INTERSECTION THEOREMS FOR SYSTEMS OF SETS (II)

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1. *Introduction*

In this paper we present the complete solution of the problem which was considered in [**1**], with the exception of the case in which both the given cardinal numbers are finite. The results of [**1**] will not be assumed. We begin by introducing some definitions.†

A system $\Sigma_1 = (B_\nu : \nu \in N)$ of sets B_ν, where ν ranges over the index set N, is said to *contain* the system $\Sigma_0 = (A_\mu : \mu \in M)$ if, for $\mu_0 \in M$, the set A_{μ_0} occurs in Σ_1 at least as often as in Σ_0, i.e. if

$$|\{\nu : \nu \in N;\ B_\nu = A_{\mu_0}\}| \geqslant |\{\mu : \mu \in M;\ A_\mu = A_{\mu_0}\}| \quad (\mu_0 \in M).$$

If Σ_1 contains Σ_0 and, at the same time, Σ_0 contains Σ_1 then we do not distinguish between the systems Σ_0 and Σ_1. The system Σ_1 is called a *$(a, < b)$-system* if $|N| = a$ and $|B_\nu| < b$ for $\nu \in N$. The system Σ_0 is called a *$\Delta(c)$-system* if $|M| = c$ and $A_{\mu_0} A_{\mu_1} = A_{\mu_2} A_{\mu_3}$ whenever $\mu_0, \mu_1, \mu_2, \mu_3 \in M$; $\mu_0 \neq \mu_1$; $\mu_2 \neq \mu_3$. The relation

$$a \to \Delta(b, c) \tag{1}$$

means, by definition, that every $(a, < b)$-system contains a $\Delta(c)$-system. Clearly, (1) implies $a_0 \to \Delta(b_0, c_0)$ whenever $a \leqslant a_0$; $b \geqslant b_0$; $c \geqslant c_0$. The logical negation of (1) is denoted by $a \nrightarrow \Delta(b, c)$.

In [**1**] the following results were established.

THEOREM I. (i) *If $a, b \geqslant 1$ then*

$$(b^+ b^b a^{b+1})^+ \to \Delta(b^+, a^+).$$

(ii) *If $a \geqslant 2$; $b \geqslant 1$; $a+b \geqslant \aleph_0$, then*

$$(a^b)^+ \to \Delta(b^+, a^+).$$

THEOREM II. *If $a, b \geqslant 1$ then $a^{b+1} \nrightarrow \Delta(b^+, a^+)$.*

THEOREM III. *If $1 \leqslant a, b < \aleph_0$ then*

$$c^+ \to \Delta(b^+, a^+),$$

where
$$c = b!\, a^{b+1} \left(1 - \frac{1}{2!\,a} - \frac{2}{3!\,a^2} - \dots - \frac{b-1}{b!\,a^{b-1}}\right).$$

R. O. Davies [**4**] has found a very simple proof of Theorem I (ii).‡

S. Michael [**3**] has found, independently of [**1**], a proof of Theorem I (ii).

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† The cardinal of the set A is denoted by $|A|$, and set union by $A \dotplus B$ or $\Sigma(\nu \in N)\, A_\nu$, and set intersection by AB or $\Pi(\nu \in N)\, A_\nu$. $A \subset B$ denotes inclusion, in the wide sense. We use the obliteration operator ^ whose effect consists in removing from a well-ordered series the term above which it is placed. Unless the contrary is stated all sets are allowed to be empty. For every cardinal a the symbol a^+ denotes the least cardinal exceeding a.

‡ [added 9-10-1968] Karel Prikry has proved a general theorem which implies the case $a = \aleph_1$; $b = \aleph_0$; of Theorem II (ii).

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It follows from Theorem I that, given any cardinals b, $c \geqslant 1$, there always is a cardinal a such that (1) holds. We shall determine, for any given b, c such that $b+c \geqslant \aleph_0$, the least a such that (1) holds. We denote this cardinal by $f_\Delta(b, c)$. The results of [1] will not be used. Indeed, by means of lemmas 1 and 2 below we shall obtain proofs of Theorem I (ii) and of Theorem II which are simpler than those in [1]. We shall express $f_\Delta(b, c)$ in terms of sums or upper bounds of sequences of cardinals which in their turn are given explicitly in terms of b and c. Our result is stated as Theorem IV. We shall also give simpler expressions for $f_\Delta(b, c)$ which are valid when the generalized continumm hypothesis

$$2^a = a^+ \quad (a \geqslant \aleph_0) \tag{H}$$

is assumed. Our results will show that the cardinal number $f_\Delta(b, c)$ is always regular (disregarding the degenerate cases mentioned at the beginning of section 3). We should like to thank the referee for his helpful suggestions and for having pointed out some omissions in our original argument.

2. *Lemmas*

For every cardinal a we denote by $\omega(a)$ the least ordinal n whose cardinal $|n|$ equals a, i.e. the initial ordinal belonging to the cardinal a. For $a \geqslant \aleph_0$ we denote by a' the least cardinal b such that a can be expressed as the sum of b cardinals less than a. If $a = a'$ then a is called *regular*, and if $a > a'$ then a is *singular*. All our arguments are based on the " naive set theory ". Unless it is stated otherwise, small letters denote ordinals or cardinals.

LEMMA 1. *Let $c = c' > b$. Suppose that $c_0{}^{b_0} < c$ whenever $b_0 < b$ and $c_0 < c$. Then $c \to \Delta(b, c)$.*

Proof. Let $(A_\nu : \nu \in N)$ be a $(c, < b)$-system which contains no $\Delta(c)$-system. Put $\sigma(M) = \Sigma(\nu \in M)\, A_\nu (M \subset N)$. We define subsets $N_0, \ldots, \hat{N}_{\omega(b)}$ of N. Let $\lambda_0 < \omega(b)$; $N_0, \ldots, \hat{N}_{\lambda_0} \subset N$ and assume that $|N_\lambda| < c (\lambda < \lambda_0)$. Put

$$M = N_0 + \ldots + \hat{N}_{\lambda_0}.$$

We then take as N_{λ_0} a maximal subset of $N - M$ such that $A_\mu A_\nu \subset \sigma(M)$ whenever† $\{\mu, \nu\}_{\neq} \subset N_{\lambda_0}$.
It follows that

$$A_\nu \sigma(N_{\lambda_0}) \not\subset \sigma(M) \quad \text{for} \quad \nu \in N - (N_{\lambda_0} + M). \tag{2}$$

If $|N_{\lambda_0}| = c$ then we obtain a contradiction. For we have‡

$$|\lambda_0| < b < c'; \quad |M| < c;$$
$$|\sigma(M)| \leqslant \Sigma(\nu \in M)\, |A_\nu| \leqslant |M|\, b < c;$$
$$\Sigma(b_0 < b)\, |[\sigma(M)]^{b_0}| < c.$$

It follows from $c = c'$ that there are a set $M_0 \subset N_{\lambda_0}$ and a set X such that $|M_0| = c$ and $A_\nu \sigma(M) = X$ $(\nu \in M_0)$. But then $(A_\nu : \nu \in M_0)$ is a $\Delta(c)$-system which is the desired contradiction. Hence $|N_{\lambda_0}| < c$, and we have defined sets N_λ such that

† $\{x_0, \ldots, \hat{x}_n\}_{\neq}$ denotes the set $\{x_0, \ldots, \hat{x}_n\}$ and, at the same time, expresses the fact that $x_\mu \neq x_\nu$ for $\mu < \nu < n$.

‡ For every set A and every cardinal b we put $[A]^b = \{X : X \subset A;\ |X| = b\}$.

$|N_\lambda| < c(\lambda < \omega(b))$. Then $|N_0 + \ldots + \hat{N}_{\omega(b)}| < c$, and we can choose

$$v_0 \in N - (N_0 + \ldots + \hat{N}_{\omega(b)}).$$

Then, using (2), we obtain the required contradiction

$$b > |A_{v_0}| \geqslant \Sigma(\lambda_0 < \omega(b)) |A_{v_0} \sigma(N_{\lambda_0}) - A_{v_0} \sigma(N_0 + \ldots + \hat{N}_{\lambda_0})| \geqslant \Sigma(\lambda_0 < \omega(b)) \, 1 = b.$$

The following version of Lemma 1, although not required in the present paper, might be of interest. In it c need not be regular, and the conclusion is weaker than that in Lemma 1.

LEMMA 1A. *Let $c' > b$ and suppose that $c_0^{b_0} < c$ whenever $b_0 < b$ and $c_0 < c$. Then $c \to \Delta(b, c_0^+)$ for $c_0 < c$.*

The proof is very similar to that of Lemma 1 and is omitted. R. O. Davies has found an alternative proof of Lemma 1. His method seems to yield Lemma 1A as well.

LEMMA 2. *Let $n = \omega(b) \geqslant 1$ and $1 + c_\nu < c(\nu < n)$. Then*

$$\Sigma(\nu < n)\, c_0 \ldots \hat{c}_\nu \nrightarrow \Delta(b, c).$$

Proof. Let $|A_\nu| = c_\nu(\nu < n)$ and $A_\mu A_\nu = \varnothing(\mu < \nu < n)$. Let† $\{X_\lambda : \lambda \in L\}_{\neq}$ be the set of all sets $X \subset A_0 + \ldots + \hat{A}_n$ such that there is $m(X) < n$ with

$$|XA_\nu| = 1(\nu < m(X)); \; XA_\nu = \varnothing \quad (m(X) \leqslant \nu < n).$$

Then $|L| = \Sigma(\nu < n)\, c_0 \ldots \hat{c}_\nu$. Assume that

$$L' \subset L; \; |L'| = c; \; X_\lambda X_\mu = X \quad (\{\lambda, \mu\}_{\neq} \subset L').$$

We have to deduce a contradiction. We have $|X_\lambda| = |m(X_\lambda)| < b \quad (\lambda \in L')$. Let

$$X = \{x_{\mu_0}, \ldots, \hat{x}_{\mu_r}\}_{\neq}; \; \mu_0 < \ldots < \hat{\mu}_r < n; \; x_{\mu_\rho} \in A_{\mu_\rho} \quad (\rho < r).$$

The assumption‡ $\{\mu_0, \ldots, \hat{\mu}_r\} = [0, n)$ implies $|L'| \leqslant 1$ which contradicts $|L'| = c > 1$. Hence there exists the ordinal

$$\bar{\mu} = \min([0, n) - \{\mu_0, \ldots, \hat{\mu}_r\}).$$

Case 1. There is $\rho_0 < r$ such that $\bar{\mu} \leqslant \mu_{\rho_0}$. Then

$$X_\lambda A_{\bar{\mu}} \neq \varnothing \; (\lambda \in L'); \quad X_{\lambda_0} X_{\lambda_1} A_{\bar{\mu}} = \varnothing \quad (\{\lambda_0, \lambda_1\}_{\neq} \subset L').$$

Hence we obtain $|L'| \leqslant |A_{\bar{\mu}}| = c_{\bar{\mu}} < c = |L'|$, i.e. a contradiction.

Case 2. $\mu_0, \ldots, \hat{\mu}_r < \bar{\mu}$. Then $\{\mu_0, \ldots, \hat{\mu}_r\} = [0, r)$ and $r < n$. Then $X_\lambda A_r \neq \varnothing$ if $\lambda \in L'$ and $X_\lambda \neq X$; $X_{\lambda_0} X_{\lambda_1} A_r = \varnothing \quad (\{\lambda_0, \lambda_1\}_{\neq} \subset L')$. This implies

$$|L'| \leqslant 1 + |A_r| = 1 + c_r < c = |L'|,$$

a contradiction which proves Lemma 2.

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† The symbol $\{X_\lambda : \lambda \in L\}_{\neq}$ denotes the set $\{X_\lambda : \lambda \in L\}$ and, at the same time, expresses the fact that $X_\lambda \neq X_\mu$ whenever $\{\lambda, \mu\}_{\neq} \subset L$.

‡ For ordinals m, n such that $m \leqslant n$ we put $[m, n) = \{\nu : m \leqslant \nu < n\}$.

LEMMA 3. *For all cardinals $p, q \geqslant 1$ we have $q^p \nrightarrow \Delta(p^+, q^+)$.*

Proof. Let

$$n = \omega(p); \ |A_\nu| = q \ (\nu < n); \ A_\mu A_\nu = \varnothing \ (\mu < \nu < n).$$

Let $\{X_\lambda : \lambda \in L\}_{\neq}$ be the set of all $X \subset A_0 + \dots + \hat{A}_n$ such that $|XA_\nu| = 1$ $(\nu < n)$. Then the $(q^p, < p^+)$-system $(X_\lambda : \lambda \in L)$ contains no $\Delta(q^+)$-system. For, let $(X_\lambda : \lambda \in L')$ be a $\Delta(q^+)$-system, for some $L' \subset L$. Then we can choose $\{\alpha, \beta\}_{\neq} \subset L'$. Since $X_\alpha \neq X_\beta$ we have $N' = \{\nu : X_\alpha X_\beta A_\nu \neq \varnothing\} \neq [0, n)$, and there is $\nu_0 \in [0, n) - N'$. Then $X_\lambda A_{\nu_0} \neq \varnothing$ for $\lambda \in L'$, and $X_\lambda X_\mu A_{\nu_0} = X_\alpha X_\beta A_{\nu_0} = \varnothing$ for $\{\lambda, \mu\}_{\neq} \subset L'$, so that $|L'| \leqslant |A_{\nu_0}| = q < q^+ = |L'|$ which is a contradiction.

Remark. If $q \geqslant \aleph_0$ then the conclusion of Lemma 3 follows, of course, from Lemma 2.

LEMMA 4. *Let $x_0, \dots, \hat{x}_l$ be cardinals, $x_0 < \dots < \hat{x}_l$, $l = \omega(|l|)$ and $t < |l|'$. Then*

$$\left(\Sigma(\lambda < l)\, x_\lambda\right)^t \leqslant \Sigma(\lambda < l)\, x_\lambda^{2^t}.$$

Proof. Let

$$|X_\lambda| = x_\lambda (\lambda < l); \quad X_\lambda X_\mu = \varnothing \ (\lambda < \mu < l); \quad |T| = t.$$

Let $f \in (X_0 + \dots + \hat{X}_l)^T$, i.e. let f be a mapping $f : T \to X_0 + \dots + \hat{X}_l$. Then there is $\lambda_0(f) < l$ such that $f(T) \subset X_0 + \dots + \hat{X}_{\lambda 0(f)}$. Hence

$$\left(\sum_\lambda x_\lambda\right)^t \leqslant \sum_\lambda (x_0 + \dots + \hat{x}_\lambda)^t \leqslant \Sigma(|\lambda|\, x_\lambda)^t \leqslant \Sigma(x_\lambda^{\,2})^t.$$

LEMMA 5. *Let $a \geqslant \aleph_0$; $n = \omega(a)$; $m = \omega(a')$. Let $x_0, \dots, \hat{x}_n$ be cardinals such that $x_0 \leqslant \dots \leqslant \hat{x}_n$. Then there are an ordinal k and ordinals $\nu_0 < \dots < \hat{\nu}_k < n$ such that either*

$$k = m \quad \text{and} \quad x_{\nu_0} < \dots < \hat{x}_{\nu_m} \tag{3}$$

or

$$k = n \quad \text{and} \quad x_{\nu_0} = \dots = \hat{x}_{\nu_n}. \tag{4}$$

Proof. For $\mu, \nu < n$ put $\mu \equiv \nu$ whenever $x_\mu = x_\nu$. Let the equivalence classes of the relation $\mu \equiv \nu$ be $N_0, \dots, \hat{N}_p$, where p is an ordinal, $1 \leqslant p \leqslant n$. We can number the N_π in such a way that whenever $r < s < p$; $\mu \in N_r$; $\nu \in N_s$, then $\mu < \nu$ and $x_\mu < x_\nu$. If $p \geqslant m$ then we can choose $\nu_\lambda \in N_\lambda$ for $\lambda < m$, and (3) holds. Now let $p < m$. Then there is $\pi < p$ such that $|N_\pi| = a$. Then, if† $N_\pi = \{\nu_0, \dots, \hat{\nu}_n\}_<$, we have (4).

LEMMA 6. *Let‡ $a' > b'$; $b = b^-$, and suppose that $a \to \Delta(b_0, c)$ for all $b_0 < b$. Then $a \to \Delta(b, c)$.*

Proof. There is a sequence§ $b_0 < \dots < \hat{b}_m \to b$, where $m = \omega(b')$.
Let $|N| = a$ and $|A_\nu| < b$ $(\nu \in N)$. Then $N = N_0 + \dots + \hat{N}_m$, where

$$N_\mu = \{\nu : |A_\nu| < b_\mu\} \quad (\mu < m).$$

† The symbol $\{\nu_0, \dots, \hat{\nu}_n\}_<$ denotes the set $\{\nu_0, \dots, \hat{\nu}_n\}$ and expresses the fact that $\nu_\alpha < \nu_\beta$ for $\alpha < \beta < n$.

‡ We put $x^- = y$ if $x = y^+$, and $x^- = x$ if x is not of the form y^+.

§ The relation $b_0 < \dots < \hat{b}_m \to b$ means that $b_0 < \dots < \hat{b}_m$ and $\sup(\mu < m)\, b_\mu = b$.

By definition of m there is $\mu < m$ such that $|N_\mu| = |N| = a$. Since $a \to \Delta(b_\mu, c)$, the system $(A_\nu : \nu \in N_\mu)$ contains a $\Delta(c)$-system.

LEMMA 7. *If* $c > c'$ *then* $c \nrightarrow \Delta(2, c)$.

Proof. We have $c = c_0 + \ldots + \hat{c}_m$, where $m = \omega(c')$ and $c_0, \ldots, \hat{c}_m < c$. Let

$$S = S_0 + \ldots + \hat{S}_m; \quad |S_\mu| = c_\mu \ (\mu < m); \quad S_\mu S_\nu = \varnothing \ (\mu < \nu < m).$$

Put $A_{\nu x} = \{\nu\}$ $(\nu < m; x \in S_\nu)$, so that $A_{\nu x}$ is independent of x. Then the $(c, < 2)$-system $(A_{\nu x} : \nu < m;\ x \in S_\nu)$ contains no $\Delta(c)$-system.

3. *Determination of* $f_\Delta(b, c)$

For cardinals b, c we denote by $f_\Delta(b, c)$ the least a such that $a \to \Delta(b, c)$.

For the sake of the completeness of the discussion we begin by stating the values of f_Δ in the degenerate cases, which are, of course, of little interest.

$$f_\Delta(0, 0) = 0; \quad f_\Delta(0, c) = 1 \ (c \geqslant 1); \quad f_\Delta(1, c) = c \ (c \geqslant 0).$$

If $b \geqslant 2$ then $f_\Delta(b, 0) = 0$; $f_\Delta(b, 1) = 1$; $f_\Delta(b, 2) = 2$. Next, if $1 \leqslant b$, $c < \aleph_0$ then Theorem III gives what seems to be the best known upper estimate for f_Δ. In this case the determination of the exact value of $f_\Delta(b, c)$ is beyond the scope of methods known at present.

For the remainder of this paper we shall assume that

$$b \geqslant 2; \quad c \geqslant 3; \quad b + c \geqslant \aleph_0. \tag{5}$$

The number $f_\Delta(b, c)$ will turn out to be closely related to the number $s(b, c)$ defined by the equation

$$s(b, c) = \sup\,(c_0, \ldots, \hat{c}_{\omega(b)} < c)\ \Sigma\big(\nu < \omega(b)\big)\ c_0 \ldots \hat{c}_\nu. \tag{6}$$

In fact, for every choice of b and c the number $f_\Delta(b, c)$ has one of the values $s(b, c)$, $s^+(b, c)$. This means that our analysis will show that $s^+(b, c) \to \Delta(b, c)$, and $s_0 \nrightarrow \Delta(b, c)$ for every $s_0 < s(b, c)$.

Our results are summarized in the following theorem†

THEOREM IV. *Let the cardinal numbers* b, c *satisfy* (5) *and let the cardinal number* $s(b, c)$ *be defined by* (6). *Then*

(a) $$f_\Delta(b, c) = s(b, c)$$

if either (i) $b < \aleph_0 \leqslant c' = c$,

or (ii) $\aleph_0 \leqslant b^- = b < c' = c^- = c$

and $\sup\,(b_0 < b;\ c_0 < c)\, c_0^{b_0} > \sup\,(b_0 < b)\, c_1^{b_0}$ *for every* $c_1 < c$,

or (iii) $\aleph_0 \leqslant b = b_0^+ < c' \leqslant c^- = c$

and $\sup\,(c_0 < c)\, c_0^{b_0} = \big(\sup\,(c_0 < c)\, c_0^{b_0}\big)' > c_1^{b_0}$ *for every* $c_1 < c$.

(b) *In all other cases*

$$f_\Delta(b, c) = \big(s(b, c)\big)^+.$$

We note that

$$s(b, c) \geqslant \max\,(b, c). \tag{7}$$

† (a) (i) is Case 1a, (ii) is Case 2b 2c 2b 1a, and (iii) is Case 2b 1c 2a.

For, if all $c_\nu = 1$ then $\Sigma c_0 \ldots \hat{c}_\nu = b$, and if c_0 is arbitrary such that $c_0 < c$, and $c_\nu = 0$ $(\nu \geqslant 1)$ then $\Sigma c_0 \ldots \hat{c}_\nu = 1 + c_0$.

If $c \geqslant \aleph_0$ then

$$s_0 \nrightarrow \Delta(b, c) \qquad (s_0 < s). \tag{8}$$

For we can choose $c_0, \ldots, \hat{c}_{\omega(b)} < c$ such that, by Lemma 2,

$$s_0 \leqslant \Sigma c_0 \ldots \hat{c}_\nu \nrightarrow \Delta(b, c),$$

and then appeal to the monotoneity of our relation.

We shall evaluate $f_\Delta(b, c)$ without assuming the generalized continuum hypothesis (H). We shall also compute $f_\Delta(b, c)$ under the assumption of (H). To avoid tiresome repetition we shall use the relation

$$s(b, c) \overset{H}{=} d$$

to express the fact that if (H) is assumed then $s(b, c) = d$. Such relations will be stated without proof. The reader can easily supply the proofs, e.g. by referring to **[2]** § 36. Whenever the arguments of the functions s or f_Δ are the given cardinals b, c we shall write s and f_Δ instead of $s(b, c)$ and $f_\Delta(b, c)$ respectively. The symbols b_ν, c_ν, where ν is an ordinal, will always denote cardinals such that $b_\nu < b$ and $c_\nu < c$. We put

$$n = \omega(b).$$

Our discussion will follow a highly ramified scheme of classification which in the interest of clarity is presented in detail.

We use the notation

$$a^{(n} = \Sigma(\nu < n)\, a^{|\nu|},$$

where a denotes a cardinal and n an ordinal number.

Case 1. $b < \aleph_0$. Then $s = c$. For we have, for any c_ν, $\Sigma(\nu < n)\, c_0 \ldots \hat{c}_\nu \leqslant c$ and (7) completes the proof.

Case 1a. $c = c'$. Then $f_\Delta = s$.

Proof. By Lemma 1, $c \rightarrow \Delta(b, c)$. For, we have $c' \geqslant \aleph_0 > b$ and $c_0^{\,b_0} < c$.

Case 1b. $c > c'$. Then $f_\Delta = s^+$.

Proof. By Lemma 7, $c \nrightarrow \Delta(2, c)$ and hence $c \nrightarrow \Delta(b, c)$. Also, by Case 1a, $c^+ \rightarrow \Delta(b, c^+)$ and therefore $c^+ \rightarrow \Delta(b, c)$.

Case 2. $b \geqslant \aleph_0$.

Case 2a. $c = c_0^{\,+}$. Then, clearly, $s = c_0^{(n}$. Also, $s \overset{H}{=} b$ if $c_0 < \aleph_0$; $s \overset{H}{=} c_0$ if $b \leqslant c_0'$; $s \overset{H}{=} c$ if $c_0' < b \leqslant c_0$; $s \overset{H}{=} b$ if $\aleph_0 \leqslant c_0 < b$.

Case 2a1. $b = b^-$. Then $f_\Delta = s^+$.

Proof. We begin by showing that

$$s \nrightarrow \Delta(b, c). \tag{9}$$

If $c \geqslant \aleph_0$ then, by Lemma 2, $s = c_0^{(n} \nrightarrow \Delta(b, c)$. Now let $c < \aleph_0$. By Lemma 3 we have, for $\nu < n$, $2^{|\nu|} \nrightarrow \Delta(|\nu|^+, 3)$. Hence there is a $(2^{|\nu|}, < |\nu|^+)$-system

$(A_{\nu\lambda} : \lambda \in L_\nu)$ which does not contain any $\Delta(3)$-system. Choose any distinct objects $x_0, \ldots, \hat{x}_n, y_0, \ldots, \hat{y}_n$ outside $\Sigma(\nu < n; \lambda \in L_\nu) A_{\nu\lambda}$ and put

$$B_{\nu\lambda} = \{x_0, \ldots, \hat{x}_\nu, y_\nu\} + A_{\nu\lambda} \quad (\nu < n; \lambda \in L_\nu).$$

Then (9) follows if we can show that $(B_{\nu\lambda} : \nu < n; \lambda \in L_\nu)$ is a $(s, < b)$-system which does not contain any $\Delta(3)$-system. Clearly

$$|\{(\nu, \lambda) : \nu < n; \lambda \in L_\nu\}| = 2^{(n} = s.$$

Also, $|B_{\nu\lambda}| < b$. Now let $\nu_0 \leqslant \nu_1 \leqslant \nu_2 < n$, and $\lambda_\rho \in L_{\nu_\rho}$ for $\rho < 3$. Suppose that the three pairs (ν_ρ, λ_ρ) are distinct and that (D_0, D_1, D_2) is a $\Delta(3)$-system, where $D_\rho = B_{\nu_\rho \lambda_\rho}$ for $\rho < 3$. If $\nu_0 = \nu_1 = \nu_2$, then $(A_{\nu_0 \lambda_\rho} : \rho < 3)$ is a $\Delta(3)$-system which contradicts the definition of the $A_{\nu_0 \lambda}$.

If $\nu_0 = \nu_1 < \nu_2$, then $y_{\nu_0} \in D_0 D_1 - D_1 D_2$ which is false.

If $\nu_0 < \nu_1 = \nu_2$, then $y_{\nu_1} \in D_1 D_2 - D_0 D_1$ which is false.

Hence $\nu_0 < \nu_1 < \nu_2$. But then $x_{\nu_0} \in D_1 D_2 - D_0 D_1$ which is false. This proves (9). Next, we prove that

$$s^+ \rightarrow \Delta(b, c). \tag{10}$$

Case 2a1a. $b \geqslant \aleph_1$. By (7), $(s^+)' = s^+ > b \geqslant b'$. Hence, by Lemma 6, (10) follows from

$$s^+ \rightarrow \Delta({b_0}^+, c) \qquad (\aleph_0 \leqslant b_0 < b). \tag{11}$$

Since $s \geqslant {c_0}^{b_0}$, (11) follows from

$$({c_0}^{b_0})^+ \rightarrow \Delta({b_0}^+, c) \qquad (\aleph_0 \leqslant b_0 < b). \tag{12}$$

Since $({c_0}^{b_0})^+ \geqslant {c_0}^+ = c$, (12) follows from

$$({c_0}^{b_0})^+ \rightarrow \Delta({b_0}^+, ({c_0}^{b_0})^+) \qquad (\aleph_0 \leqslant b_0 < b). \tag{13}$$

But (13) follows from Lemma 1. For we have, if $\aleph_0 \leqslant b_0 < b$,

$$(({c_0}^{b_0})^+)' = ({c_0}^{b_0})^+ > {c_0}^{b_0} \geqslant {b_0}^+$$

and

$$({c_0}^{b_0})^{b_0} = {c_0}^{b_0} < ({c_0}^{b_0})^+.$$

This proves (10).

Case 2a1b. $b = \aleph_0$.

Case 2a1b1. $c < \aleph_0$. Then $s = \aleph_0$. By Lemma 1, $s^+ = \aleph_1 \rightarrow \Delta(\aleph_0, \aleph_1)$, and this implies (10).

Case 2a1b2. $c \geqslant \aleph_0$. Then $s = c_0$. By Lemma 1,

$$s^+ = {c_0}^+ \rightarrow \Delta(\aleph_0, {c_0}^+),$$

which is (10).

Case 2a2. $b = {b_0}^+$. Then $s = {c_0}^{b_0}$ and $f_\Delta = s^+$. Also, $s \overset{H}{=} c_0$ if $b_0 < c_0'$; $s \overset{H}{=} c$ if $c_0' \leqslant b_0 \leqslant c_0$; $s \overset{H}{=} b$ if $c_0 < b_0$.

Proof. Clearly, $s = {c_0}^{(n} \leqslant b{c_0}^{b_0} = {c_0}^{b_0} \leqslant s$. By Lemma 3, $s = {c_0}^{b_0} \nrightarrow \Delta(b, c)$. By Lemma 1,

$$({c_0}^{b_0})^+ \rightarrow \Delta(b, ({c_0}^{b_0})^+). \tag{14}$$

For, we have $(c_0^{b_0})^+ > c_0^{b_0} \geqslant b$ and $(c_0^{b_0})^{b_0} = c_0^{b_0} < (c_0^{b_0})^+$. By (7), (14) implies $s^+ \to \Delta(b, c)$.

Case 2b. $c = c^-$.

Case 2b1. $b = b_0^+$.

Case 2b1a. $c' = b$. Then we can choose a sequence $0 < c_0 < \ldots < \hat{c}_n \to c$. Then $s = \Sigma(\nu < n)\, c_0 \ldots \hat{c}_\nu$; $f_\Delta = s^+$; $s \overset{H}{=} c$.

Proof. Let $x_0, \ldots, \hat{x}_n < c$. Then we can find inductively numbers

$$f(0) < \ldots < \hat{f}(n) < n$$

such that $x_\nu < c_{f(\nu)}$ $(\nu < n)$. Then $\Sigma(\nu < n)\, x_0 \ldots \hat{x}_\nu \leqslant \Sigma c_0 \ldots \hat{c}_\nu$. Hence, by Lemma 2,

$$s = \Sigma c_0 \ldots \hat{c}_\nu \nrightarrow \Delta(b, c).$$

We now prove

$$s^+ \to \Delta(b, c). \tag{15}$$

By (7), (15) follows from

$$s^+ \to \Delta(b, s^+). \tag{16}$$

We have $(s^+)' = s^+ > s \geqslant b$. Hence, by Lemma 1, (16) follows from

$$s^{b_0} \leqslant s. \tag{17}$$

Put $c_0 \ldots \hat{c}_\nu = p_\nu$ $(\nu < n)$.

Case 2b1a1. There is a sequence $\nu_0 < \ldots < \hat{\nu}_n < n$ with $p_{\nu_0} < \ldots < \hat{p}_{\nu_n}$. Then

$$s = \Sigma(\nu < n)\, p_\nu \leqslant \Sigma(\lambda < n)\, |\nu_\lambda|\, p_{\nu_\lambda} \leqslant \Sigma(\lambda < n)\, p_{\nu_\lambda} \leqslant s.$$

By Lemma 4,

$$s^{b_0} = (\Sigma p_{\nu_\lambda})^{b_0} \leqslant \Sigma p_{\nu_\lambda}^{2b_0} \leqslant \Sigma p_\nu{}^{b_0}. \tag{18}$$

Put $\omega(b_0) = m$ and consider the sequence $d_0, \ldots, \hat{d}_n$, where $d_{m\rho+\mu} = c_\rho (\mu < m;\ \rho < n)$. By definition of s,

$$\Sigma(\nu < n)\, p_\nu{}^{b_0} \leqslant \Sigma(\nu < n)\, d_0 \ldots \hat{d}_\nu \leqslant s.$$

Hence (18) implies (17).

Case 2b1a2. There is no sequence $\nu_0 < \ldots < \hat{\nu}_n < n$ with $p_{\nu_0} < \ldots < \hat{p}_{\nu_n}$. Then, by Lemma 5, there is a sequence $0 < \nu_0 < \ldots < \hat{\nu}_n < n$ such that $p_{\nu_0} = \ldots = \hat{p}_{\nu_n} = p$, say. Then $p_\nu = p$ $(\nu_0 \leqslant \nu < n)$;

$$s = \Sigma(\nu < \nu_0)\, p_\nu + \Sigma(\nu_0 \leqslant \nu < n)\, p \leqslant bp.$$

Also, $bp = \Sigma(\nu_0 \leqslant \nu < n)\, p_\nu \leqslant s$. Hence $s = bp$. We have $p \geqslant c_\nu$ $(\nu < n)$ and so $p \geqslant c \geqslant c' = b$; $s = p$.
Consider the sequence $d_0, \ldots, \hat{d}_n$ defined by

$$d_{\nu_0 \nu + \mu} = c_\mu \qquad (\mu < \nu_0;\ \nu < n).$$

By definition of s,

$$s^{b_0} = (c_0 \ldots \hat{c}_{\nu_0})^{b_0} \leqslant \Sigma(\nu < n)\, d_0 \ldots \hat{d}_\nu \leqslant s.$$

This again proves (17).

Case 2b1b. $c' < b$. Then

$$s = c^{(n}; \quad f_\Delta = s^+; \quad s \overset{H}{=} c^+.$$

Proof. Choose $0 < c_0 < \ldots < \hat{c}_m \to c$, where $m = \omega(c')$, and put

$$d_{m\nu+\mu} = c_\mu \quad (\mu < m;\ \nu < n).$$

Let $x_0, \ldots, \hat{x}_n < c$, and choose any $\nu < n$. Then we can find inductively numbers $f_\nu(0) < \ldots < \hat{f}_\nu(m) < m$ such that $x_{m\nu+\mu} \leqslant d_{m\nu+f_\nu(\mu)}$ $(\mu < m;\ \nu < n)$. We define f by putting $f(m\nu+\mu) = m\nu+f_\nu(\mu)$ $(\mu < m;\ \nu < n)$. Then $f(0) < \ldots < \hat{f}(n) < n$ and $x_\nu \leqslant d_{f(\nu)}$ $(\nu < n)$. Hence

$$\Sigma(\nu < n)\, x_0 \ldots \hat{x}_\nu \leqslant \Sigma d_0 \ldots \hat{d}_\nu.$$

This proves $s = \Sigma d_0 \ldots \hat{d}_\nu$. Put $p_\nu = d_0 \ldots \hat{d}_\nu$ $(\nu < n)$. We shall make use of the fact that $p_m = c^{c'}$ ([**2**], p. 141, Satz 6). We have

$$\begin{aligned} s &= \Sigma(\nu < n)\, \Sigma(m\nu \leqslant \lambda < m\nu+m)\, p_\lambda \leqslant \Sigma(\nu < n)\, |m|\, p_{m\nu+m} \\ &= \Sigma(\nu < n)\, c'(c^{c'})^{|\nu+1|} \leqslant \Sigma(\nu < n)\, c^{c'|\nu|} \\ &= \Sigma(\nu < m)\, c^{c'} + \Sigma(m \leqslant \nu < n)\, c^{|\nu|} \leqslant c^{(n} \\ &\leqslant \Sigma(\nu < n)(c_0 \ldots \hat{c}_m)^{|\nu|} \leqslant \Sigma(\nu < n)\, d_0 \ldots \hat{d}_\nu \leqslant s. \end{aligned}$$

This shows that, by Lemma 2,

$$s = c^{(n} = \Sigma d_0 \ldots \hat{d}_\nu \nrightarrow \Delta(b, c).$$

Finally, by Lemma 1, $s^+ \to \Delta(b, s^+)$. For, we have $(s^+)' > s \geqslant b$;

$$s = c^{(n} \leqslant bc^{b_0} = c^{b_0} \leqslant s,$$

$$s^{b_0} = c^{b_0\, b_0} = c^{b_0} = s < s^+.$$

Now (7) yields $s^+ \to \Delta(b, c)$.

Case 2b1c. $c' > b$. Then $s = \sup\,(c_* < c) c_*^{\ b_0} \overset{H}{=} c$.

Proof. Let $c_0, \ldots, \hat{c}_n < c$. Then there is $\bar{c}$ such that $2, c_0, \ldots, \hat{c}_n \leqslant \bar{c} < c$ and hence $\Sigma(\nu < n)\, c_0 \ldots \hat{c}_\nu \leqslant b\bar{c}^{b_0} = \bar{c}^{b_0}$. Therefore $s \leqslant \sup\,(c_* < c)\, c_*^{b_0} = \sigma$, say. If $x_\nu = c_* < c$ $(\nu < n)$, then $s \geqslant \Sigma x_0 \ldots \hat{x}_\nu \geqslant c_*^{b_0}$. Hence $s \geqslant \sigma$ and so $s = \sigma$.

Case 2b1c1. There is $2 \leqslant c_1 < c$ such that $s = c_1^{\ b_0}$. Then $f_\Delta = s^+$.

Proof. By Lemma 3, $c_1^{\ b_0} \nrightarrow \Delta(b, c_1^{\ +})$. Hence $s \nrightarrow \Delta(b, c)$. By Lemma 1, $s^+ \to \Delta(b, s^+)$. For, we have $(s^+)' > s \geqslant b$; $s^{b_0} = c_1^{\ b_0\, b_0} = s < s^+$. By (7) we deduce $s^+ \to \Delta(b, c)$.

Case 2b1c2. $s > c_0^{\ b_0}$ $(c_0 < c)$.

Case 2b1c2a. $s = s'$. Then $f_\Delta = s$.

Proof. By Lemma 1, $s \to \Delta(b, s)$. For, if $s_0 < s$ then there is $c_0 < c$ such that $s_0^{\ b_0} \leqslant (c_0^{\ b_0})^{b_0} = c_0^{\ b_0} < s$. Also, using (7) we find $s' = s \geqslant c \geqslant c' > b$, so that Lemma 1 applies and gives $s \to \Delta(b, s)$. By (7) we deduce $s \to \Delta(b. c)$, and (8) completes the proof.

Case 2b1c2b. $s > s'$. Then $f_\Delta = s^+$.

Proof. If we assume that $s > c$ then there is $c_0 < c$ such that $c_0^{\ b_0} \geqslant c$. Then, for every $c_1 < c$, we have $c_1^{\ b_0} \leqslant c^{b_0} \leqslant (c_0^{\ b_0})^{b_0} = c_0^{\ b_0}$ and hence $s \leqslant c_0^{\ b_0} < s$ which is a contradiction. Hence, by Lemma 7, $s = c \nrightarrow \Delta(2, c)$ and therefore $s \nrightarrow \Delta(b, c)$. Also,

$$(c^+)' = c^+ > c' > b; \quad c = x_0 + \ldots + \hat{x}_l; \quad l = \omega(c'); \quad x_0 < \ldots + \hat{x}_l < c;$$
$$b_0 < b < c' = |l| = |l|'.$$

Hence, by Lemma 4,

$$c^{b_0} \leqslant \Sigma(\lambda < l)\, x_\lambda^{2b_0} \leqslant |l|\, s = c < c^+.$$

Now Lemma 1 gives $c^+ \rightarrow \Delta(b, c^+)$ and so $s^+ \rightarrow \Delta(b, c)$.

Case 2b2. $b = b^-$.

Case 2b2a. $c' = b$. Choose $0 < c_0 < \ldots < \hat{c}_n \rightarrow c$. Then

$$s = \Sigma(\nu < n)\, c_0 \ldots \hat{c}_\nu; \quad f_\Delta = s^+; \quad s \overset{H}{=} c.$$

Proof. If $x_0, \ldots, \hat{x}_n < c$ then there is a sequence $f(0) < \ldots < \hat{f}(n) < n$ such that $x_\nu \leqslant c_{f(\nu)}$ $(\nu < n)$. Then $\Sigma x_0 \ldots \hat{x}_\nu \leqslant \Sigma c_0 \ldots \hat{c}_\nu$ and therefore, by Lemma 2, $s = \Sigma c_0 \ldots \hat{c}_\nu \nrightarrow \Delta(b, c)$. We now prove

$$s^+ \rightarrow \Delta(b, c). \tag{19}$$

By (7), (19) follows from

$$s^+ \rightarrow \Delta(b, s^+). \tag{20}$$

By Lemma 1, (20) follows from

$$s^{b_0} \leqslant s \qquad (b_0 < b). \tag{21}$$

Let $b_0 < b$ and put $c_0 \ldots \hat{c}_\nu = p_\nu$ $(\nu < n)$.

Case 2b2a1. There is $\nu_0 < \ldots < \hat{\nu}_n < n$ such that $p_{\nu_0} < \ldots \hat{p}_{\nu_n}$. Then

$$s = \Sigma(\nu < n)\, p_\nu \leqslant \Sigma(\lambda < n)\, |\nu_\lambda|\, p_{\nu_\lambda} \leqslant \Sigma(\lambda < n)\, p_{\nu_\lambda} \leqslant s.$$

By Lemma 4, which applies since $b_0 < b = c' = c'' = b' = |n|'$,

$$s^{b_0} = (\Sigma p_{\nu_\lambda})^{b_0} \leqslant \Sigma p_{\nu_\lambda}^{2b_0} \leqslant \Sigma p_\nu^{b_0}. \tag{22}$$

Put $\omega(b_0) = m$ and $d_{m\nu+\mu} = c_\nu$ $(\mu < m;\ \nu < n)$. Then, by definition of s,

$$\Sigma(\nu < n)\, p_\nu{}^{b_0} \leqslant \Sigma d_0 \ldots \hat{d}_\nu \leqslant s.$$

Hence (22) implies (21), and (19) is proved.

Case 2b2a2. There is no $\nu_0 < \ldots < \hat{\nu}_n < n$ such that $p_{\nu_0} < \ldots < \hat{p}_{\nu_n}$. Then, by Lemma 5, there is $0 < \nu_0 < \ldots < \hat{\nu}_n < n$ such that $p_{\nu_0} = \ldots = \hat{p}_{\nu_n} = p$, say. Then

$$\begin{aligned} &p_\nu = p \quad (\nu_0 \leqslant \nu < n); \\ &s = \Sigma(\nu < \nu_0)\, p_\nu + \Sigma(\nu_0 \leqslant \nu < n)\, p_\nu \leqslant |\nu_0|\, p + |n|\, p \\ &\quad = bp = \Sigma(\nu_0 \leqslant \nu < n)\, p_\nu \leqslant s; \\ &s = bp; \quad p \geqslant c_\nu \ (\nu < n); \quad p \geqslant c \geqslant c' = b; \quad s = p. \end{aligned}$$

Put $d_{\nu_0 \nu + \lambda} = c_\lambda$ $(\nu < n;\ \lambda < \nu_0)$. Then, by definition of s,

$$(c_0 \ldots \hat{c}_{\nu_0})^{b_0} \leqslant \Sigma(\nu < n)\, d_0 \ldots \hat{d}_\nu \leqslant s; \quad s^{b_0} = p^{b_0} \leqslant s.$$

Hence (21) follows and (19) is proved.

Case 2b2b. $c' < b$. Then

$$s = c^{(n}; \quad f_\Delta = s^+; \quad s \overset{H}{=} c^+.$$

Proof. Let $m = \omega(c')$, and choose a sequence $0 < c_0 < \ldots < \hat{c}_m \to c$. Put $d_{m\nu+\mu} = c_\mu \quad (\mu < m;\ \nu < n)$. Let $x_0, \ldots, \hat{x}_n < c$. Then, for every $\nu < n$, we can find inductively numbers $f_\nu(0) < \ldots < \hat{f}_\nu(m) < m$ such that

$$x_{m\nu+\mu} \leqslant d_{m\nu+f_\nu(\mu)} \quad (\mu < m;\ \nu < n).$$

Define f by putting $f(m\nu+\mu) = m\nu+f_\nu(\mu) \quad (\mu < m;\ \nu < n)$. Then

$$f(0) < \ldots < \hat{f}(n) < n$$

and $$x_\nu \leqslant d_{f(\nu)} \quad (\nu < n); \qquad \Sigma(\nu < n)\, x_0 \ldots \hat{x}_\nu \leqslant \Sigma d_0 \ldots \hat{d}_\nu.$$

Thus $s = \Sigma d_0 \ldots \hat{d}_\nu$. Put $p_\nu = d_0 \ldots \hat{d}_\nu \ (\nu < n)$. Then

$$\begin{aligned} s &= \Sigma(\nu < n)\, \Sigma(m\nu \leqslant \lambda < m\nu+m)\, p_\lambda \leqslant \Sigma(\nu < n)\, |m|\, p_{m\nu+m} \\ &= \Sigma(\nu < n)\, c'(c^{c'})^{|\nu+1|} \leqslant \Sigma(\nu < n)\, c^{|\nu| c'} \\ &= \Sigma(\nu < m)\, c^{c'} + \Sigma(m \leqslant \nu < n)\, c^{|\nu|} \leqslant c^{(n}. \end{aligned}$$

On the other hand, using $c_0 \ldots \hat{c}_m = c^{c'} > c$, we find

$$c^{(n} \leqslant \Sigma(\nu < n)(c_0 \ldots \hat{c}_m)^{|\nu|} \leqslant \Sigma(\nu < n)\, d_0 \ldots \hat{d}_\nu = s.$$

Hence, by Lemma 2, $s = c^{(n} = \Sigma d_0 \ldots \hat{d}_\nu \nrightarrow \Delta(b, c)$. We now prove

$$s^+ \to \Delta(b, c). \tag{23}$$

We recall that s always stands for the number $s(b, c)$. By Lemma 6, (23) follows from

$$s^+ \to \Delta(b_1{}^+, c) \quad (c' < b_1 < b). \tag{24}$$

Next, by the monotoneity property of s, (24) follows from

$$s^+(b_1{}^+, c) \to \Delta(b_1{}^+, c) \quad (c' < b_1 < b). \tag{25}$$

But (25) follows from case 2b1b, and (23) is established.

Case 2b2c. $c' > b$. Then†

$$s = \sup(b_0 < b;\ c_0 < c)\, c_0{}^{b_0} \overset{H}{=} c.$$

Proof. Let $x_0, \ldots, \hat{x}_n < c$. Then there is c_1 such that $x_0, \ldots, \hat{x}_n < c_1 < c$. Put $\sigma = \sup(b_0 < b;\ c_0 < c)\, c_0{}^{b_0}$. Then

$$\Sigma(\nu < n)\, x_0 \ldots \hat{x}_\nu \leqslant c_1{}^{(n} \leqslant b\sigma.$$

Also, $\sigma \geqslant 2^{|\nu|} > |\nu| \quad (\nu < n)$; $\sigma \geqslant b$, so that $s \leqslant \sigma$. If $s < \sigma$ then there are b_0 and c_0 such that $s < c_0{}^{b_0}$. Put $y_\nu = c_0 \ (\nu < n)$. Then $s < c_0{}^{b_0} \leqslant \Sigma(\nu < n)\, y_0 \ldots \hat{y}_\nu \leqslant s$ which is a contradiction. Hence $s = \sigma$.

Case 2b2c1. There are b_0 and c_0 such that $s = c_0{}^{b_0}$. Then $f_\Delta = s^+$.

Proof. By Lemma 3, $s = c_0{}^{b_0} \nrightarrow \Delta(b_0{}^+, c_0{}^+)$ and hence $s \nrightarrow \Delta(b, c)$. By Lemma 1, $s^+ \to \Delta(b, s^+)$. For, we have $(s^+)' > s \geqslant b$, and if $b_1 < b$ then

$$s^{b_1} = c_0{}^{b_0 b_1} \leqslant s < s^+.$$

† This value of s remains valid for the remainder of the paper.

We now conclude that $s^+ \to \Delta(b, c)$.

Case 2b2c2. $s > c_0{}^{b_0}$ $(b_0 < b;\ c_0 < c)$. Put $k = \omega(s')$. Then there are sequences $b_0 \leqslant \ldots \leqslant \hat{b}_k < b$ and $c_0 \leqslant \ldots \leqslant \hat{c}_k < c$ such that $c_0{}^{b_0} < \ldots < \hat{c}_k{}^{b_k} \to s$.

Case 2b2c2a. There is $\kappa_0 < k$ such that $c_{\kappa_0} = \ldots = \hat{c}_k = \bar{c}$, say. Then $f_\Delta = s^+$.

Proof. $\sup(\kappa < k)\, \bar{c}^{b_\kappa} = s$. There is $\bar{b}$ such that

$$b_{\kappa_0} < b_{\kappa_0+1} < \ldots < \hat{b}_k \to \bar{b} \leqslant b.$$

If $\bar{b} < b$ then $\bar{c}^{b_\kappa} \leqslant \bar{c}^{\bar{b}} < s$ $(\kappa < k)$ which is false. Hence $\bar{b} = b$; $|k| = b'$; $s' = b'$.
We have $\bar{c}^{(n} \geqslant \bar{c}^{|\nu|}$ $(\nu < n)$ and hence $\bar{c}^{(n} \geqslant s$.
On the other hand, $\bar{c}^{(n} \leqslant \Sigma(\nu < n)\, s = bs = s$.
Hence, by Lemma 2, $s = \bar{c}^{(n} \nrightarrow \Delta(b, c)$.
We now prove

$$s^+ \to \Delta(b, c). \tag{26}$$

We have $(s^+)' > s' = b'$. Hence, by Lemma 6, (26) follows from

$$s^+ \to \Delta(b_*^+, c) \quad (b_* < b). \tag{27}$$

Choose any $b_* < b$. Then, by Case 1 if $b_* < \aleph_0$ and by Case 2b1c if $b_* \geqslant \aleph_0$, $f_\Delta(b_*^+, c) \leqslant s^+(b_*^+, c) \leqslant s^+$. This implies (27) and therefore (26).

Case 2b2c2b. There is no $\kappa_0 < k$ such that $c_{\kappa_0} = \ldots = \hat{c}_k$. Then, by Lemma 5, there is $\kappa_0 < \ldots < \hat{\kappa}_k < k$ such that

$$c_{\kappa_0} < \ldots < \hat{c}_{\kappa_k} \to \bar{c} \leqslant c.$$

Case 2b2c2b1. $\bar{c} = c$. Then $s = c$.

Proof. We have $|k| \geqslant c'$. If $s > c$, then there are d_0, e_0 such that

$$d_0 < b; \quad e_0 < c; \quad e_0{}^{d_0} \geqslant c.$$

Then, for $d_1 < b$ and $e_1 < c$, we have

$$e_1{}^{d_1} \leqslant c^{d_1} \leqslant e_0{}^{d_0 d_1} \leqslant \sup(d_2 < b)\, e_0{}^{d_2} \leqslant s$$

and therefore $s = \sup(d_2 < b)\, e_0{}^{d_2}$. This implies the contradiction

$$s' \leqslant b' \leqslant b < c' \leqslant |k| = s'.$$

We have thus proved that $s = c$. Let $d_0 < b$. Then, by Case 1 or Case 2b1,

$$f_\Delta(d_0{}^+, c) \leqslant s^+(d_0{}^+, c) \leqslant s^+.$$

Hence $c^+ \to \Delta(d_0{}^+, c)$ $(d_0 < b)$, and we deduce from Lemma 6 that

$$c^+ \to \Delta(b, c). \tag{28}$$

Case 2b2c2b1a. $c = c'$. Then $f_\Delta = s = c$.

Proof. $c = c' > b$. If $d_0 < b$ and $e_0 < c$ then $e_0{}^{d_0} < s = c$. Hence, by Lemma 1, $c \to \Delta(b, c)$.

Case 2b2c2b1b. $c > c'$. Then $f_\Delta = s^+ = c^+$.

Proof. By Lemma 7, $c \nrightarrow \Delta(2, c)$ and so $c \nrightarrow \Delta(b, c)$. Now (28) completes the proof.

Case 2b2c2b2. $\bar{c} < c$. Then

$$s = \sup (b_* < b)\, \bar{c}^{b_*}; \quad f_\Delta = s^+.$$

Proof. By Lemma 2,

$$s = \sup (\kappa < k)\, c_\kappa{}^{b_\kappa} \leqslant \sup (\kappa < k)\, \bar{c}^{b_\kappa} \leqslant \bar{c}^{(n} \nrightarrow \Delta(b, c).$$

Hence $s \nrightarrow \Delta(b, c)$. We now prove

$$s^+ \rightarrow \Delta(b, c). \tag{29}$$

In view of Lemma 6 and the relations $(s^+)' > s \geqslant b$, (29) follows from

$$s^+ \rightarrow \Delta(b_*^+, c) \qquad (b_* < b). \tag{30}$$

By Case 1 or Case 2b1,

$$f_\Delta(b_*^+, c) \leqslant s^+(b_*^+, c) \leqslant s^+ \qquad (b_* < b).$$

This proves (30) and hence (29).

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