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A PROBLEM ON WELL QRDERED SETS

By

P. ERDŐS (Budapest), member of the Academy, A. HAJNAL (Budapest) and

E. C. MTLNER (Calgary)

To Professor G. Alexits on his 70th birthday

1. Introduction. In this paper we settle one of the questions left open in [1] concerning the symbol

$$\alpha \Rightarrow [\beta, \gamma]_m. \tag{1}$$

By definition, (1) means that the following statement is true: *If S is well ordered set of order type α and if $\mathscr{F} = (F_\mu : \mu \in M)$ is any family of $m = |M|$ subsets of S such that each F_μ $(\mu \in M)$ has order type less than β, then S contains a subset C of type γ which is disjoint from m sets F_μ of the family $\mathscr{F}$*, i.e.

$$|\{\mu : \mu \in M; \ F_\mu \cap C = \varnothing\}| = m.$$

The set C is said to be $(\mathscr{F}, m)$-free. The negation of (1) is written as

$$\alpha \not\Rightarrow [\beta, \gamma]_m.$$

We proved ([1], Theorem 10.0) that

$$\omega_{\nu+2}\alpha \Rightarrow [\omega_{\nu+1}^{\omega}, \omega_{\nu+2}\alpha]_{\aleph_{\nu+2}} \qquad (\alpha < \omega_{\nu+1}). \tag{2}$$

So that, in particular,

$$\omega_2 \alpha \Rightarrow [\omega_1^{\omega}, \omega_2 \alpha]_{\aleph_2} \tag{3}$$

holds for all $\alpha < \omega_1$. The condition $\alpha < \omega_{\nu+1}$ in (2) is necessary since, for example ([1], Theorem 10. 1) assuming $2^{\aleph_1} = \aleph_2$,

$$\omega_2\omega_1 \not\Rightarrow [\omega_1 + 1, \omega_2\omega_1]_{\aleph_2}.$$

By using a result of [2] on set mappings (see [1], Theorem 6.2) it is very easily seen that

$$\omega_2 n \Rightarrow [\beta, \omega_2 n]_{\aleph_2} \qquad (n < \omega; \ \beta < \omega_2),$$

and this is stronger than (3) when $\alpha < \omega$. We asked in [1] (Problem 5) whether (3) is best possible when $\alpha = \omega$, i.e. does

$$\omega_2 \omega \not\Rightarrow [\omega_1^{\omega} + 1, \omega_2 \omega]_{\aleph_2} \tag{4}$$

hold?

Using the generalized continuum hypothesis (more precisely, using $2^{\aleph_1} = \aleph_2$) we can now show that (4) holds. In fact, the following theorem shows that (3) is best possible in the sense that ω_1^{ω} cannot be replaced by any larger ordinal.

THEOREM. *If $2^{\aleph_1} = \aleph_2$ and $\omega \leqq \alpha < \omega_1$, then*

$$\omega_2 \alpha \not\to [\omega_1^\omega + 1, \omega_2 \alpha]_{\aleph_2}. \tag{5}$$

2. Notation and preliminary results. Capital letters denote sets and small letters denote ordinal numbers unless stated otherwise. The cardinal of X is $|X|$. The obliterator sign ^ written above a symbol means that that symbol should be disregarded. For example,

$$\{x_0, \ldots, \hat{x}_\alpha\} = \{x_\nu : \nu < \alpha\}.$$

We write $S = \{x_0, \ldots, \hat{x}_\alpha\}_\lhd$ if the set $S = \{x_0, \ldots, \hat{x}_\alpha\}$ is simply ordered by $\lhd$ so that $x_\mu \lhd x_\nu$ for $\mu < \nu < \alpha$. For any α, β we write $[\alpha, \beta) = \{\nu : \alpha \leqq \nu < \beta\}$.

The order type of the well ordered set A is denoted by tp A. If the sets A_ν $(\nu < \alpha)$ are disjoint and ordered, we write

$$S = A_0 \cup \ldots \cup \hat{A}_\alpha \, (\mathrm{tp})$$

to indicate that S is the union of the A_ν and also that S is ordered in such a way that the order relations in each A_ν are preserved and $x \lhd y$ if $x \in A_\mu$, $y \in A_\nu$ and $\mu < \nu < \alpha$. T is a *cofinal* subset of the ordered set S if for each $x \in S$ there is some $y \in T$ so that $x \leqq y$. For $\alpha > 0$, co (α) denotes the smallest ordinal β such that $[0, \alpha)$ contains a cofinal subset of type β. Thus co (α) is either 1 or an initial ordinal. If α is such that $\beta + \gamma < \alpha$ whenever $\beta < \alpha$ and $\gamma < \alpha$, then α is said to be ***indecomposable.*** The indecomposable ordinals are 0, 1 and powers of ω.

An ordinal valued function f defined on the set of ordinal numbers A is *regressive* if $f(\alpha) < \alpha$ $(\alpha \in A;\ \alpha \neq 0)$. $B \subset A$ is ***closed*** (w.r.t. A) if B contains the limit of any increasing sequence of elements of B which is also in A. $S \subset [0, \omega_\alpha)$ is ***stationary*** if $[0, \omega_\alpha) - S$ does not contain a closed subset cofinal with $[0, \omega_\alpha)$. It is easily seen (see [3]) that the set

$$\{\alpha : \alpha < \omega_2;\ \mathrm{co}(\alpha) = \omega_1\}$$

is stationary. It is well known that if $\aleph_\alpha (> \aleph_0)$ is regular and f is a regressive function defined on the stationary set $S \subset [0, \omega_\alpha)$, then f has a ***stationary value,*** i.e. there is some θ such that

$$|\{\alpha : \alpha \in S;\ f(\alpha) = \theta\}| = \aleph_\alpha.$$

It has been proved in [4] that if S is a well ordered set and tp $S < \omega_{\alpha+1}$, then there is a partition of S into countably many (small) sets,

$$S = S_0 \cup S_1 \cup \ldots \cup \hat{S}_\omega \tag{6}$$

with tp $S_n \leqq \omega_\alpha^n \, (n < \omega)$. We shall use this in the special case $\alpha = 1$ and refer to (6) as a ***paradoxical decomposition*** of S.

3. Lemmas. To prove our theorem we need the following two lemmas.

LEMMA 1. *Let $A = [0, \alpha_0)$, where $\omega \leqq \alpha_0 < \omega_1$ and α_0 is indecomposable. Let $S_\nu^\gamma = \{(\nu, \delta) : \delta < \gamma\}$ $(\nu \in A;\ \gamma < \omega_2)$ and let*

$$S = \bigcup_{\nu \in A} \bigcup_{\gamma < \omega_2} S_\nu^\gamma$$

be ordered lexicographically. If $S' \subset S$ *and* tp $S' = \omega_2\alpha_0$, *then there are* $\gamma < \omega_2$ *and* $N \subset A$ *such that* co $(\gamma) = \omega_1$, *N is cofinal with A and* $S' \cap S_\nu^\gamma$ *is cofinal with* S_ν^γ *for all* $\nu \in N$.

PROOF. Suppose the lemma is false. Then for each

$$\gamma \in M = \{\varrho : \varrho < \omega_2 ; \mathrm{co}(\varrho) = \omega_1\}$$

the set

$$N_\gamma = \{\nu : \nu \in A;\ S' \cap S_\nu^\gamma \text{ is cofinal with } S_\nu^\gamma\}$$

is not cofinal with A. Therefore, for $\gamma \in M$, there is $\nu_\gamma \in A$ so that

$$S' \cap S_\nu^\gamma \text{ is not cofinal with } S_\nu^\gamma \qquad (\nu_\gamma \leqq \nu < \alpha_0).$$

Thus for $\gamma \in M$ and $\nu_\gamma \leqq \nu < \alpha_0$, there is $\theta_\nu < \gamma$ such that

$$S' \cap \{(\nu, \delta) : \theta_\nu < \delta < \gamma\} = \emptyset .$$

Also, since $|A| = \aleph_0$ and co $(\gamma) = \omega_1$ for $\gamma \in M$, it follows that there is $f(\gamma) < \gamma$ such that

$$\theta_\nu < f(\gamma) \qquad (\gamma \in M;\ \nu_\gamma \leqq \nu < \alpha_0).$$

Since by NEUMER's Theorem M is stationary, the regressive function f has a stationary value $\theta < \omega_2$, i.e. there is $M_1 \subset M$ such that $|M_1| = \aleph_2$ and

$$f(\gamma) = \theta \qquad (\gamma \in M_1).$$

Since $\nu_\gamma < \alpha_0$ $(\gamma \in M)$, there is $M_2 \subset M_1$ such that $|M_2| = \aleph_2$ and

$$\nu_\gamma = \xi \qquad (\gamma \in M_2).$$

If $\gamma \in M_2$ and $\xi \leqq \nu < \alpha_0$, then

$$S' \cap \{(\nu, \delta) : \theta \leqq \delta < \gamma\} = 0.$$

This holds for each $\gamma \in M$, and as $|M_2| = \aleph_2$, it follows that

$$S' \cap \{(\nu, \delta) : \theta \leqq \delta < \omega_2\} = 0 \qquad (\xi \leqq \nu < \alpha_0).$$

We now have the contradiction

$$\text{tp } S' \leqq \omega_2\xi + \theta\alpha_0 < \omega_2\alpha_0 .$$

This proves Lemma 1.

LEMMA 2. *Let* $1 \leqq n < \omega$ *and let* $P = \{\alpha_\varrho : \varrho < \omega_1^n\}$ *be a set of ordinal numbers with*

$$\alpha_\varrho < \omega_2,\ \mathrm{co}(\alpha_\varrho) = \omega_1 \qquad (\varrho < \omega_1^n).$$

For $\varrho < \omega_1^n$, let $C_{\varrho 0}, C_{\varrho 1}, \ldots, C_{\varrho\omega}, \ldots$ *be* $\aleph_1$ *sets which are all cofinal subsets of* $[0, \alpha_\varrho)$. *Then there is a set* C^* *such that* tp $C^* \leqq \omega_1^{n+1}$ *and*

$$C^* \cap C_{\varrho\nu} \neq \emptyset \quad (\varrho < \omega_1^n;\ \nu < \omega_1).$$

PROOF. For $\varrho < \omega_1^n$, we define β_ϱ in the following way. $\beta_0 = 0$. If $\varrho = \sigma + 1$, put $\beta_\varrho = \alpha_\sigma$; if ϱ is a limit number put

$$\beta_\varrho = \lim_{\sigma < \varrho} \alpha_\sigma .$$

Note that $\beta_\varrho < \alpha_\varrho$ if $\mathrm{co}(\varrho) = 1$ or ω, since $\mathrm{co}(\alpha_\varrho) = \omega_1$.

We will first prove, by induction on n, that there is a regressive function f defined on P so that

(7) $$|\{\varrho : \varrho_0 < \varrho < \omega_1^n ; f(\alpha_\varrho) < \alpha_{\varrho_0}\}| \leqq \aleph_0 \qquad (\varrho_0 < \omega_1^n).$$

If $n = 1$, the function $f(\alpha_\varrho) = \beta_\varrho$ $(\varrho < \omega_1)$ obviously satisfies (7). Now suppose $n > 1$. Let $Q = \{\alpha_\sigma : \sigma < \omega_1^n ; \mathrm{co}(\sigma) = \omega_1\}$. Then

$$\{\alpha_{\omega_1(\sigma+1)} : \sigma < \omega_1^{n-1}\} \subset Q \subset \{\alpha_{\omega_1\sigma} : \sigma < \omega_1^{n-1}\}$$

and so Q has order type ω_1^{n-1}. By the induction hypothesis, there is a regressive function g defined on Q so that

$$|\{\sigma : \sigma_0 < \sigma ; \alpha_\sigma \in Q ; g(\alpha_\sigma) < \alpha_{\sigma_0}\}| \leqq \aleph_0 \qquad (\alpha_{\sigma_0} \in Q).$$

Now define f in the following way:

$$f(\alpha_\varrho) = g(\alpha_\varrho) \qquad (\alpha_\varrho \in Q),$$
$$f(\alpha_\varrho) = \beta_\varrho \qquad (\alpha_\varrho \in P - Q).$$

Clearly f is regressive. We have to verify that (7) holds. Let $\varrho_0 < \omega_1^n$. It follows from the definition of the β_ϱ that, if $\varrho_0 < \varrho < \omega_1^n$ and $\alpha_\varrho \in P - Q$, then $\alpha_{\varrho_0} \leqq f(\alpha_\varrho)$. Therefore,

$$R = \{\varrho : \varrho_0 < \varrho < \omega_1^n ; f(\alpha_\varrho) < \alpha_{\varrho_0}\} = \{\varrho : \varrho_0 < \varrho, \alpha_\varrho \in Q, f(\alpha_\sigma) < \alpha_{\varrho_0}\}.$$

Let σ_0 be the least ordinal such that $\varrho_0 \leqq \omega_1 \sigma_0$. Then

$$R \subset \{\sigma : \sigma_0 < \sigma ; \alpha_\sigma \in Q, g(\alpha_\sigma) < \alpha_{\sigma_0}\}$$

which is countable. Therefore (7) holds.

We now prove the substantive part of the lemma.

Let $\varrho < \omega_1^n$ and suppose we have already defined $x_{\sigma\nu}$ for $\sigma < \varrho$ and $\nu < \omega_1$. Since $C_{\varrho 0}$ is cofinal with $[0, \alpha_\varrho)$, we can choose $x_{\varrho 0} \in C_{\varrho 0}$ so that

$$x_{\varrho 0} > f(\alpha_\varrho).$$

More generally, by induction on ν, since $C_{\varrho\nu}$ is cofinal with $[0, \alpha_\varrho)$ we can define elements $x_{\varrho\nu} \in C_{\varrho\nu}$ $(\nu < \omega_1)$ so that

$$f(\alpha_\varrho) < x_{\varrho\nu} < x_{\varrho\mu} \qquad (\nu < \mu < \omega_1)$$

and $C_\varrho^* = \{x_{\varrho\nu} : \nu < \omega_1\}$ is a cofinal subset of $[f(\alpha_\varrho), \alpha_\varrho)$. Now put

$$C^* = \bigcup_{\varrho < \omega_1^n} C_\varrho^* .$$

Then $C^* \cap C_{\varrho\nu} \neq \emptyset$ $(\varrho < \omega_1^n ; \nu < \omega_1)$. To prove the lemma we must show that tp $C^* \leqq \omega_1^{n+1}$.

For $\sigma < \omega_1^n$ put $B_\sigma = [\beta_\sigma, \alpha_\sigma)$. Then

$$\bigcup_{\sigma<\omega_1^n} [0, \alpha_\sigma) = \bigcup_{\sigma<\omega_1^n} B_\sigma \text{ (tp)}.$$

If $\varrho<\sigma$, then $C_\varrho^* \cap B_\sigma = \emptyset$. If $\varrho=\sigma$, then $C_\varrho^* \cap B_\sigma$ is either empty (if $\beta_\sigma = \alpha_\sigma$) or it is a cofinal subset of B_σ of order type ω_1. By (7) there are only countably many values of $\varrho > \sigma$ such that $C_\varrho^* \cap B_\sigma \neq \emptyset$ and for every such ϱ, $C_\varrho^* \cap B_\sigma$ is countable since C_ϱ^* is cofinal with α_ϱ $(>\alpha_\sigma)$ and has order type ω_1. Thus we see that, if $D_\sigma = C^* \cap B_\sigma$, then

$$\operatorname{tp} D_\sigma \leqq \omega_1 \qquad (\sigma < \omega_1^n).$$

Since $C^* = \bigcup_{\sigma<\omega_1^n} D_\sigma$(tp), we have the desired conclusion that tp $C^* \leqq \omega_1^{n+1}$.

4. **Proof of Theorem.** First we observe that it is enough to prove (5) in the case of indecomposable ordinals, i.e. that

$$(8) \qquad \omega_2 \alpha_0 \nrightarrow [\omega_1^\omega + 1, \omega_2 \alpha_0]_{\aleph_2}$$

holds if α_0 is indecomposable and $\omega \leqq \alpha_0 < \omega_1$. Let $\omega \leqq \alpha < \omega_1$. Then $\alpha = \alpha_0 + \alpha_1$, where α_0 is indecomposable and $\alpha_1 < \alpha$. Let $S = S_0 \cup S_1$ (tp), tp $S_i = \omega_2 \alpha_i$ $(i<2)$. If (8) holds, then there is a family $\mathscr{F} = (F_\mu : \mu < \omega_2)$ of subsets of S_0 such that tp $\mathscr{F}_\mu \leqq \omega_1^\omega$ $(\mu < \omega_2)$ and such that S_0 does not contain any $(\mathscr{F}, \aleph_2)$-free subset of type $\omega_2\alpha_0$. Therefore, if S' is any $(\mathscr{F}, \aleph_2)$-free subset of S, we have that

$$\operatorname{tp} S' = \operatorname{tp}(S' \cap S_0) + \operatorname{tp}(S' \cap S_1) \leqq \gamma + \omega_2\alpha_1,$$

where $\gamma < \omega_2\alpha_0$. Therefore, tp $S' < \omega_2\alpha$. Thus (5) follows from (8).

We now assume that α_0 is indecomposable and that $\omega \leqq \alpha_0 < \omega_1$. Let $A = [0, \alpha_0)$,

$$S_\nu^\gamma = \{(\nu, \delta) : \delta < \gamma\} \qquad (\nu \in A;\ \gamma < \omega_2),$$

and let $S_\nu = \bigcup_{\gamma<\omega_2} S_\nu^\gamma$. Then the set

$$S = \bigcup_{\nu \in A} S_\nu,$$

ordered lexicographically has order type $\omega_2\alpha_0$. Since α_0 is indecomposable and $\omega \leqq \alpha_0 < \omega_1$, there are sets $A_n \neq \emptyset$ $(n < \omega)$ such that

$$A = A_0 \cup A_1 \cup \dots \cup \hat{A}_\omega \text{ (tp)}.$$

If $\gamma < \omega_2$ and N is cofinal with A, the set $\bigcup_{\nu \in N} S_\nu^\gamma$ has power $\aleph_1$. Therefore, by the hypothesis $2^{\aleph_1} = \aleph_2$, it follows that there are only $\aleph_2$ sets $B \subset S$ which are such that

$$B \subset \bigcup_{\nu \in N} S_\nu^\gamma$$

for some $\gamma = \gamma(B) < \omega_2$ and $N = N(B) \subset A$ with cf $(\gamma) = \omega_1$ and N cofinal with A, and which have the further property that

$$B \cap S_\nu^\gamma \text{ is cofinal with } S_\nu^\gamma \qquad (\nu \in N(B)).$$

Let $B_0, B_1, \dots, \hat{B}_{\omega_2}$ be a well ordering of all such sets B.

We are going to define a family $\mathscr{F} = (F_\mu : \mu < \omega_2)$ of subsets of S such that

(9) $$\text{tp } F_\mu \leqq \omega_1^\omega \qquad (\mu < \omega_2),$$

(10) $$F_\mu \cap B_\nu \neq \emptyset \qquad (\nu < \mu < \omega_2).$$

This will prove (8). For suppose the F_μ $(\mu < \omega_2)$ satisfy (9) and (10). If $S' \subset S$ and tp $S' = \omega_2 \alpha_0$, then by Lemma 1, $S' \supset B_\nu$ for some $\nu < \omega_2$. Therefore, by (10),

$$\{\mu : F_\mu \cap S' = \emptyset\} \subset [0, \nu]$$

and so S' is not $(\mathscr{F}, \aleph_2)$-free.

Let $\mu < \omega_2$.

Put $C_\mu = \{\gamma(B_\nu) : \nu < \mu\}$. Since tp $C_\mu < \omega_2$, there is a paradoxical decomposition of C_μ,

$$C_\mu = C_{\mu 0} \cup \dots \cup C_{\mu\omega},$$

so that tp $C_{\mu n} \leqq \omega_1^n$ $(n < \omega)$. Thus we may write

$$C_{\mu n} = \{\gamma_{\mu n\delta} : \delta < \delta_{\mu n}\}_<,$$

where

$$\delta_{\mu n} \leqq \omega_1^n \qquad (n < \omega).$$

For $\delta < \delta_{\mu n}$ the set $M_{\mu n\delta} = \{\nu : \nu < \mu;\ \gamma(B_\nu) = \gamma_{\mu n\delta}\}$ is nonempty and has cardinal power less than or equal to $\aleph_1$. Therefore, there is a sequence $(\nu_{\mu n\delta\sigma})_{\sigma < \omega_1}$ (whose terms are not necessarily distinct) such that

$$M_{\mu n\delta} = \{\nu_{\mu n\delta\sigma} : \sigma < \omega_1\}.$$

Let $C_{\mu n\delta\sigma} = \{\gamma : (\varrho, \gamma) \in B_{\nu_{\mu n\delta\sigma}}$ *for some* $\varrho \in A - (A_0 \cup \dots \cup A_n)\}$. Then the sets $C_{\mu n\delta\sigma}$ are cofinal with $[0, \gamma_{\mu n\delta})$ for $\sigma < \omega_1$ and $\delta < \delta_{\mu n} \leqq \omega_1^n$. By Lemma 2, there is a set $C^*_{\mu n}$ such that

(11) $$C^*_{\mu n} \cap C_{\mu n\delta\sigma} \neq \emptyset \qquad (\sigma < \omega_1;\ \delta < \delta_{\mu n})$$

and

(12) $$\text{tp } C^*_{\mu n} \leqq \omega_1^{n+1}$$

Put $G_{\mu n} = \{(\varrho, \gamma) : \gamma \in C^*_{\mu n},\ \varrho \in A - (A_0 \cup \dots \cup A_n)\}$. Then

(13) $$\text{tp } (G_{\mu n} \cap S_\varrho) \leqq \omega_1^{n+1} \qquad (\varrho \in A_m,\ n < m < \omega),$$

(14) $$G_{\mu n} \cap S_\varrho = \emptyset \qquad (\varrho \in A_m,\ m \leqq n < \omega).$$

Also, by (11),

(15) $$G_{\mu n} \cap B_\nu \neq \emptyset \qquad (n < \omega;\ \nu \in M_{\mu n\delta};\ \delta < \delta_{\mu n}).$$

Now put $F_\mu = \bigcup_{n<\omega} G_{\mu n}$. Then, by (15) and the definition of the sets $M_{\mu n\delta}$, we have that

$$F_\mu \cap B_\nu \neq \emptyset \qquad (\nu < \mu).$$

i.e. (10) holds. If $m<\omega$ and $\varrho\in A_m$, then by (13) and (14)

$$\operatorname{tp}(F_\mu\cap S_\varrho)=\operatorname{tp}\Big(\bigcup_{n<m}G_{\mu n}\cap S_\varrho\Big)\leqq\omega_1^{m+1}.$$

Therefore

$$\operatorname{tp}\Big(F_\mu\cap\bigcup_{\varrho\in A_m}S_\varrho\Big)\lhd\omega_1^{m+2}\qquad(m\lhd\omega).$$

Since $A=A_0\cup A_1\cup\ldots\cup\hat{A}_\omega$ (tp), it follows that

$$\operatorname{tp}F_\mu\leqq\sum_{m<\omega}\omega_1^{m+2}=\omega_1^\omega.$$

This proves (9) and completes the proof of the theorem.

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MTA MATEMATIKAI KUTATÓ INTÉZETE,
BUDAPEST, V., REÁLTANODA U. 13—15

ANALÍZIS I. TANSZÉK,
EÖTVÖS LORÁND TUDOMÁNYEGYETEM,
BUDAPEST, VIII., MÚZEUM KRT. 6-8

DEPARTMENT OFMATHEMATICS,
UNIVERSITY OF CALGARY,
CALGARY, ALBERTA,
CANADA

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