P. ERDÖS and A. RÉNYI (Budapest)

ON RANDOM ENTIRE FUNCTIONS

This paper deals with random power series, a topic concerning which the first important results are due to H. Steinhaus [4].

While in [4] power series with a finite radius of convergence were considered, we study random entire functions.

Let

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an arbitrary entire function; let

$$(2) \qquad M(r) = \max_{|z|=r} |f(z)|$$

denote the maximum-modulus function of $f(z)$ and

$$(3) \qquad \mu(r) = \max_n |a_n| r^n$$

the maximal term of the series (1).

According to Wiman's well known theorem, for every $\delta > 0$ there exists a set E_δ of finite logarithmic measure (i.e. such that $\int_{E_\delta} \dfrac{dr}{r} < \infty$) such that if $r \notin E_\delta$ one has

$$(4) \qquad M(r) < \mu(r) \big(\log \mu(r)\big)^{\frac{1}{2}+\delta}.$$

The simplest proof of this theorem is the probabilistic proof given by Rosenbloom [2], which deduces (4) from Chebishev's inequality.

It is known that the number $\frac{1}{2}$ in the exponent of $\log \mu(r)$ on the right hand side of (4) is best possible, as there exist entire functions $f(z)$ for which there exists a constant $c > 0$ such that

$$M(r) > c\mu(r) \big(\log \mu(r)\big)^{\frac{1}{2}} \quad \text{for all} \quad r \geqslant 0.$$

As a matter of fact, if $f(z) = e^z$, $M(r) = e^r$ and $\mu(r) = \dfrac{r^{[r]}}{[r]!}$ for $r > 0$ where $[r]$ denotes the integral part of r; thus by Stirliug's formula

(5) $$\lim_{r \to \infty} \frac{M(r)}{\mu(r)\left(\log \mu(r)\right)^{1/2}} = \sqrt{2\pi}.$$

The first-named author has stated without proof some years ago — in a paper [1] which was dedicated to Professor Steinhaus — that if we give random signs to the terms of the power series of e^z then for almost all choices of the sequence of signs the exponent $\frac{1}{2}$ in (4) can be replaced by $\frac{1}{4}$ but by no smaller number. In the present paper we consider the same question for an arbitrary entire function. By other words we consider the class of entire functions obtained by giving random signs to the terms of the series (1), i.e. we consider the entire functions

(6) $$f(z, t) = \sum_{n=0}^{\infty} a_n R_n(t) z^n \qquad (0 \leqslant t < 1)$$

where $R_n(t)$ is the n-th Rademacher function, $R_n(t) = \operatorname{sign} \sin(2^n \pi t)$.

We shall prove that for almost all values of t in the inequality (4) for $f(z, t)$ the exponent $\frac{1}{2}$ of $\log \mu(r)$ can be replaced by $\frac{1}{4}$.

We shall prove even more. Rosenbloom in his above mentioned paper([1]) [2] has proved the following sharper form of Wiman's theorem: for every $\delta > 0$ there exists a set E_δ of finite logarithmic measure, such that if $r \notin E_\delta$ one has

(4*) $$M(r) < \mu(r)\left(\log \mu(r)\right)^{1/2}\left(\log\log \mu(r)\right)^{1+\delta}.$$

We shall prove that for almost all values of t, in the corresponding inequality for $f(z, t)$ the exponent $\frac{1}{2}$ of $\log \mu(r)$ can be replaced by $\frac{1}{4}$. Thus we prove the following

THEOREM 1. *Let (1) be an arbitrary entire function and let $\mu(r)$ be defined by (3). Let the entire function $f(z, t)$ be defined by (6) and put*

(7) $$M(r, t) = \max_{|z| = r} |f(z, t)| \qquad (0 \leqslant t < 1).$$

Then for every $\delta > 0$, for almost all values of t there exists a subset $E_\delta(t)$ (depending on t) of the half line $r \geqslant 0$ of finite logarithmic measure, such that for $r \notin E_\delta(t)$ one has

(8) $$M(r, t) < \mu(r)\left(\log \mu(r)\right)^{1/4}\left(\log\log \mu(r)\right)^{1+\delta}.$$

([1]) The formulation of Theorem 1 in paper [2] contains some misprints: In formula (2) (p. 327) the integral sign is missing; in row 9 of p. 327 the sign $=$ has to be replaced by $<$; in row 10 of p. 327 instead of "inequality (2)" one should read "inequality (3)".

Proof. Let us put

$$(9) \qquad f_1(z) = \sum_{n=0}^{\infty} |a_n| z^n.$$

We may suppose that $a_0 = 1$, which implies $\mu(0) = 1$. As (4^*) is valid for $f_1(z)$, there exists for every $\delta > 0$ a set E_δ of finite logarithmic measure such that for $r \notin E_\delta$ the inequality

$$(10) \qquad f_1(r) < \mu(r) \big(\log \mu(r)\big)^{1/2} \big(\log\log \mu(r)\big)^{1+\delta}$$

is valid. We may suppose that the set E_δ is the union of a denumerable set of disjoint open intervals, the endpoints of which have no finite limit point. Let us define the sequence r_n of nonnegative numbers as follows. We put $r_0 = 0$; if r_k is already defined for $k \leqslant n$, let $r_n^* > r_n$ be defined by

$$(11) \qquad \log \mu(r_n^*) = \log \mu(r_n) + 1.$$

If $r_n^* \notin E_\delta$, put $r_{n+1} = r_n^*$. If however $r_n^* \in E_\delta$, then r_n^* is contained in one of the open intervals of E_δ; in this case let r_{n+1} be the lower end-point and r_{n+2} the upper end-point of this interval.

The increasing sequence r_n defined in this way has the following properties:

a) $r_n \notin E_\delta$ for $n \geqslant 1$.

b) If the open interval (r_n, r_{n+1}) contains a number r not belonging to the set E_δ, then

$$\log \mu(r_{n+1}) = \log \mu(r_n) + 1.$$

c) $\log \mu(r_n) \geqslant [n/2]$ where $[n/2]$ denotes the integral part of $n/2$.

Now let us suppose that for some t one has for $n \geqslant n_0(t)$

$$(12) \qquad M(r_n, t) \leqslant \tfrac{1}{3} \mu(r_n) \big(\log \mu(r_n)\big)^{1/4} \big(\log\log \mu(r_n)\big)^{1+\delta}.$$

As $M(r, t)$ and $\mu(r)$ are both increasing functions of r, it follows (in view of property b) of the sequence r_n) that for $n \geqslant n_0(t)$ and for $r_n < r < r_{n+1}$ and $r \notin E_\delta$ one has

$$M(r, t) \leqslant \frac{e}{3} \mu(r) \big(\log \mu(r) + 1\big)^{1/4} \Big(\log\big(\log \mu(r) + 1\big)\Big)^{1+\delta},$$

and thus one can find an $n_1(t)$ such that

$$(13) \qquad M(r, t) < \mu(r) \big(\log \mu(r)\big)^{1/4} \big(\log\log \mu(r)\big)^{1+\delta}$$

if $r \notin E_\delta(t)$ where $E_\delta(t)$ denotes the union of the set E_δ and the set $r \leqslant r_{n_1(t)}$.

Thus to prove our theorem it is sufficient to prove that for almost all values of t there exists a number $n_0(t)$ such that (12) holds for $n \geqslant n_0(t)$.

Now let us put

$$(14) \qquad g(x) = \log f_1(e^x),$$

$$(15) \qquad A(r) = g'(\log r) = \frac{\sum\limits_{n=1}^{\infty} n\,|a_n|\,r^n}{f_1(r)}$$

and

$$(16) \qquad B^2(r) = g''(\log r) = \frac{\sum\limits_{n=1}^{\infty} n^2\,|a_n|\,r^n}{f_1(r)} - A^2(r).$$

If ξ_r is a random variable such that [2]

$$(17) \qquad \mathbf{P}(\xi_r = n) = \frac{|a_n|\,r^n}{f_1(r)} \qquad (n = 0, 1, \ldots)$$

then clearly $A(r)$ is the expectation and $B^2(r)$ the variance of ξ_r and thus by Chebishev's inequality one gets [3] for every $T > 1$

$$(18) \qquad \sum_{|n-A(r)| \geqslant B(r)T} |a_n|\,r^n \leqslant \frac{f_1(r)}{T^2}$$

and thus, choosing $T = \left(\log \mu(r)\right)^{1/8} \left(\log\log \mu(r)\right)^{(1+\delta)/2}$ we get from (10) for every $r \notin E_\delta$, putting

$$(19) \qquad C(r) = B(r)\left(\log \mu(r)\right)^{1/8}\left(\log\log \mu(r)\right)^{(1+\delta)/2},$$

$$(20) \qquad \sum_{|n-A(r)| \geqslant C(r)} |a_n|\,r^n \leqslant \mu(r)\left(\log \mu(r)\right)^{1/4}.$$

Now, in Rosenbloom's proof of (4^*) the set E_δ is defined as the set, on which $B^2(r) > \left(\log f_1(r)\right)\left(\log\log f_1(r)\right)^{2+2\delta}$; thus for $r \notin E_\delta$ and $\delta < 1/4$, using again (10) we have

$$(21) \qquad B^2(r) < 4\left(\log \mu(r)\right)^{1+\delta}$$

and therefore, in view of (20), we obtain, putting

$$(22) \qquad C_1(r) = \left(\log \mu(r)\right)^{3/4}$$

that

$$\sum_{|n-A(r)| \geqslant C_1(r)} |a_n|\,r^n < \mu(r)\left(\log \mu(r)\right)^{1/4}.$$

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[2] $\mathbf{P}(\ldots)$ denotes the probability of the event in the brackets.
[3] This is the main step in Rosenbloom's proof of (4^*).

It follows that for $r \notin E_\delta$

$$(23) \qquad M(r, t) \leqslant \mu(r) \big(\log \mu(r)\big)^{1/4} + \underset{0 \leqslant \varphi \leqslant 2\pi}{\mathrm{Max}} \Big| \sum_{|k - A(r)| < C_1(r)} a_k R_k(t) r^k e^{ik\varphi} \Big|.$$

Now we need the following lemma:

LEMMA. *Let* $b_1, b_2, \ldots, b_D$ $(D \geqslant e^{10})$ *be arbitrary complex numbers,* $\xi_1, \xi_2, \ldots, \xi_D$ *independent random variables, taking on the values* ± 1 *with probability* $\frac{1}{2}$. *Then, putting*

$$S = \sum_{k=1}^{D} |b_k|^2$$

one has

$$(24) \qquad \mathbf{P}\Big(\underset{0 \leqslant \varphi \leqslant 2\pi}{\mathrm{Max}} \Big| \sum_{k=1}^{D} b_k \xi_k e^{ik\varphi} \Big| > 2\sqrt{22 S \log D} \Big) \leqslant \frac{1}{D^4}.$$

Proof. The idea of the proof of this lemma is well known, and is due originally to S. Bernstein. Clearly, if $\gamma_1, \gamma_2, \ldots, \gamma_D$ are arbitrary complex numbers, $\gamma_k = a_k + i\beta_k$ for every $A > 0$ and $\varepsilon > 0$, in view of $|x + iy| \leqslant \sqrt{2} \max(|x|, |y|)$, one has

$$\mathbf{P}\Big(\Big| \sum_{k=1}^{D} \gamma_k \xi_k \Big| > A\Big) \leqslant 2\mathbf{P}\Big(e^{\varepsilon \sum_{a=1}^{D} a_k \xi_k} > e^{\frac{\varepsilon A}{\sqrt{2}}}\Big) + 2\mathbf{P}\Big(e^{\varepsilon \sum_{k=1}^{D} \beta_k \xi_k} > e^{\frac{\varepsilon A}{\sqrt{2}}}\Big)$$

and thus, by the Markov inequality $\mathbf{P}(\zeta > B) \leqslant E(\zeta)/B$ (valid for $B > 0$ and for every nonnegative random variable ζ with finite expectation $E(\zeta)$),

$$\mathbf{P}\Big(\Big| \sum_{k=1}^{D} \gamma_k \xi_k \Big| > A\Big) \leqslant 2e^{-\frac{\varepsilon A}{\sqrt{2}}} \Big[E\Big(e^{\varepsilon \sum_{k=1}^{D} a_k \xi_k}\Big) + E\Big(e^{\varepsilon \sum_{k=1}^{D} \beta_k \xi_k}\Big)\Big]$$

where $E(\ldots)$ denotes the expectation of the random variable in the brackets. Now clearly, using the inequality $\frac{1}{2}(e^x + e^{-x}) \leqslant e^{x^2/2}$,

$$E\Big(e^{\varepsilon \sum_{k=1}^{D} a_k \xi_k}\Big) = \prod_{k=1}^{D} \Big(\frac{e^{\varepsilon a_k} + e^{-\varepsilon a_k}}{2}\Big) \leqslant e^{\frac{\varepsilon^2}{2}(\sum_{k=1}^{D} a_k^2)}$$

and similarly

$$E\Big(e^{\varepsilon \sum_{k=1}^{D} \beta_k \xi_k}\Big) \leqslant e^{\frac{\varepsilon^2}{2}(\sum_{k=1}^{D} \beta_k^2)}$$

and thus

$$\mathbf{P}\Big(\Big| \sum_{k=1}^{D} \gamma_k \xi_k \Big| > A\Big) \leqslant 4e^{-\frac{\varepsilon A}{\sqrt{2}} + \frac{\varepsilon^2}{2}\sum_{k=1}^{D} |\gamma_k|^2}$$

Choosing

$$\mathbf{A} = \lambda \left(\sum_{k=1}^{D} |\gamma_k|^2 \right)^{1/2} \quad \text{and} \quad \varepsilon = \frac{\lambda}{\sqrt{2} \left(\sum_{k=1}^{D} |\gamma_k|^2 \right)^{1/2}}$$

it follows

$$(25) \qquad \mathbf{P} \left(\left| \sum_{k=1}^{D} \gamma_k \xi_k \right| > \lambda \left(\sum_{k=1}^{D} |\gamma_k|^2 \right)^{1/2} \right) \leqslant 4 e^{-\lambda^2/4}.$$

Now let us apply (25) for $\gamma_k = b_k e^{ik\varphi}$ with an arbitrary real value of φ; it follows

$$(26) \qquad \mathbf{P} \left(\left| \sum_{k=1}^{D} b_k \xi_k e^{ik\varphi} \right| > \lambda \sqrt{S} \right) \leqslant 4 e^{-\lambda^2/4}.$$

Let us substitute in place of φ successively the values $\varphi_j = 2\pi j/N$ $(j = 0, 1, \ldots, N-1)$ where the positive integer N is defined by the inequality

$$(27) \qquad \frac{2\pi D^{3/2}}{\lambda} \leqslant N < \frac{2\pi D^{3/2}}{\lambda} + 1.$$

It follows

$$(28) \qquad \mathbf{P} \left(\max_{0 \leqslant j \leqslant N-1} \left| \sum_{k=1}^{D} b_k \xi_k e^{jk\varphi_j} \right| > \lambda \sqrt{S} \right) \leqslant 4 \left(\frac{2\pi D^{3/2}}{\lambda} + 1 \right) e^{-\lambda^2/4}.$$

On the other hand, if $\varphi_j \leqslant \varphi < \varphi_{j+1}$

$$(29) \qquad \left| \sum_{k=1}^{D} b_k \xi_k e^{ik\varphi} - \sum b_k \xi_k e^{ik\varphi_j} \right| \leqslant \lambda \sqrt{S}.$$

From (28) and (29), putting $\lambda = \sqrt{22 \log D}$, we obtain (24).

Let us now apply the lemma to the estimation of the second term on the right of (23). As the Rademacher functions are independent with respect to the Lebesgue measure and take on the values ± 1 with probability $\frac{1}{2}$, and for $r \notin E_\delta$

$$(30) \qquad \sum_{|k-A(r)|<C_1(r)} |a_k|^2 r^{2k} \leqslant \sum_{k=0}^{\infty} |a_k|^2 r^{2k} \leqslant \mu^2(r) \left(\log \mu(r) \right)^{1/2} \left(\log\log \mu(r) \right)^{1+\delta},$$

it follows for $r \notin E_\delta$ (denoting by $V(\ldots)$ the Lebesgue measure of the set of values of t for which the condition in the brackets holds), and, taking (22) into account, that, if r is sufficiently large,

$$(31) \quad \mathbf{P} \left(\max_{0 \leqslant \varphi \leqslant 2\pi} \left| \sum_{|k-A(r)|<C_1(r)} a_k r^k R_k(t) e^{ik\varphi} \right| \geqslant \frac{1}{3} \mu(r) \left(\log \mu(r) \right)^{1/4} \left(\log\log \mu(r) \right)^{1+\delta} \right)$$

$$\leqslant \frac{1}{2^8 \left(\log \mu(r) \right)^3}.$$

Now let us apply (31) for $r = r_n$. It follows that denoting by $\mathbf{A}_n$ the set of those t for which

$$(32) \quad \underset{0 \leqslant \varphi \leqslant 2\pi}{\text{Max}} \left| \sum_{|k - A(r_n)| < C_1(r_n)} a_k r_n^k R_k(t) e^{ik\varphi} \right| > \tfrac{1}{3} \mu(r_n) \big(\log \mu(r_n)\big)^{1/4} \big(\log\log \mu(r_n)\big)^{1+\delta}$$

in view of property c) of the sequence r_n

$$\sum_{n=1}^{\infty} V(\mathbf{A}_n) < +\infty.$$

Thus by the Borel-Cantelli lemma, for almost all values of t ($0 \leqslant t \leqslant 1$) the inequality (32) can hold only for a finite number of values of n. Thus with respect to (23) for almost all t (12) holds for $n \geqslant n_0(t)$.

As was pointed out earlier, this proves our theorem.

As mentioned above, our result is best possible as regards the exponent of $\log \mu(r)$; however, it is not best possible as regards the exponent of $\log\log \mu(r)$. For instance for the case of $f(z) = e^z$ one can prove more, namely that there exist positive constants c_1 and c_2 such that for almost all values of t one has([4])

$$(33) \quad c_1 \leqslant \limsup_{r \to \infty} \frac{M(r, t)}{\mu(r) \big(\log \mu(r)\big)^{1/4} \big(\log\log \mu(r)\big)^{1/2}} \leqslant c_2.$$

To prove the lower inequality of (33), one needs the results of R. Salem and A. Zygmund contained in their paper [3], dedicated to Professor Steinhaus at his 65th birthday.

To get the upper inequality of (33) one has to notice that the proof of Theorem 1 yields also the following result, which is slightly stronger than Theorem 1.

THEOREM 2. *Let $f(z)$ be an arbitrary entire function, having the power series (1). Let $\mu(r)$ be defined by (3) and put*

$$(34) \quad S^2(r) = \sum_{n=1}^{\infty} |a_n|^2 r^{2n}$$

Let $f(z, t)$ be defined by (6) and $M(r, t)$ by (7). Then for almost all t and for $r \notin E_\delta(t)$ where $E_\delta(t)$ is a set of finite logarithmic measure, one has

$$(35) \quad M(r, t) < c_3 S(r) \big(\log\log \mu(r)\big)^{1/2}$$

where $c_3 > 0$ is a constant, not depending on r or t.

REMARK. For the case $f(z) = e^z$ we have $S(r) = O(e^r/r^{1/4})$ and $\log \mu(r) \sim r$, and thus we get for almost all t

$$(36) \quad M(r, t) < c_4 \frac{e^r \sqrt{\log r}}{\sqrt[4]{r}} \quad \text{for} \quad r \notin E_\delta(t).$$

([4]) In what follows $c_1, c_2, c_3, \ldots$ denote positive constants, not depending on r or t, but they may depend on the function $f(z)$ considered.

Moreover in this case the set $E_\delta(t)$ is an interval $0 \leqslant r \leqslant r_1(t)$. Thus the upper inequality of (33) follows. We can write (33) also in the form: for almost all t

$$(37) \qquad 0 < c_5 \leqslant \limsup_{r \to \infty} \frac{M(r, t)\sqrt[4]{r}}{e^r \sqrt{\log r}} \leqslant c_6$$

where c_5 and c_6 are positive constants.

Note that (37) is sharper than the corresponding statement in [3].

Proof of Theorem 2. The proof of Theorem 2 follows that of Theorem 1 step by step, only instead of estimating $S^2(r)$ as in (30) we express our result in terms of $S(r)$. As by (30) we have for $r \notin E_\delta$

$$S(r) < \mu(r)\big(\log\mu(r)\big)^{1/4}\big(\log\log\mu(r)\big)^{(1+\delta)2}$$

it is clear that Theorem 1 is contained in Theorem 2.

Using the mentioned results of Salem and Zygmund one can prove that for all those entire functions $f(z)$ for which there exists an $\varepsilon > 0$ such that

$$(38) \qquad S(r) \geqslant \mu(r)\big(\log\mu(r)\big)^\varepsilon \quad \text{for} \quad r \geqslant c_7$$

one has for almost all t

$$(39) \qquad \limsup_{r \to \infty} \frac{M(r, t)}{S(r)\big(\log\log\mu(r)\big)^{1/2}} \geqslant c_8 > 0.$$

It should be however mentioned that (39) does not mean that the statement of Theorem 2 cannot be improved for those functions $f(z)$ for which (38) is valid, because (39) does not exclude the possibility that those values of r for which

$$\frac{M(r, t)}{S(r)\big(\log\log\mu(r)\big)^{1/2}} > c > 0$$

are contained in a set of finite logarithmic measure. However, by restricting ourselves to the class of those entire functions for which the statement of Wiman's theorem is valid for sufficiently large values of r, our result is best possible.

Thus the following result is valid:

THEOREM 3. *Let $f(z)$ be an entire function with the power series* (1) *such that defining $M(r)$ by* (2) *and $\mu(r)$ by* (3) *one has for some $c_9 > 0$ and $c_{10} > 0$*

$$M(r) < c_9\mu(r)\big(\log\mu(r)\big)^{c_{10}}$$

for all $r \geqslant 1$. *Let* $S(r)$ *be defined by* (34) *and suppose that* (38) *holds. Let* $f(z, t)$ *be defined by* (6) *and* $M(r, t)$ *by* (7). *Then for almost all values of* t *we have*

$$(40) \qquad c_{11} \leqslant \limsup_{r \to \infty} \frac{M(r, t)}{S(r) \{\log \log \mu(r)\}^{1/2}} \leqslant c_{12}.$$

REMARK 1. Clearly Theorem 3 includes our statement (37) about the function e^z, as all the assumptions of Theorem 3 are satisfied for $f(z) = e^z$.

REMARK 2. Let us point out that there exist entire functions, having a lacunary power series such that $M(r)/\mu(r)$ is bounded; for such functions for all values of t $M(r, t)/\mu(r)$ is of course bounded too.

Finally we should like to mention that our results remain valid if instead of the Rademacher functions we multiply the terms of the series (1) by the *Steinhaus-functions*, i.e. by the functions $e^{2\pi i \vartheta_n(t)}$ where $\vartheta_1(t), \ldots, \vartheta_n(t), \ldots$ are independent functions, uniformly distributed in the interval $(0, 1)$.

REMARK ADDED IN PROOF. The random entire functions obtained by using Seinhaus factors have been studied first by P. Lévy (*Sur la croissance des fonctions entières*, Bull. Soc. Math. France 58 (1930), pp. 29-59, 127-149), who has proved for a class of entire functions, the coefficients of which satisfy certain conditions of regularity, the inequality corresponding to (35). The class considered by P. Lévy includes the function e^z. Thus the inequality corresponding to (36) (if instead of random signs Steinhaus factors are used) is due to P. Lévy who conjectured also for this variant of the problem that the lower inequality of (37) holds.

References

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