$

which proves the lemma.

Let now s satisfy (5) and (13). Then for $k > k_0$

$$\sum_{j>k} \frac{V_2^*(y+j)}{t^{y+j}} < \sum_{j>k} \frac{j^2}{t^{y+k}} < \frac{1}{t^{k/2}} \qquad \ldots(14)$$

By (8) and our Lemma there are values of s which satisfy (5) and (13). By (12) and (14) these s also satisfy (6) (the left side of (6) is trivially satisfied), hence the proof of our theorem is complete.

If we do not assume $(n_i, n_j)=1$ the proof becomes more complicated. We have to use the result that if the fractional part of $t^n\alpha$ takes on infinitely many different values, then α is irrational.

If we assume $(n_i, n_j)=1$, then by using Brun's method $\sum_i \frac{1}{n_i} < \infty$ could probably be replaced by $\sum_{n_i<x} \frac{1}{n_i} = o\,(\log\log x)$ but I do not see how to handle the case if the n's are the set of all primes.

REFERENCE

1. P. Erdös: *On arithmetical properties of Lambert series*, J. Indian Math. Soc. **12**, 1948 pp. 63-66.

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