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ON SETS OF ALMOST DISJOINT SUBSETS OF A SET

By

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1. Introduction

The cardinal power of a set A is denoted by $|A|$. Two sets A_1, A_2 are said to e *almost disjoint* if

$$|A_1 \cap A_2| < |A_i| \quad (i = 1, 2).$$

\ e call B a *transversal* of the disjoint non-empty sets A_ν $(\nu \in M)$ if $B \subset \bigcup_{\nu \in M} A_\nu$ and B intersects each A_ν $(\nu \in M)$ in a singleton.

An old and well known theorem of W. SIERPINSKI is that an infinite set of po'ver $\mathbf{m}$ contains more than $\mathbf{m}$ subsets of power $\mathbf{m}$ which are pairwise almost disjoint and A. .ARSKI obtained various generalizations and extensions of this in [1] and [2]. It is easy to see that Sierpinski's result is equivalent to the following statement: *If A_ν $(\nu \in M)$ are* $\mathbf{m}$ *disjoint sets of power* $\mathbf{m}$, *then there are more than* $\mathbf{m}$ *almost disjoint transversals of the* A_ν. In § 3 we prove some new results which are analogous to this formulation of Sierpinski's theorem.

Ve will denote the following statement by $\mathscr{H}$: There are $\aleph_2$ almost disjoint transversals of $\aleph_1$ disjoint denumerable sets. In view of recent axiomatic results $\mathscr{H}$ is independent of the usual axioms of set theory and the generalized continuum hypothesis. In § 4 we show that $\mathscr{H}$ implies a certain unsolved problem of [3].

In § 5 we consider another question about sets of almost disjoint subsets of a set which was raised by F. S. CATER [4].

2. Notation

Ca ital letters always denote sets and $\mathscr{F}$ denotes a set whose members are sets. We writ $\bigcup \mathscr{F}$ to denote the union of all the members of $\mathscr{F}$. The set-theoretic difference of A and B is $A - B$. Bold lower case latin letters denote cardinals and greek letters denote ordinal numbers. If S is a well-ordered set of type α, then the cardinal of α is the same as the cardinal of S and is denoted by $|\alpha|$. The smallest ordinal number v ith cardinal $\mathbf{m}$ is denoted by $\omega(\mathbf{m})$. As is customary we write ω_α instead of $\omega(\aleph_\alpha)$ nd ω instead of ω_0. The set of ordinal numbers $\{\nu: \alpha \leqq \nu < \beta\}$ is denoted by $[\alpha, \beta)$. he obliterator sign $\hat{}$ written above any symbol indicates that that symbol is to be disregarded. For example, we sometimes write $A_0 \cup \ldots \cup \hat{A}_\lambda$ instead of $\bigcup_{\nu < \lambda} A_\nu$.

The smallest cardinal greater than $\mathbf{m}$ is called the *successor* of $\mathbf{m}$ and is denoted by $\mathbf{m}^+$. If $\mathbf{a}$ is not a successor cardinal (i. e. $\mathbf{a} \neq \mathbf{b}^+$ for any $\mathbf{b}$), then $\mathbf{a}$ is called a *limit* cardinal. The *cofinality cardinal* of $\mathbf{a}$, denoted by $\mathbf{a}'$, is the smallest cardinal $\mathbf{m}$ which is such that $\mathbf{a}$ can be expressed as the sum of $\mathbf{m}$ cardinals each less than $\mathbf{a}$. In the notation of Tarski, $\aleph'_\alpha = \aleph_{\mathrm{cf}(\alpha)}$. $\mathbf{a}$ is *regular* if $\mathbf{a}' = \mathbf{a}$ and *singular* if $\mathbf{a}' < \mathbf{a}$. A cardinal is *inaccessible* if it is a regular limit number. It is not known if there are inaccessible cardinals greater than $\aleph_0$ but the assumption that there are not is known to be consistent with the axioms of set theory.

If B is a set of ordinal numbers we call β a limit point of B if β is the limit of an increasing sequence of members of B. B is *closed in* A if all the limit points of B which are in A are also in B. B is a *cofinal* subset of $[0, \lambda)$ if for any $\nu < \lambda$ there is $\beta \in B$ such that $\nu \leqq \beta < \lambda$. B is a *band* in $[0, \lambda)$ if it is a closed cofinal subset. If S is a set of ordinal numbers and f is an ordinal-valued function on S such that $f(\nu) < \nu$ for all arguments $\nu(\neq 0)$ in S, then f is called a *regressive* function on S. A *stationary value* of such a function is an ordinal number Θ such that $|\{\nu : \nu \in S, f(\nu) = \Theta\}| = |S|$. A well known result of ALEXANDROFF and URYSOHN is that, if $\mathbf{m}$ is a regular cardinal greater than $\aleph_0$, then any regressive function on $[0, \omega(\mathbf{m}))$ has a stationary value. A more general theorem of W. NEUMER [5] is the following: *Let* $\mathbf{m} = \mathbf{m}' > \aleph_0$ *and let* S *be a subset of* $[0, \omega(\mathbf{m}))$ *of power* $\mathbf{m}$. *Then every regressive function on* S *has a stationary value if and only if the complement* $[0, \omega(\mathbf{m})) - S$ *contains no band of* $[0, \omega(\mathbf{m}))$. A set satisfying this condition is said to be *stationary*.

The theorem of Sierpinski stated in § 1 does not depend for its proof on the generalized continuum hypothesis (g. c. h.) that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ — in fact, not even the axiom of choice is required in the case $\mathbf{m} = \aleph_0$. In this paper we always assume the axiom of choice and sometimes we use the g. c. h. or some weaker hypothesis, but we always indicate when this hypothesis is employed.

3. Transversals of disjoint sets

THEOREM 1. *Let* A_ν $(\nu < \omega_{\alpha+1})$ *be* $\aleph_{\alpha+1}$ *disjoint sets each of power* $\aleph_\alpha$. *Then there is a set*, $\mathscr{F}$, *of transversals of the* A_ν *such that* $|\mathscr{F}| = \aleph_{\alpha+1}$ *and*

$$(1) \qquad |F \cap F'| < \aleph_\alpha \quad (F \neq F'; \quad F, F' \in \mathscr{F}).$$

PROOF. We can assume that $A_\nu = \{\xi_{\nu\mu} : \mu \leqq \max\{\nu, \omega_\alpha\}\}$ $(\nu < \omega_{\alpha+1})$. Let $\lambda < \omega_{\alpha+1}$ and suppose the transversals F_ϱ have already been defined for $\varrho < \lambda$. Put $\pi = \min\{\lambda, \omega_\alpha\}$ and let f be a 1—1 map of $[0, \pi)$ onto $[0, \lambda)$. If $\varrho < \omega_\alpha$, then $T_\varrho = A_{f(\varrho)} - \bigcup_{\sigma < \varrho} F_{f(\sigma)} \neq \emptyset$ and we can choose $x_{f(\varrho)} \in T_\varrho$. Now put

$$F_\lambda = \{x_{f(\varrho)} : \varrho < \pi\} \cup \{\xi_{\nu\lambda} : \nu \in [\lambda, \omega_{\alpha+1})\}.$$

This defines F_λ for $\lambda < \omega_{\alpha+1}$ by induction. It is clear from the construction that F_λ is a transversal of the A_ν. Also, if $\mu < \lambda < \omega_{\alpha+1}$ then $\mu = f(\varrho)$ for some $\varrho < \pi$ and

$$F_\lambda \cap F_\mu \subset \{x_{f(0)}, \ldots, x_{f(\varrho)}\},$$

i. e. $|F_\lambda \cap F_\mu| < \aleph_\alpha$. Thus the set $\mathscr{F} = \{F_\lambda : \lambda < \omega_{\alpha+1}\}$ has the properties described.

A. TARSKI [1] proved: *If* $\mathscr{F}$ *is a set of subsets of a set of power* $\mathbf{m}$ *and if* $|F \cap F'| < \mathfrak{p}$ *for distinct members* $F, F' \in \mathscr{F}$, *then* $|\mathscr{F}| \leqq \mathbf{m}^{\mathfrak{p}}$. It follows from this and the g. c. h. that if $\mathscr{F}$ is any set of transversals of $\aleph_{\alpha+1}$ sets of power $\aleph_\alpha$ such that (1) holds, then $|\mathscr{F}| \leqq \aleph_{\alpha+1}^{\aleph_\alpha} = \aleph_{\alpha+1}$. In this sense Theorem 1 is the best possible result. The total number of different transversals of the A_ν is $\aleph_\alpha^{\aleph_{\alpha+1}} \geqq \aleph_{\alpha+2}$.

Our next theorem has some relevance to problem $\mathscr{H}$.

THEOREM 2. *Let A_ν $(\nu<\omega_{\alpha+1})$ be $\aleph_{\alpha+1}$ disjoint sets of power $\aleph_\alpha$. Then there is a maximal set, $\mathscr{F}$, of $\aleph_{\alpha+1}$ almost disjoint transversals of the A_ν, i.e. if B is any transversal of the A_ν, then there is some $F\in\mathscr{F}$ such that $|F\cap B|=\aleph_{\alpha+1}$.*

PROOF. Since $|A_\nu|=\aleph_\alpha$ we can assume that

$$A_\nu=\{\xi_{\nu\mu}:\mu<\max\{\nu,\omega_\alpha\}\}\quad(\nu<\omega_{\alpha+1}).$$

For $\lambda<\omega_{\alpha+1}$ put

$$F_\lambda=\{\xi_{\nu 0}:\nu\leqq\lambda\}\cup\{\xi_{\nu\lambda}:\lambda<\nu<\omega_{\alpha+1}\}.$$

Then $\mathscr{F}=\{F_\lambda:\lambda<\omega_{\alpha+1}\}$ is a maximal set of almost disjoint transversals of the A_ν. To see this consider any transversal B. By the definition of the F_λ we have that

$$A_\nu\subset\bigcup_{\lambda<\nu}F_\lambda\quad\textit{if}\quad \nu\in S=[\omega_\alpha,\omega_{\alpha+1}).$$

Therefore, for each $\nu\in S$, there is $f(\nu)<\nu$ such that

$$B\cap F_{f(\nu)}\cap A_\nu\neq\emptyset.$$

Since f is regressive on S, there is $\gamma<\omega_{\alpha+1}$ such that $N_\gamma=\{\nu:\nu\in S, f(\nu)=\gamma\}$ has power $\aleph_{\alpha+1}$. Since the A_ν are disjoint it follows that $|B\cap F_\gamma|\geqq|N_\gamma|=\aleph_{\alpha+1}$.

The remaining theorems in this section are concerned with almost disjoint transversals of sets A_ν which do not necessarily have the same power. Let $\lambda=\omega(\mathbf{m}')$ and let $A_0,\dots,\hat{A}_\lambda$ be $\mathbf{m}'$ disjoint sets which satisfy

$$(2)\qquad 0<|A_0|\leqq|A_1|\leqq\dots\leqq|\hat{A}_\lambda|\leqq\mathbf{m}=\lim_{\nu<\lambda}|A_\nu|.$$

By Kőnig's theorem, the total number of transversals is

$$|A_0|\cdot|A_1|\dots|\hat{A}_\lambda|>|A_0\cup A_1\cup\dots\cup\hat{A}_\lambda|=\mathbf{m}.$$

In Theorem 3 we show that there are $\mathbf{m}^+$ almost disjoint transversals *if* $\mathbf{m}'=\aleph_0$. The corresponding statement in the case $\mathbf{m}'>\aleph_0$ is not true. In this case the existence or non-existence of $\mathbf{m}^+$ almost disjoint transversals depends upon whether or not some extra condition on the cardinals $|A_\nu|$ $(\nu<\lambda)$ is satisfied (Theorems 5, 6).

THEOREM 3. *Let $\mathbf{m}'=\aleph_0$ and let A_ν $(\nu<\omega)$ be $\aleph_0$ disjoint sets which satisfy the condition* (2). *Then there are $\mathbf{m}^+$ almost disjoint transversals of the A_ν.*

PROOF. There are cardinals $\mathbf{m}_\nu<\mathbf{m}$ $(\nu<\omega)$ such that

$$1\leqq\mathbf{m}_0\leqq\mathbf{m}_1\leqq\dots<\mathbf{m}=\mathbf{m}_0+\mathbf{m}_1+\dots.$$

If $\theta<\omega$, then by (2) there is $\nu_\theta<\omega$ such that

$$(3)\qquad |A_\nu|>\mathbf{m}_0+\dots+\hat{\mathbf{m}}_\theta=\mathbf{n}_\theta\quad(\nu_\theta\leqq\nu<\omega),$$

and we can assume that $0=\nu_0<\nu_1<\dots$.

We shall define $\mathbf{m}^+$ almost disjoint transversals F_μ $(\mu<\omega(\mathbf{m}^+))$ by induction. Let $\mu<\omega(\mathbf{m}^+)$ and suppose that we have already defined the transversals F_ϱ $(\varrho<\mu)$.

Since $|\mu| \leqq \mathbf{m}$, we may write

$$\{F_0, \ldots, \hat{F}_\mu\} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \ldots \cup \hat{\mathscr{F}}_\omega,$$

where $|\mathscr{F}_\nu| \leqq \mathbf{m}_\nu$ $(\nu < \omega)$. By (3) there is

$$x_\nu \in A_\nu - \bigcup_{\varphi < \theta} \cup \mathscr{F}_\varphi \quad (\nu_\theta \leqq \nu < \nu_{\theta+1}; \quad \theta < \omega).$$

Then $F_\mu = \{x_\nu : \nu < \omega\}$ is a transversal which intersects each F_ϱ $(\varrho < \mu)$ in a finite set. This completes the proof.

Assuming the hypothesis (4) (which is true if $\mathbf{m} = \aleph_0$ and is implied by the g. c. h. if $\mathbf{m} > \aleph_0$), the next theorem shows that, if $\mathbf{m}' = \aleph_0$ then no set of $\mathbf{m}$ almost disjoint transversals of denumerably many disjoint sets is maximal. Theorem 3 can easily be deduced from this. We cannot prove Theorem 4 without the hypothesis (4).

THEOREM 4. *Suppose that* $\mathbf{m}' = \aleph_0$ *and*

$$2^{\mathbf{n}} < \mathbf{m} \quad \text{if} \quad \mathbf{n} < \mathbf{m}. \tag{4}$$

Let $\mathscr{F}$ be any set of power $\mathbf{m}$ of almost disjoint transversals of $\aleph_0$ disjoint sets A_ν $(\nu < \omega)$. Then $\mathscr{F}$ is not maximal, i. e. there is a transversal which is almost disjoint from each member of $\mathscr{F}$.

PROOF. If $\theta < \omega$ we will show that there is $\nu_\theta < \omega$ such that (3) holds. Suppose this is false. Then there is an infinite set $I \subset [0, \omega)$ such that

$$|A_\nu| \leqq \mathbf{n}_\theta \quad (\nu \in I). \tag{5}$$

Case 1. $\mathbf{m} = \aleph_0$. Then n_θ is finite. Let $\mathscr{F}^*$ be a subset of $\mathbf{n}_{\theta+1}$ members of $\mathscr{F}$. Since $\mathscr{F}^*$ is a finite set of almost disjoint transversals, there is $\pi < \omega$ such that

$$F \cap A_\nu \neq F' \cap A_\nu$$

if $\pi \leqq \nu < \omega$ and F, F' are different members of $\mathscr{F}^*$. There is $\gamma \in I$ such that $\gamma > \pi$ and we have the contradiction that

$$|A_\gamma| \geqq |A_\gamma \cap \cup \mathscr{F}^*| > \mathbf{n}_\theta.$$

Case 2. $\mathbf{m} > \aleph_0$. Since $\mathbf{n}_\theta < \mathbf{m}$, it follows from (4) and (5) that the total number of distinct transversals of the sets A_ν $(\nu \in I)$ is at most $\mathbf{n}_\theta^{\aleph_0} \leqq 2^{\mathbf{n}_\theta \aleph_0} < \mathbf{m}$. Therefore, there are distinct members $F, F' \in \mathscr{F}$ such that $F \cap A_\nu = F' \cap A_\nu$ $(\nu \in I)$ and this contradicts the fact that the members of $\mathscr{F}$ are almost disjoint. The theorem now follows by precisely the same argument used in the last paragraph of the proof of Theorem 3.

There is no analogue of Theorem 4 for cardinals not cofinal with $\aleph_0$. For example, if $A_\nu = \{\xi_{\nu\mu} : \mu < \omega_\nu\}$ $(\nu < \omega_1)$ and

$$F_\sigma = \{\xi_{\nu 0} : \nu \leqq \varrho\} \cup \{\xi_{\nu\sigma} : \varrho < \nu < \omega_1\} \quad (\omega_\varrho \leqq \sigma < \omega_{\varrho+1}; \quad \varrho < \omega_1),$$

then there is a maximal set of almost disjoint transversals, $\mathscr{F}$, which contains the set $\{F_\sigma : \omega \leqq \sigma < \omega_{\omega_1}\}$, i. e. $|\mathscr{F}| \geqq \aleph_{\omega_1}$. The sets A_ν $(\nu < \omega_1)$ do not have the property $\mathscr{P}(\aleph_1)$ defined below and so, by Theorem 6 and the g. c. h., $|\mathscr{F}| \leqq \aleph_{\omega_1}$. On the other hand, if $|B_\nu| = \aleph_{\nu+1}$ $(\nu < \omega_1)$, then the sets B_ν do have the property $\mathscr{P}(\aleph_1)$ and therefore, by Theorem 5, there is a set of $\aleph_{\omega_1+1}$ almost disjoint transversals of the B_ν.

A set of cardinal numbers $M = \{\mathbf{m}_0, \ldots, \mathbf{m}_\lambda\}_<$* is *closed* if it contains the limits of all increasing sequences in M, i. e. if ϱ is a limit number and $\nu_\sigma < \lambda\,(\sigma < \varrho)$, then

$$\lim_{\sigma<\varrho} \mathbf{m}_{\nu_\sigma} = \mathbf{m}_\nu,$$

where $\nu = \lim\limits_{\sigma<\varrho} \nu_\sigma$. We prove the following simple lemma.

LEMMA. *Let* $\mathbf{m}$ *be a limit cardinal not cofinal with* $\aleph_0$, *let* $\lambda = \omega(\mathbf{m}')$ *and let* $\{\mathbf{m}_0, \ldots, \hat{\mathbf{m}}_\lambda, \mathbf{m}\}_<$ *and* $\{\mathbf{n}_0, \ldots, \hat{\mathbf{n}}_\lambda, \mathbf{m}\}_<$ *be closed sets of cardinals. Then*

$$B = \{\nu : \nu < \lambda, \quad \mathbf{m}_\nu = \mathbf{n}_\nu\}$$

is a band in $[0, \lambda)$.

PROOF. It is clear that B is closed in $[0, \lambda)$. Suppose there is $\nu_0 < \lambda$ such that $\mathbf{m}_\nu \neq \mathbf{n}_\nu$ for $\nu > \nu_0$. There are ordinals $\nu_\varrho < \lambda$ $(\varrho < \omega)$ such that $\nu_0 < \nu_1 < \nu_2 < \ldots$ and

$$\mathbf{m}_{\nu_0} < \mathbf{n}_{\nu_1} < \mathbf{m}_{\nu_2} < \mathbf{n}_{\nu_3} < \ldots.$$

If $\nu = \lim\limits_{\varrho<\omega} \nu_\varrho$, then $\nu < \lambda$ since $\mathbf{m}' > \aleph_0$ and

$$\mathbf{m}_\nu = \lim_{\varrho<\omega} \mathbf{m}_{\nu_\varrho} = \lim_{\varrho<\omega} \mathbf{n}_{\nu_\varrho} = \mathbf{n}_\nu.$$

This contradiction proves that B is cofinal, and hence a band, in $[0, \lambda)$.

Let $\mathbf{m}$ be any limit cardinal and let $\lambda = \omega(\mathbf{m}')$. Then there are cardinals $\mathbf{m}_0, \ldots, \hat{\mathbf{m}}_\lambda$ such that

$$\mathbf{m}_0 < \mathbf{m}_1 < \ldots < \hat{\mathbf{m}}_\lambda < \mathbf{m} = \lim_{\nu<\lambda} \mathbf{m}_\nu$$

and we can assume that the set $\{\mathbf{m}_0, \ldots, \hat{\mathbf{m}}_\lambda, \mathbf{m}\}_<$ is closed (if not, the closure of the set also contains $\mathbf{m}'$ cardinals). Let A_ν $(\nu < \lambda)$ be $\mathbf{m}'$ disjoint sets. We say that these sets have the property $\mathscr{P}(\mathbf{m})$ if (2) holds and the set

$$C = \{\nu : 1 \leqq \nu < \lambda, \quad |A_\nu| \leqq \mathbf{m}_\nu\}$$

is non-stationary in $[0, \lambda)$. This definition of $\mathscr{P}(\mathbf{m})$ does not depend upon the particular choice of the $\mathbf{m}_\nu$ $(\nu < \lambda)$. For suppose that $\{\mathbf{n}_0, \ldots, \hat{\mathbf{n}}_\lambda, \mathbf{m}\}_<$ is another closed set of cardinals and

$$C' = \{\nu : 1 \leqq \nu < \lambda, \quad |A_\nu| \leqq \mathbf{n}_\nu\}.$$

If C is stationary then so is C' since $C' \supset C \cap B$, where B is the set defined in the lemma, and the intersection of a stationary set and a band is also stationary.** Conversely, C is stationary if C' is.

* The symbol $\{\mathbf{m}_0, \ldots, \mathbf{m}_\lambda\}_<$ indicates that $\mathbf{m}_0 < \mathbf{m}_1 < \ldots < \mathbf{m}_\lambda$.

** See H. BACHMANN [6] page 41.

THEOREM 5. *Let* $\mathbf{m}$ *be any limit cardinal not cofinal with* $\aleph_0$ *and let* $\lambda = \omega(\mathbf{m}')$. *If the disjoint sets* A_ν $(\nu < \lambda)$ *have the property* $\mathscr{P}(\mathbf{m})$, *then there are* $\mathbf{m}^+$ *almost disjoint transversals of the* A_ν.

PROOF. Since $C = \{\nu : \nu < \lambda, |A_\nu| \leqq \mathbf{m}_\nu\}$ is non-stationary, there is a band $\{\nu_0, \nu_1, \ldots, \hat{\nu}_\lambda\}_<$ in the complement $[0, \lambda) - C$. We can assume that $\nu_0 = 0$. Then, if $\mu < \lambda$, there is $\varrho(\mu) < \lambda$ such that

$$\nu_{\varrho(\mu)} \leqq \mu < \nu_{\varrho(\mu)+1}.$$

Let $\theta < \omega(\mathbf{m}^+)$ and suppose the transversals F_φ $(\varphi < \theta)$ of the A_ν have already been defined. Since $|\theta| \leqq \mathbf{m}$, we may write

$$\{F_0, \ldots, \hat{F}_\theta\} = \mathscr{F}_0 \cup \ldots \cup \hat{\mathscr{F}}_\lambda,$$

where $|\mathscr{F}_\nu| \leqq \mathbf{m}_\nu$ $(\nu < \lambda)$. If $\mu < \lambda$, then

$$|A_\mu| \geqq |A_{\nu_{\varrho(\mu)}}| > \mathbf{m}_{\nu_{\varrho(\mu)}}$$

since $\nu_{\varrho(\mu)} \notin C$. Also,

$$\Bigl| A_\mu \cap \bigcup_{\nu < \nu_{\varrho(\mu)}} \cup \mathscr{F}_\nu \Bigr| \leqq |\nu_{\varrho(\mu)}| \cdot \mathbf{m}_{\nu_{\varrho(\mu)}} = \mathbf{m}_{\nu_{\varrho(\mu)}}.$$

Therefore, we can choose

$$x_\mu \in A_\mu - \bigcup_{\nu < \nu_{\varrho(\mu)}} \cup \mathscr{F}_\nu \quad (\mu < \lambda).$$

Then $F_\theta = \{x_0, \ldots, \hat{x}_\lambda\}$ is a transversal of the A_ν. If $\varphi < \theta$, then there is $\sigma < \lambda$ such that $F_\varphi \in \mathscr{F}_\sigma$ and, by the definition of F_θ,

$$F_\theta \cap F_\varphi \subset \{x_\mu : \mu < \nu_{\varrho(\sigma)+1}\}.$$

Therefore, F_θ is almost disjoint from all the sets F_φ $(\varphi < \theta)$. Theorem 5 now follows by induction.

The next theorem shows that (if we assume (6) which is weaker than the g. c. h.) the condition $\mathscr{P}(\mathbf{m})$ in Theorem 5 is a necessary one for the existence of $\mathbf{m}^+$ almost disjoint transversals in the case when $\mathbf{m}$ is a singular limit number. We do not know if a similar result holds for inaccessible cardinals.

THEOREM 6. *Let* $\mathbf{m} > \mathbf{m}' > \aleph_0$ *and suppose that*

$$\mathbf{n}^{\mathbf{m}'} < \mathbf{m}^+ \quad (\mathbf{n} < \mathbf{m}). \tag{6}$$

Let $\lambda = \omega(\mathbf{m}')$ *and suppose that the disjoint sets* A_ν $(\nu < \lambda)$ *satisfy* (2) *but do not have the property* $\mathscr{P}(\mathbf{m})$. *Then any set of almost disjoint transversals of the* A_ν *has power less than or equal to* $\mathbf{m}$.

PROOF. We will assume that $\mathscr{F}$ is a set of $\mathbf{m}^+$ almost disjoint transversals of the A_ν and deduce a contradiction.

By hypothesis, the set $C = \{\nu : 1 \leqq \nu < \lambda, |A_\nu| \leqq \mathbf{m}_\nu\}$ is stationary in $[0, \lambda)$. Therefore L, the set of limit ordinals in C, is also stationary. For each $\nu \in L$ we can assume

$$A_\nu \subset \{(\nu, \varrho) : \varrho < \omega(\mathbf{m}_\nu)\}.$$

If $F \in \mathscr{F}$ and $\nu \in L$, then there is $\varrho(F, \nu) < \omega(\mathbf{m}_\nu)$ such that

$$(\nu, \varrho(F, \nu)) \in F.$$

Since ν is a limit ordinal less than λ and $\{\mathbf{m}_0, \ldots, \hat{\mathbf{m}}_\lambda, \mathbf{m}\}_<$ is closed, there is $f_F(\nu) < \nu$ such that

$$|\varrho(F, \nu)| < \mathbf{m}_{f_F(\nu)}.$$

Since f_F is regressive on the stationary set L, there are $\theta_F < \lambda$ and $N_F \subset L$ such that $|N_F| = \mathbf{m}'$ and

$$f_F(\nu) = \theta_F \quad (\nu \in N_F).$$

It follows from (6) that there are $\theta < \lambda$, $N \subset L$ and $\mathscr{F}^* \subset \mathscr{F}$ such that $|\mathscr{F}^*| = \mathbf{m}^+$ and

$$\theta_F = \theta, \quad N_F = N \quad (F \in \mathscr{F}^*).$$

Put

$$A_\nu^* = A_\nu \cap \{(\nu, \sigma) : \sigma < \omega(\mathbf{m}_\theta)\} \quad (\nu \in N).$$

Then each $F \in \mathscr{F}^*$ meets each A_ν^* $(\nu \in N)$ in a singleton. By (6) the number of distinct transversals of the A_ν^* $(\nu \in N)$ is at most $\mathbf{m}_\theta^{\mathbf{m}'} < \mathbf{m}^+$. Therefore, there are distinct members $F, F' \in \mathscr{F}^*$ such that

$$F \cap A_\nu^* = F' \cap A_\nu^* \quad (\nu \in N).$$

This is a contradiction since $|N| = \mathbf{m}'$ and the members of $\mathscr{F}$ are pairwise almost disjoint.

4. A deduction from $\mathscr{H}$

One of the unsolved problems mentioned in [3] is to to prove or disprove the following statement. $\mathscr{S}$: *Let S be a set of power $\aleph_2$ and let E be the set of all unordered distinct pairs in S.* Then there is a partition of E*

$$E = E_0 \cup \ldots \cup \hat{E}_{\omega_1} \tag{7}$$

into $\aleph_1$ disjoint sets E_ν $(\nu < \omega_1)$ such that for every subset $S' \subset S$ of power $\aleph_1$

$$|\{\nu : \nu < \omega_1, \quad E' \cap E_\nu \neq \emptyset\}| = \aleph_1, \tag{8}$$

where E' is the set of all pairs in S'.

Let A_ν $(\nu < \omega_1)$ be $\aleph_1$ disjoint denumerable sets. Assuming that $\mathscr{H}$ is true, we take S to be a set of $\aleph_2$ almost disjoint transversals of the A_ν. If F, F' are distinct elements of S, then there is $\varrho_{FF'} < \omega_1$ such that

$$F \cap A_\nu \neq F' \cap A_\nu \quad (\varrho_{FF'} \leqq \nu < \omega_1).$$

If E is the set of all unordered pairs of S, put

$$E_\nu = E \cap \{\{F, F'\} : \varrho_{FF'} = \nu\} \quad (\nu < \omega_1).$$

* i.e. (S, E) is a complete graph.

Then (7) holds and $E_\mu \cap E_\nu = \emptyset$ $(\mu < \nu < \omega_1)$. Let S' be any subset of S of power $\aleph_1$ and let E' be the set of pairs in S'. If (8) is false, then there is $\varrho < \omega_1$ such that

$$E' \subset E_0 \cup \dots \cup \hat{E}_\varrho,$$

i. e. $\varrho_{FF'} < \varrho$ for all distinct pairs $\{F, F'\} \subset S'$. This implies that different members of S' meet A_ϱ in different points, and therefore $|A_\varrho| \geqq |S'| = \aleph_1$. This contradiction proves that $\mathscr{H}$ implies $\mathscr{S}$.

We do not know if $\mathscr{S}$ also implies $\mathscr{H}$.

5. A problem of F. S. Cater

F. S. Cater [4] noted the following extension of Sierpinski's theorem. *If* $\mathbf{m}$ *is a regular cardinal and* $|S| = \mathbf{m}$, *then there is a set,* $\mathscr{F}$, *of almost disjoint subsets of power* $\mathbf{m}$ *of* S *such that* $|\mathscr{F}| > \mathbf{m}$ *and, in addition,*

(9) *if* $\mathscr{F}' \subset \mathscr{F}$ *and* $|\mathscr{F}'| = \mathbf{m}$, *then there is* $F \in \mathscr{F} - \mathscr{F}'$ *such that* $|F \cap \bigcup \mathscr{F}'| = \mathbf{m}$.

If $\mathscr{F}$ is any *maximal* set of almost disjoint subsets of power $\mathbf{m}$ of S and $|\mathscr{F}| > \mathbf{m}$, then it can easily be seen that (9) follows if $\mathbf{m}$ is regular. The g. c. h. is not used in the proof just outlined but it only works for regular $\mathbf{m}$. Cater asked if the result is true for singular numbers. We will show (Theorem 8) with the aid of the g. c. h. that Cater's result holds for arbitrary $\mathbf{m} \geqq \aleph_0$ and that (9) can be replaced by the stronger condition

(10) *if* $\mathscr{F}' \subset \mathscr{F}$ *and* $|\mathscr{F}'| = \mathbf{m}$, *then there is* $F \in \mathscr{F} - \mathscr{F}'$ *such that* $F \subset \bigcup \mathscr{F}'$.

We cannot prove this result without the g. c. h. even in the case of regular $\mathbf{m}$.

We call B a *weak transversal* of the $\mathbf{m}$ disjoint sets A_ν $(\nu \in M)$ if

$$B \subset \bigcup_{\nu \in M} A_\nu, \quad |B| = \mathbf{m} \quad \text{and} \quad |B \cap A_\nu| \leqq 1 \quad (\nu \in M).$$

The next theorem shows that Theorem 4 can be extended to arbitrary cardinals if we consider weak transversals instead of transversals.

THEOREM 7. *Suppose that* $\mathbf{m}$ *is an infinite cardinal and that**

$$\mathbf{n}^{\mathbf{m}'} < \mathbf{m} \quad \text{if} \quad \mathbf{m}' < \mathbf{n} < \mathbf{m}. \tag{11}$$

If $\mathscr{F}$ *is any set of almost disjoint weak transversals of* $\mathbf{m}'$ *disjoint sets* A_ν $(\nu < \omega(\mathbf{m}'))$ *such that* $|\mathscr{F}| = \mathbf{m}$, *then* $\mathscr{F}$ *is not maximal.*

PROOF. Put $\lambda = \omega(\mathbf{m}')$.

Case 1. $\mathbf{m} = \mathbf{m}'$. Let $\mathscr{F} = \{F_0, \dots, \hat{F}_\lambda\}$. Let $\varrho < \lambda$ and suppose that $\nu_\sigma < \lambda$ and $x_\sigma \in A_{\nu_\sigma}$ have been defined for $\sigma < \varrho$. Let $N = [0, \lambda) - \{\nu_0, \dots, \hat{\nu}_\varrho\}$. If

$$A_\nu \subset F_0 \cup \dots \cup \hat{F}_\varrho \quad \textit{for all} \quad \nu \in N,$$

* (11) is satisfied vacuously if $\mathbf{m}$ is regular.

then $|F_\varrho \cap F_\sigma| = \mathbf{m}$ for some $\sigma < \varrho$ since $\mathbf{m} = \mathbf{m}'$ and $|F_\varrho \cap \bigcup_{v \in N} A_v| = \mathbf{m}$. This contradiction shows that there is $v_\varrho \in N$ such that $A_{v_\varrho} \not\subset F_0 \cup \ldots \cup \hat{F}_\varrho$ and we can choose $x_\varrho \in A_{v_\varrho} - F_0 \cup \ldots \cup \hat{F}_\varrho$. The set $\{x_\varrho : \varrho < \lambda\}$ defined by induction is a weak transversal of the A_v which is almost disjoint from all the members of $\mathscr{F}$.

Case 2. $\mathbf{m} > \mathbf{m}'$. Let $\mathscr{F} = \mathscr{F}_0 \cup \ldots \cup \hat{\mathscr{F}}_\lambda$, where $|\mathscr{F}_v| = \mathbf{m}_v < \mathbf{m}$ $(v < \lambda)$. Put $\mathscr{F}_v^* = \mathscr{F}_0 \cup \ldots \cup \hat{\mathscr{F}}_v$ $(v < \lambda)$ and let

$$N_v = \{\mu : \mu < \lambda, A_\mu \not\subset \cup \mathscr{F}_v^*\} \quad (v < \lambda).$$

If $|N_v| < \mathbf{m}'$ for some $v < \lambda$, then there is $\theta < \lambda$ such that $A_\mu \subset \cup \mathscr{F}_v^*$ $(\mu \in [\theta, \lambda))$. Therefore, $|A_\mu| \leqq \mathbf{m}_0 + \ldots + \hat{\mathbf{m}}_v = \mathbf{n}_v < \mathbf{m}$ and the number of distinct weak transversals of the A_μ $(\mu \in [\theta, \lambda))$ is at most $2^{\mathbf{m}'} \cdot \mathbf{n}_v^{\mathbf{m}'} < \mathbf{m}$. Therefore, there are distinct elements $F, F' \in \mathscr{F}$ which meet in a common weak transversal of the A_μ $(\mu \in [\theta, \lambda))$. This contradiction proves that

$$|N_v| = \mathbf{m}' \quad (v < \lambda).$$

Let $\varrho < \lambda$ and suppose $v_\sigma < \lambda$ and $x_\sigma \in A_{v_\sigma}$ have been defined for $\sigma < \varrho$. Then we can choose $v_\varrho \in N_\varrho - \{v_0, \ldots, \hat{v}_\varrho\}$ and $x_\varrho \in A_{v_\varrho} - \cup \mathscr{F}_\varrho^*$. The set $B = \{x_\varrho : \varrho < \lambda\}$ is a weak transversal of the A_v $(v < \lambda)$. Also, if $F \in \mathscr{F}_\gamma$, then $B \cap F \subset \{x_0, \ldots, x_\gamma\}$, i. e. B is almost disjoint from all members of $\mathscr{F}$. This completes the proof of Theorem 7.

THEOREM 8. *Let* $\mathbf{m} \geqq \aleph_0$ *and assume that*

$$2^{\mathbf{m}} = \mathbf{m}^+ \quad \textit{and} \quad \mathbf{n}^{\mathbf{m}'} < \mathbf{m}^+ \quad \textit{if} \quad \mathbf{m}' < \mathbf{n} < \mathbf{m}.$$

Let $|S| = \mathbf{m} \geqq \mathbf{p} \geqq \mathbf{p}' = \mathbf{m}'$. *Then there is a set,* $\mathscr{F}$, *of almost disjoint subsets of* S *each of power* $\mathbf{p}$ *such that* $|\mathscr{F}| = \mathbf{m}^+$ *and* (10) *holds.*

PROOF. Throughout the proof we write $\lambda = \omega(\mathbf{m}')$, $\alpha = \omega(\mathbf{m})$ and $\beta = \omega(\mathbf{m}^+)$.

Case 1. $\mathbf{p} = \mathbf{m}'$. The number of distinct subsets of $[0, \beta)$ of power $\mathbf{m}$ is $(\mathbf{m}^+)^{\mathbf{m}} = 2^{\mathbf{m} \cdot \mathbf{m}} = \mathbf{m}^+$ and we assume that these are the sets $N_0, \ldots, \hat{N}_\beta$. Let A_v $(v < \lambda)$ be $\mathbf{m}'$ disjoint subsets of S of power $\mathbf{m}$. Also, let $F_0, \ldots, \hat{F}_\alpha$ be any $\mathbf{m}$ mutually disjoint transversals of the A_v.

Let $\theta \in [\alpha, \beta)$. Suppose that we have already defined ordinals $\tau_\varphi < \beta$ for $\alpha \leqq \varphi < \beta$ and also the weak transversals of the A_v $(v < \lambda)$ $F_0, \ldots, \hat{F}_\theta$ which are pairwise almost disjoint. Since $[0, \theta)$ contains $\mathbf{m}^{\mathbf{m}} = \mathbf{m}^+$ subsets of power $\mathbf{m}$, we can choose τ_θ to be the least ordinal $\tau < \beta$ such that $\tau \neq \tau_\varphi$ $(\varphi < \theta)$ and such that $N_\tau \subset [0, \theta)$. Put $\mathscr{F}_\theta = \{F_\varrho : \varrho \in N_{\tau_\theta}\}$, $A_v^* = A_v \cap \cup \mathscr{F}_\theta$ $(v < \lambda)$ and let $F_\varrho^* = F_\varrho \cap \bigcup_{v < \lambda} A_v^*$ $(\varrho < \theta)$. Now put $\mathscr{F}_\theta^* = \{F_\varrho^* : \varrho < \theta, |F_\varrho^*| = \mathbf{m}'\}$. Then $\mathscr{F}_\theta^*$ is a set of almost disjoint weak transversals of the A_v^* $(v < \lambda)$ and $|\mathscr{F}_\theta^*| = \mathbf{m}$ since $\mathscr{F}_\theta \subset \mathscr{F}_\theta^*$. By Theorem 7 there is a weak transversal F_θ of the A_v^* which is almost disjoint from each member of $\mathscr{F}_\theta^*$. Since

$$F_\theta \subset \bigcup \mathscr{F}_\theta = \bigcup_{\varrho \in N_{\tau_\theta}} F_\varrho \tag{12}$$

it follows that F_θ is also almost disjoint from each F_φ $(\varphi < \theta)$. This defines F_θ and τ_θ for $\alpha \leqq \theta < \beta$. By the construction, it is clear that $\mathscr{F} = \{F_\theta : \theta < \beta\}$ is a set of $\mathbf{m}^+$ almost disjoint weak transversals of the A_v $(v < \lambda)$.

In order to complete the proof in this case we only need to observe that as θ ranges from α to β so τ_θ assumes all ordinal values less than β. If this were not the case there is a least $\varrho < \beta$ such that $\tau_\theta \neq \varrho$ $(\alpha \leqq \theta < \beta)$. There is $\pi < \beta$ such that $N_\varrho \subset [0, \pi)$ and the definition of τ_θ implies that $\tau_\theta < \varrho$ for $\pi \leqq \theta < \beta$. But this is impossible since $\tau_\theta \neq \tau_\varphi$ if $\theta \neq \varphi$. It follows that if $\mathscr{F}'$ is any subset of $\mathscr{F}$ of power $\mathbf{m}$, then there is some $\theta \in [\alpha, \beta)$ such that $\mathscr{F}' = \{F_\varrho : \varrho \in N_{\tau_\theta}\}$, i. e. $\mathscr{F}' = \mathscr{F}_\theta$ and (10) follows from (12) and the fact that $\theta \notin N_{\tau_\theta}$.

Case 2. $\mathbf{p} > \mathbf{m}'$. There are cardinals $\mathbf{p}_\nu$ $(\nu < \lambda)$ such that $\mathbf{p}_0 < \mathbf{p}_1 < \ldots < \hat{\mathbf{p}}_\lambda < \mathbf{p} = \lim_{\nu < \lambda} \mathbf{p}_\nu$. The result in this case easily follows from the last case if we replace each element of A_ν $(\nu < \lambda)$ by a subset of S of cardinal $\mathbf{p}_\nu$. Since the weak transversals meet $\mathbf{m}'$ different A_ν they will, in this case, be subsets of S of power $\mathbf{p}$.

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