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ON THE COMPLETE SUBGRAPHS OF GRAPHS DEFINED BY SYSTEMS OF SETS*

By

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Professor R. RADO *on his 60th birthday*

§ 1. Introduction. A system of sets is an ordered triple $\mathscr{I}=(F, M, S)$ where

$$F: M \to \{X: X \subset S\} \tag{1. 1}$$

is a mapping from the set M into the set of all subsets of the set S. If $\mu \in M$, the image of μ by the mapping (1. 1) is F_μ. In particular, if M is a set of power m and S is an ordered set of order type α, we call $\mathscr{I}$ an (m, α)-system. The system of sets $\mathscr{I}=(F, M, S)$ defines a graph $\mathscr{G}=\mathscr{G}(\mathscr{I})$ on the set S in which two distinct vertices $x, y \in S$ are joined by an edge of the graph if and only if $x, y \in F_\mu$ for some $\mu \in M$. A set $B \subset S$ is a *complete subgraph* of $\mathscr{G}(\mathscr{I})$ if each pair of vertices in B are joined by an edge of the graph. Thus $\mathscr{G}(\mathscr{I})$ is defined by the complete subgraphs $F_\mu (\mu \in M)$. A set $C \subset S$ is said to be $(\mathscr{I}, n)$*-free* if C is disjoint from at least n of the sets $F_\mu (\mu \in M)$, i. e. the cardinal number of the set $\{\mu \in M: F_\mu \cap C = 0\}$ is not less than n.

The problems studied in this paper have the following general combinatorial character. We seek conditions which enable us to assert that (in a precise sense) if $\mathscr{I}=(F, M, S)$ is any (m, α)-system, then either there is a large complete subgraph of $\mathscr{G}(\mathscr{I})$ or there is a large $(\mathscr{I}, n)$-free subset of S. For example, we study relations of the form

$$\alpha \to [\beta, \gamma]^2_m. \tag{1. 2}$$

By definition, (1. 2) means that the following statement is true. *Let $\mathscr{I}=(F, M, S)$ be any (m, α)-system. Then either there is a set $B \subset S$ of order type β which is a complete subgraph of $\mathscr{G}(\mathscr{I})$, or there is a $(\mathscr{I}, m)$-free subset of S of order type γ.* We investigate under what conditions (1. 2) is true or false. We study also a number of related problems. The main results of the paper and the outstanding problems are summarized in § 3 after we have introduced some notation.

§ 2. Notation and definitions. Capital letters $A, B, \dots, Z$ always denote sets. As usual, the symbols $\in, \subset, \cup, \cap$ denote respectively the membership relation, inclusion in the wide sense, and the binary operations of forming the union and intersection of sets. We use $\bigcirc$ to denote both zero and the empty set. $A \sim B$ denotes the set theoretical difference of A and B. If A is a set of sets then[1] $\bigcup(X \in A)X$

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[1] For typographical convenience we write $\bigcup (X \in A) X$ instead of $\bigcup_{X \in A} X$. We adopt a similar convention with other symbols. E. g. we write $\sum(\nu \leqq r)\alpha_\nu$ instead of $\sum_{\nu=0}^{r} \alpha_\nu$.

and $\bigcap(X \in A)X$ denote the union and intersection of all the sets $X \in A$. The cartesian product of A and B is $A \times B$. A^B denotes the set of all functions which map B into A. If Φ is any property which the elements of a set A may or may not possess, then $\{x \in A: \Phi(x)\}$ denotes the set of all those elements of A which possess the property Φ. In some cases, when it is clear that Φ refers only to the elements of A, we omit A from the symbol.

Small greek letters denote ordinal numbers *(ordinals)* and small latin letters denote cardinal numbers *(cardinals)*. If S is simply ordered by the binary relation R, then $\mathrm{tp}_R S$ denotes the order type of S with this ordering. In most cases that we consider there is only one order relation defined on a set S which we invariably denote by $<$. In this case we write simply $\mathrm{tp} S$ instead of $\mathrm{tp}_< S$. The cardinal of a set S is denoted by $|S|$. If $\mathrm{tp}\, S = \alpha$, then the *cardinal of* α is $|\alpha| = |S|$. We make no distinction between finite ordinals and the corresponding cardinals. Consequently, in view of the convention mentioned above, a small latin letter may also denote a *finite* ordinal, but this will always be made clear in any particular context. As usual, the strictly increasing sequence of infinite cardinals is denoted by $\aleph_0, \aleph_1, \ldots, \aleph_\alpha, \ldots$. The *initial ordinal* of cardinal a is the least ordinal with cardinal a and is denoted by $\omega(a)$. We write ω_α instead of $\omega(\aleph_\alpha)$, also we write just ω instead of ω_0. If r is finite, by our convention, $\omega(r) = r$. For any α, β the set $\{\nu: \alpha \leqq \nu < \beta\}$ is denoted by $[\alpha, \beta)$.

The obliterator sign $\hat{}$ written above any symbol means that that symbol is to be disregarded. Thus $A_0 \cup A_1 \cup \ldots \cup \hat{A}_\alpha$ denotes $\bigcup(\nu < \alpha)A_\nu$. The symbol $\{x_0, \ldots, \hat{x}_\alpha\}_<$ is used to indicate that the set $S = \{x_\nu: \nu < \alpha\}$ is simply ordered by $<$ and that $x_\lambda < x_\mu$ if $\lambda < \mu < \alpha$. In a similar way, $\{x_0, \ldots, \hat{x}_\alpha\}_{\neq}$ means that $x_\lambda \neq x_\mu$ if $\lambda < \mu < \alpha$.

If the sets $A_\nu (\nu < \alpha)$ are *disjoint* and *ordered*, then either of the symbols

$$S = \bigcup(\nu < \alpha)A_\nu(\mathrm{tp}) \quad \text{or} \quad S = A_0 \cup \ldots \cup \hat{A}_\alpha(\mathrm{tp}) \tag{2. 1}$$

mean that S is the union of the $A_\nu (\nu < \alpha)$ and that S is ordered in a natural way, i. e. the order relations in each A_ν are unchanged and elements of A_λ precede elements of A_μ if $\lambda < \mu < \alpha$. Thus, if $\mathrm{tp}\, A_\nu = \beta_\nu (\nu < \alpha)$, then (2. 1) implies that

$$\mathrm{tp}\, S = \beta_0 + \ldots + \hat{\beta}_\alpha.$$

We say that T is *cofinal* with the simply ordered set S if $T \subset S$ and for each $x \in S$ there is $y \in T$ such that $x \leqq y$. Also, if $\alpha = \mathrm{tp}\, S$ and $\beta = \mathrm{tp}\, T$, we say that β is cofinal with α. For any α, $\mathrm{cf}(\alpha)$ denotes the least ordinal cofinal with α — this is the cofinality index of α introduced by TARSKI [10]. For any α, $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$. If $\alpha = \beta + 1$, we write $\alpha^- = \beta$. If $\alpha > 0$ and $\alpha \neq \beta + 1$ for any β, then α is called a *limit ordinal* and we write $\alpha^- = \alpha$. Thus α is a limit ordinal if and only if $\mathrm{cf}(\alpha) > 1$. If α is such that $\beta + \gamma < \alpha$ whenever $\beta, \gamma < \alpha$, then α is *indecomposable*. It is well known that the only indecomposable ordinals are 0,1 and the powers of ω. Every ordinal $\alpha > 0$ can be expressed in a unique way as a finite sum of positive, non-increasing indecomposable ordinals, i. e.

$$\alpha = \alpha_0 + \ldots + \alpha_\varrho, \tag{2. 2}$$

where $\varrho < \omega$, α_λ is indecomposable $(\lambda \leqq \varrho)$ and $\alpha_0 \geqq \ldots \geqq \alpha_\varrho > 0$. We refer to (2. 2) as the *standard representation* of α. We say that α is *even* if $\alpha = 2\beta$, otherwise α is *odd*.

The next largest cardinal to a is a^+. If $a=b^+$, then we write $a^-=b$. If a is not the successor of any cardinal b (i. e. $a\neq b^+$), then a is a *limit cardinal*, and we write $a^-=a$. If $a\geqq\aleph_0$ then a' denotes the *cofinality cardinal* of a, i. e. the least cardinal b such that a is the sum of b smaller cardinals. If $a=\aleph_\alpha$, then $a'=\aleph_{\mathrm{cf}(\alpha)}$. For every a, $a'\leqq a$. If $a'=a$, a is said to be *regular*; otherwise, if $a'<a$, a is a *singular* number.

The set of all subsets of a set S is denoted by $[S]$. We write

$$[S]^a=\{X\colon X\subset S,\ |X|=a\}.$$

In particular, $[S]^a=0$ if $|S|<a$. In a similar way, $[S]^{<a}$ denotes the set $\{X\colon X\subset S, |X|<a\}$. If A, B are disjoint sets, then $[A,B]^{1,1}=\{\{x,y\}\colon x\in A, y\in B\}$.

A graph is an ordered pair $\mathcal{G}=(S,E)$ with $E\subset[S]^2$. More generally, if $E\subset[S]^k$, $\mathcal{G}=(S,E)$ is called a k-graph. S is the set of *vertices* of $\mathcal{G}$ and E is the set of *edges* (k-edges) of $\mathcal{G}$. The *complementary* graph of the k-graph $\mathcal{G}=(S,E)$ is $\mathcal{G}^*=(S,E^*)$, where $E^*=[S]^k\sim E$. A set $T\subset S$ is a *complete subgraph* of the k-graph $\mathcal{G}=(S,E)$ if $[T]^k\subset E$. If $S_1\subset S$ and $E_1=[S_1]^k\cap E$, then $\mathcal{G}_1=(S_1,E_1)$ is the *restriction* of $\mathcal{G}$ on S_1. A set $S'\subset S$ is a *connected component* of the graph $\mathcal{G}=(S,E)$ if for each pair of points $x,y\in S$ there is a finite integer r and edges $\Pi_0,\Pi_1,\ldots,\Pi_r\in E$ such that $x\in\Pi_0$, $y\in\Pi_r$ and $\Pi_\varrho\cap\Pi_{\varrho+1}\neq0$ ($\varrho<r$). The graph $\mathcal{G}=(S,E)$ is *p-chromatic* if S can be expressed as the union of p sets $X_\nu(\nu<\omega(p))$ which do not contain any edge of $\mathcal{G}$, i. e. $S=\bigcup(\nu<\omega(p))X_\nu$ and X_ν $(\nu<\omega(p))$ is a complete subgraph of the complementary graph $\mathcal{G}^*$.

An (m,α)-*system* of sets is an ordered triple $\mathcal{J}=(F,M,S)$, where S is an ordered set of type α, $|M|=m$ and

$$F\colon M\to[S] \tag{2. 3}$$

is a mapping on M into $[S]$, the set of subsets of S. The image of $\mu\in M$ under the mapping (2. 3) is denoted by F_μ. If we do not take the ordering of S into account, we call $\mathcal{J}=(F,M,S)$ an (m,a)-system, where $a=|S|$. The number of sets of the system $\mathcal{J}$ (i. e. $|M|$) is also denoted by $|\mathcal{J}|$. If $M_1\subset M$, then $\mathcal{J}_1=(F,M_1,S)$ is a *sub-system* of $\mathcal{J}$ and we write $\mathcal{J}_1\subset\mathcal{J}$. The set-system $\mathcal{J}=(F,M,S)$ defines a k-graph $\mathcal{G}_k(\mathcal{J})=(S,E_k)$ on the set S with edges

$$E_k=\bigcup(\mu\in M)[F_\mu]^k.$$

We are mostly concerned with ordinary 2-graphs and we usually write $\mathcal{G}(\mathcal{J})$ instead of $\mathcal{G}_2(\mathcal{J})$. A set $T\subset S$ is said to be $(\mathcal{J},k)$-*complete* if T is a complete subgraph of $\mathcal{G}_k(\mathcal{J})$. Similarly, we say $T\subset S$ is $(\mathcal{J},<k)$-*complete* if $[T]^{<k}\subset\bigcup(\mu\in M)[F_\mu]^{<k}$. A set $V\subset S$ is said to be $(\mathcal{J},n)$-*free* if $|\{\mu\in M\colon F_\mu\cap V=0\}|\geqq n$, i. e. V is disjoint from at least n sets of the system. If $\mathcal{J}=(F,M,S)$ is any system of sets we define

$$P(\mathcal{J})=\{x\in S\colon |\{\mu\in M\colon x\notin F_\mu\}|<|M|\},$$

$$Q(\mathcal{J})=\{x\in S\colon |\{\mu\in M\colon x\in F_\mu\}|=|M|\}.$$

We are mainly concerned with infinite set-systems, i. e. $|\mathcal{J}|\geqq\aleph_0$. In this case $P(\mathcal{J})$ is the set of elements of S which belong to *almost all* the sets F_μ and $P(\mathcal{J})\subset Q(\mathcal{J})$. Suppose that $\mathcal{J}_1=(F,M_1,S)$ is any infinite sub-system of $\mathcal{J}=(F,M,S)$. If X is a finite set, $X\subset P(\mathcal{J}_1)$, then $|\bigcup(x\in X)\{\mu\in M_1\colon x\notin F_\mu\}|<|M_1|\leqq|M|$. Hence, there is some $\mu\in M$ such that $X\subset F_\mu$. This proves that

$$P(\mathcal{J}_1)\ \ is\ \ (\mathcal{J},<\aleph_0)\text{-}complete\ \ if\ \ \mathcal{J}_1\subset\mathcal{J},\ |\mathcal{J}_1|\geqq\aleph_0. \tag{2. 4}$$

The symbol

(2. 5) $$\alpha \to [\beta, \gamma]^k_m$$

means that the following statement is true. *Let* $\mathscr{J}=(F, M, S)$ *be any* (m, α)*-system. Then either* (i) *there is a set* $B \subset S$ *of type* β *which is* $(\mathscr{J}, k)$*-complete, or* (ii) *there is a set* $C \subset S$ *of type* γ *which is* $(\mathscr{J}, m)$*-free.* If this last statement is true when we replace the condition $(\mathscr{J}, k)$-complete in (i) by $(\mathscr{J}, <k)$-complete, we write instead

(2. 6) $$\alpha \to [\beta, \gamma]^{<k}_m.$$

The symbol

(2. 7) $$\alpha \Rightarrow [\beta, \gamma]_m$$

means the following is true. *If* $\mathscr{J}=(F, M, S)$ *is any* (m, α)*-system such that* tp $F_\mu < \beta$ $(\mu \in M)$*, then there is a set* $C \subset S$ *of type* γ *which is* $(\mathscr{J}, m)$*-free.* Since each set F_μ $(\mu \in M)$ is a $(\mathscr{J}, <k^+)$-complete set, it follows that (2. 7) implies (2. 5) and (2. 6) for any k.

For part of our discussion it is convenient to use another symbol

(2. 8) $$\alpha \rightleftarrows [\beta, \gamma]_m$$

which is weaker than (2. 7) but stronger than (2. 6) in the special case when $k=\aleph_0$ and $m \geqq \aleph_0$. By definition, (2. 8) means the following is true. *Let* $\mathscr{J}=(F, M, S)$ *be any* (m, α)*-system such that* tp $F_\mu < \beta$ $(\mu \in M)$*. Then either* (i) *there is an* (m, α)*-system* $\mathscr{J}_1 \subset \mathscr{J}$ *such that* tp $P(\mathscr{J}_1) \geqq \beta$ *or* (ii) *there is a set* $C \subset S$ *of type* γ *which is* $(\mathscr{J}, m)$*-free.* The relation (2. 8) is clearly weaker than (2. 7), but if $m \geqq \aleph_0$ then, in view of (2. 4), (2. 8) implies that $\alpha \to [\beta, \gamma]^{<\aleph_0}_m$.

Similar to the relations (2. 5), (2. 6) and (2. 7) we have analogous relations

(2. 5)′ $$a \to [b, c]^k_m,$$

(2. 6)′ $$a \to [b, c]^{<k}_m,$$

(2. 7)′ $$a \Rightarrow [b, c]_m$$

connecting cardinal numbers. For example, (2. 7)′ means the following. *If* $\mathscr{J}=(F, M, S)$ *is any* (m, a)*-system such that* $|F_\mu| < b$ $(\mu \in M)$*, then there is a set* $C \in [S]^c$ *which is* $(\mathscr{J}, m)$*-free.* By the well-ordering axiom (2. 7)′ is equivalent to the special case of (2. 7) when $\alpha=\omega(a)$, $\beta=\omega(b)$ and $\gamma=\omega(c)$. A similar remark applies in the case of (2. 5)′ and (2. 6)′.

We shall investigate relations of the forms

(2. 9) $$(a, m, n, c)^k \to s,$$

(2. 10) $$(a, m, n, c)^k \twoheadrightarrow s.$$

(2. 9) means that the following is true. *Let* $\mathscr{J}=(F, M, S)$ *be any* (m, a)*-system of sets which is such that S does not contain any* $(\mathscr{J}, n)$*-free subset of cardinal c. Then S is the union of a set of power less than a and fewer than s* $(\mathscr{J}, k)$*-complete sets, i. e.* $S=\bigcup(\nu<\delta)B_\nu \cup A$*, where* $|A|<a$, $\delta<\omega(s)$ *and* B_ν $(\nu<\delta)$ *is* $(\mathscr{J}, k)$*-complete.* The statement (2. 10) means that, under the same hypothesis, a stronger conclusion holds namely that *S is the union of fewer than s sets which are* $(\mathscr{J}, k)$*-complete* (i. e.

the last statement holds with $A=0$). The symbols (2. 9) and (2. 10) both have an obvious interpretation when k is replaced by $<k$. In some respects (2. 9) is more general than (2. 5)'. For example, if $s<a'$, then $(a, m, m, c)^k \to s$ implies that $a \to [a, c]^k_m$.

For finite integers m, n, k, s (2. 9) is closely related to

(2. 11) $$(m, n)^k \to s$$

(see Theorem 12. 1). By definition (2. 11) means the following holds. *Let* $\mathcal{J}=(F, M, S)$ *be any system of* $|M|=m$ *subsets of a set* S *such that the union of any* n *of these subsets cover* S, *i. e.*

$$S=\bigcup(\mu\in N)F_\mu \qquad (N\in[M]^n).$$

Then S *is the union of fewer than* s *complete subgraps of the* k*-graph* $\mathcal{G}_k(\mathcal{J})$, *i. e.* $S=C_1\cup C_2\cup\ldots\cup\hat{C}_s$ *and* $[C_\sigma]^k\subset\bigcup(\mu\in M)[F_\mu]^k \quad (1\leq\sigma<s)$.

In the final section of this paper we establish a few results in another direction. These are expressed in terms of the symbol

(2. 12) $$(m, \alpha, \beta)^2 \to n$$

which means the following. *Let* $\mathcal{J}=(F, M, S)$ *be any* (m, α)*-system which has no* $(\mathcal{J}, 2)$*-complete subset of type* β. *Then* S *is the union of* n *sets* S_ν $(\nu<\omega(n))$ *such that* $\operatorname{tp} S_\nu=\alpha(\nu<\omega(n))$ *and for each* $\mu\in M$ *there is some* $\nu=\nu(\mu)<\omega(n)$ *such that* $F_\mu\cap S_\nu=0$. We only consider the relation (2. 12) in the case when α is indecomposable. We use the same symbol with cardinals a, b in place of the ordinals α, β and the interpretation, in this case, is the obvious one.

In some of our proofs we employ known results (from [2], [3], [7]) about the *partition* symbol

(2. 13) $$\alpha \to (\beta_0, \ldots, \hat{\beta}_\lambda)^r.$$

This means: *If* $\operatorname{tp} S=\alpha$ *and* $[S]^r=\bigcup(\nu<\lambda)K_\nu$, *then there is a set* $S'\subset S$ *and an index* $\nu<\lambda$ *such that* $\operatorname{tp} S'=\alpha_\nu$ *and* $[S']^r\subset K_\nu$. Like the other symbols we have used, (2.13) has an obvious interpretation when the ordinals $\alpha, \beta_0, \ldots, \hat{\beta}_\lambda$ are replaced by cardinals. If $\beta_\nu=\beta$ $(\nu<\lambda)$, we write (2. 13) in the alternative form

$$\alpha \to (\beta)^r_\lambda.$$

We employ also the so-called *polarized* partition symbol

(2. 14) $$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix}^{1,1}.$$

This means: *If* A, B *are disjoint sets*, $|A|=a$, $|B|=b$ *and* $[A, B]^{1,1}=K_0+K_1$, *then there are sets* $A'\subset A$, $B'\subset B$ *and* $\varrho<2$ *such that* $|A'|=a_\varrho$, $|B'|=b_\varrho$ *and* $[A', B']^{1,1}\subset K_\varrho$.

The negation of any one of the 'arrow' relations (2. 5)—(2. 14) is conveniently expressed by striking out the arrow — thus $\alpha\nrightarrow[\beta, \gamma]^k_m$ indicates that (2. 5) is false.

The continuum hypothesis asserts that $2^{\aleph_0}=\aleph_1$. We sometimes make use of the more general hypothesis that $2^{\aleph_\nu}=\aleph_{\nu+1}$ for all ν. When the statement of a theorem is prefaced by $(*)$ this means that some form of the continuum hypothesis is used in the proof.

§ 3. Summary of results and problems. The symbol (2.6), or

(3. 1) $$\alpha \to [\beta, \gamma]^{<k}_m,$$

is slightly more general than (2. 5) in the sense that this last relation implies $\alpha \to [\beta, \gamma]^p_m$ for any $p < k$. In § 4 we establish a few general results concerning these symbols and also (2. 7). In § 5 a complete analysis of (3. 1) is given in the two simple cases (i) $k = 2$, m arbitrary and (ii) m finite, k arbitrary. In these cases (3. 1) is closely related to the partition symbol (2. 13) with $r = 1$; such relations have already been fully discussed in [7].

In § 6 we study the cardinal forms of these relations (2. 5)′, (2. 6)′ and (2. 7)′. Our main result in this section (Theorem 6. 1) shows that

(3. 2) $$m \to [m, m]^{<\aleph_0}_n$$

holds for any $m, n > \aleph_0$. In fact, if $m' \neq n'$, then the stronger relation $m \Rightarrow [m, m]_n$ holds. We have only very few results of the form (2. 5)′ when $k \geqq \aleph_0$. We prove (Theorems 6. 3, 6. 4) that

$$m \to [m, n]^p_m \quad \text{and} \quad m^+ \nrightarrow [m^+, m]^m_{m}.$$

provided $m \geqq \aleph_0$ and $p^n < m'$. The following questions which we cannot resolve represent extensions of these results in various directions.

PROBLEM 1. (?) $\aleph_2 \to [\aleph_2, a]^{\aleph_0}_{\aleph_2}$ $(a = \aleph_1 \text{ or } \aleph_2)$.

PROBLEM 2. (?) $\aleph_{\omega+1} \to [\aleph_{\omega+1}, \aleph_0]^{\aleph_\omega}_{\aleph_{\omega+1}}$.

PROBLEM 3. (?) $\aleph_{\omega_1} \to [\aleph_{\omega_1}, \aleph_0]^{\aleph_0}_{\aleph_1}$.

In § 7,8 we study (3. 1) and similar relations when $m = \aleph_0$ and α, β, γ are denumerable ordinals. As we have already remarked, (2. 8) is stronger than (3. 1) in this case with $k = \aleph_0$. Our main result in these sections (Theorem 8. 2) shows that for given $\beta, \gamma < \omega_1$, there is $\alpha_0 < \omega_1$ such that

$$\alpha_0 \rightleftarrows [\beta, \gamma]_{\aleph_0},$$

$$\alpha \nrightarrow [\beta, \gamma]^2_{\aleph_0} \qquad (\text{if } \alpha < \alpha_0).$$

We also show how to calculate α_0 in terms of the given β, γ. In § 9 we study the similar problem with denumerable α, β, γ but arbitrary m. A surprising feature of this analysis is that the relations

$$\alpha \to [\beta, \gamma]^2_m \quad \text{and} \quad \alpha \to [\beta, \gamma]^{<\aleph_0}_m$$

are equivalent for $\alpha, \beta, \gamma < \omega_1$. We do not know if this is the case for arbitrary ordinals α, β, γ.

In § 10 we study (3. 1) in the case when α is non-denumerable. Our analysis is incomplete and we mainly restrict our attention to relations of the form

$$\alpha \to [\beta, \gamma]^2_{\aleph_1}$$

with α indecomposable. With this restriction we are able to decide whether or not such relations hold provided $\alpha \leqq \omega_1^{\omega+1}$. It is a little surprising that new difficulties are encountered at this stage. The simplest question we cannot answer is

PROBLEM 4. (?) $\omega_1^{\omega+1}\omega \to [\omega_1^{\omega+1}, \omega_1\omega]^2_{\aleph_1}$.

There is another interesting problem which arises in this section. We can show (Theorem 10. 6) that $\omega_2\alpha \Rightarrow [\omega_1^\omega, \omega_2\alpha]_{\aleph_2}$ if $\alpha < \omega_1$, but we cannot solve

PROBLEM 5 (?) $\omega_2\omega \Rightarrow [\omega_1^\omega + 1, \omega_2\omega]_{\aleph_2}$.

In § 10 we also study relations of the form

$$\alpha \to [\beta, \gamma]^2_{\aleph_0}$$

in the case $\alpha = \omega_1^\lambda$ ($\lambda < \omega_2$). Again our analysis is incomplete. In Theorem 10. 9 we prove that

$$\omega_1^\lambda \to [\omega_1^\lambda, \omega_1^\omega]^2_{\aleph_0} \quad \text{if} \quad \omega \leqq \lambda < \omega_2$$

and in Theorem 10. 10 we show that this result is best possible in the case $\mathrm{cf}(\lambda) = \omega$. The stronger relation

$$\omega_1^\lambda \to [\omega_1^\lambda, \gamma]^2_{\aleph_0} \qquad (\gamma < \omega_1^{\omega+1})$$

is proved in Theorem 10. 15 for the case when $\mathrm{cf}(\lambda) \neq \omega$ and $\omega < \lambda < \omega_2$. We also prove that this last result is best possible if $\lambda = \mu + 1$ and $\mathrm{cf}(\mu) = \omega$. In Theorem 10. 14 we prove the negative relation

$$\alpha \nrightarrow [\omega_1^\omega, \omega_1^{\omega+2}]^2_{\aleph_0} \qquad (\alpha < \omega_2).$$

We cannot answer the following questions

PROBLEM 6. (?) $\omega_1^\lambda \to [\omega_1^\omega, \omega_1^{\omega+1}]^2_{\aleph_0}$ (any $\lambda < \omega_2$).

PROBLEM 7. (?) $\omega_1^{\omega+2} \to [\omega_1^{\omega+\nu}, \omega_1^{\omega+1}\alpha]^2_{\aleph_0}$ ($\nu = 1$ or 2; $\alpha < \omega_1$).

In § 11 we establish a few general results concerning (2. 9) and (2. 10). For example, we show that $(a, m, n, c)^2 \to 2$ is equivalent to the polarized partition relation

$$\begin{pmatrix} a \\ m \end{pmatrix} \to \begin{pmatrix} c & c \\ n & n \end{pmatrix}^{1,1}$$

if $a \geqq c \geqq \aleph_0$. This enables us to utilize some results from [2]. In § 12 we show that (Theorem 12. 1) for *finite* k, m, n, s

$$(a, m, n, c)^k \to s$$

is equivalent to (2. 11). We prove (Theorem 12. 2) that

$$(m, n)^k \to s$$

holds if $m \geqq k(n-s+1)+s-1$ and we show that this is best possible in a number of cases. We cannot prove this in general, i. e.

PROBLEM 8. (?) $(m, n)^k \nrightarrow s$ *if* $m < k(n-s+1)+s-1$.

In § 13, § 14 we investigate (2. 9) in the case of infinite cardinals. In Theorem 13. 1 we show that

(3. 3) $$(\aleph_0, \aleph_0, \aleph_0, \aleph_0)^{<\aleph_0} \to \aleph_0.$$

This is a stronger statement than (3. 2) when $m = n = \aleph_0$. We also show that the

$\aleph_0$ on the right-side of (3. 3) cannot be replaced by a finite number. For k-graphs $(k<\aleph_0)$ we have the much stronger result (Theorem 14. 4)

$$(m, m, m, m)^k \to 3 \quad \text{if} \quad m' = \aleph_0.$$

Also, if m is not a limit number (i. e. $m = n^+$), then (Theorem 14. 1)

$$(m, m, m, m)^2 \to m$$

holds. These partial generalizations of (3. 3) leave several questions unanswered, e. g.

PROBLEM 9. (?) $(\aleph_1, \aleph_1, \aleph_1, \aleph_1)^3 \to \aleph_1$.

PROBLEM 10. (?) $(\aleph_{\omega_1}, \aleph_{\omega_1}, \aleph_{\omega_1}, \aleph_{\omega_1})^2 \to \aleph_{\omega_1}$.

It follows from Theorem 14. 5 that

$$(\aleph_\omega, \aleph_1, \aleph_1, \aleph_\omega)^2 \nrightarrow \aleph_0,$$

but we do not know if the following is true.

PROBLEM 11. (?) $(\aleph_\omega, \aleph_1, \aleph_1, \aleph_\omega)^2 \to \aleph_1$.

We prove in Theorem 14. 6 that, if $m \geqq \aleph_0$,

$$(m^+, m^+, m^+, m^+)^2 \nrightarrow m$$

and

$$(m^+, m^+, m^+, m)^2 \nrightarrow n \quad \text{if} \quad n < m.$$

Also, in Theorem 14. 7, we strengthen the last formula, in the case when m is a regular limit number, to

$$(m^+, m^+, m^+, m)^2 \nrightarrow m.$$

We do not know if a similar result holds when m is a singular limit number.

PROBLEM 12. (?) $(\aleph_{\nu+1}, \aleph_{\nu+1}, \aleph_{\nu+1}, \aleph_\nu)^2 \to \aleph_\nu \qquad (\nu = \omega \text{ or } \omega_1)$.

In § 15 we establish a few relations of the form (2. 13). It follows from Theorem 15. 1 and the partition relations

(3. 4) $$\omega \to (\omega, \omega)^2,$$

(3. 5) $$\omega^2 \to (\omega^2, \beta)^2 \quad \text{if} \quad \beta < \omega$$

due to RAMSEY [8] and SPECKER [9], that

$$(m, \omega, \omega)^2 \to 3,$$

$$(m, \omega^2, \beta)^2 \to 3 \qquad \text{if} \qquad \beta < \omega.$$

Also, by using essentially the same construction used by SPECKER [9] to prove that $\omega^3 \nrightarrow (\omega^3, 3)^2$, it is easy to prove that

$$(m, \omega^3, 4)^2 \nrightarrow 3 \qquad (m \geqq \aleph_0).$$

We do not know if the following is true.

PROBLEM 13. (?) $(\aleph_0, \omega^3, 5)^2 \to 4$.

For infinite cardinals we prove (Theorem 15. 5) that

$$(m, a, a)^2 \to (a')^+ \quad \text{if} \quad a' < a.$$

This result is best possible since we prove also (Theorem 15. 6) that

$$(m, a, (a')^+)^2 \nrightarrow a' \quad \text{if} \quad m \geqq a^+.$$

In Theorem 15. 7 we show that

$$(\aleph_1, \aleph_1, \aleph_1)^2 \nrightarrow \aleph_0,$$

and this is obviously best possible. We conclude with the following question.

PROBLEM 14. (?) $(\aleph_2, \aleph_2, \aleph_2)^2 \to \aleph_1$.

§ 4. Some general results. The theorems of this section give connectives between different relations of the various forms (2. 5)—(2. 8). For the purposes of this section it is unnecessary to distinguish between (2. 5) and (2. 6). Consequently, we state our results only in terms of the first of these $\alpha \to [\beta, \gamma]^k_m$ — it is a trivial matter to check that similar statements remain valid if k is replaced by $<k$. Frequently the proof of a statement concerning the symbol (2. 5) requires only slight modification to establish analogous results about (2. 7). To avoid tedious repetition we merely state corresponding results.

As an immediate consequence of the definitions we have the following monotonicity relations.

THEOREM 4. 1. *Let* $\alpha \leqq \alpha', \beta \geqq \beta', \gamma \geqq \gamma', l \leqq k \leqq |\beta|$. *Then*

$$\alpha \to [\beta, \gamma]^k_m \quad \textit{implies} \quad \alpha' \to [\beta', \gamma']^l_m,$$

$$\alpha \Rightarrow [\beta, \gamma]_m \quad \textit{implies} \quad \alpha' \Rightarrow [\beta', \gamma']_m,$$

$$a \rightleftarrows [\beta, \gamma]_m \quad \textit{implies} \quad a' \rightleftarrows [\beta' \gamma']_m.$$

THEOREM 4. 2. *If* $m \geqq \aleph_0$ *and* $\alpha \to [\beta, \gamma]^k_m$, *then* $\alpha \to [\beta, \gamma]^k_{m'}$.

REMARK. Similarly, *if* $m \geqq \aleph_0$ *and* $\alpha \Rightarrow [\beta, \gamma]_m$, *then* $\alpha \Rightarrow [\beta, \gamma]_{m'}$.

PROOF OF THEOREM 4. 2. Let $\mathscr{F}' = (F', M', S)$ be an (m', α)-system. We will assume that S does not contain any $(\mathscr{F}', k)$-complete subset of type β and deduce that S contains a $(\mathscr{F}', m')$-free subset of type γ.

There are disjoint sets M_ν $(\nu \in M')$ such that $|M_\nu| < m$ $(\nu \in M')$ and such that $M = \bigcup(\nu \in M')M_\nu$ has cardinal m. Consider the (m, α)-system $\mathscr{F} = (F, M, S)$, where $F_\mu = F'_\nu$ if $\mu \in M_\nu$ $(\nu \in M')$. Since the two k-graphs $\mathscr{G}_k(\mathscr{F})$ and $\mathscr{G}_k(\mathscr{F}')$ are identical, it follows from the hypothesis $\alpha \to [\beta, \gamma]^k_m$ that there is a set $C \subset S$ of type γ which is $(\mathscr{F}, m)$-free. Hence, there is $M'_1 \subset M'$ such that $|M'_1| = m'$ and $C \cap F'_\nu = 0$ $(\nu \in M'_1)$. Therefore, C is also $(\mathscr{F}', m')$-free.

We do not know if the converse of Theorem 4. 2 is true, but in the next two theorems we show that this is the case when extra conditions are imposed on m.

THEOREM 4. 3. *Let* $m \geqq \aleph_0$; $|\alpha|^{|\gamma|} \leqq n \leqq m'$; $\alpha \to [\beta, \gamma]^k_n$. *Then* $\alpha \to [\beta, \gamma]^k_m$.

PROOF. Let $\mathcal{J}=(F, M, S)$ be any (m, α)-system. We will suppose that there is no $(\mathcal{J}, m)$-free subset of type γ and deduce that there is an $(\mathcal{J}, k)$-complete subset of S of type β.

We want to show first that there is a sub-system $\mathcal{J}'\subset\mathcal{J}$ such that $|\mathcal{J}'|=n$ and such that S does not contain any $(\mathcal{J}', n)$-free subset of type γ. Let $K=\{C: C\subset S, \text{tp } C=\gamma\}$. Then $|K|\leqq|\alpha|^{|\gamma|}\leqq n\leqq m'$. Also, by our assumption, $|\{\mu\in M: C\cap F_\mu=0\}|<m$ $(C\in K)$.

Case 1. $|K|<m'$. Then $|M^*|<m$, where $M^*=\bigcup(C\in K)\{\mu: F_\mu\cap C=0\}$. Hence there is a set of cardinal n, $N\in[M\sim M^*]^n$. In this case put $\mathcal{J}'=(F, N, S)$. Clearly, S does not contain any $(\mathcal{J}', n)$-free subset of type γ since $F_\mu\cap C\neq 0$ $(\mu\in N;\ C\in K)$.

Case 2. $|K|=m'$. Then $m'=n$, and $K=\{C_0, C_1, \ldots, \hat{C}_\lambda\}_{\neq}$ where $\lambda=\omega(n)$. It follows by induction that for $\nu<\lambda$ there is

$$\mu(\nu)\in M\sim\bigcup(\varrho<\nu)\{\mu\in M: C_\varrho\cap F_\mu=0\}\cup\{\mu(\varrho)\}.$$

Now put $N=\{\mu(0), \ldots, \hat{\mu}(\lambda)\}$ and let $\mathcal{J}'=(F, N, S)$. Then $\mathcal{J}'\subset\mathcal{J}$ and $|\mathcal{J}'|=n$. Moreover, S does not contain any $(\mathcal{J}', n)$-free subset of type γ since $C_\varrho\cap F_{\mu(\nu)}\neq 0$ $(\varrho<\nu<\lambda)$.

The hypothesis $\alpha\to[\beta, \gamma]^k_n$ now implies that there is a set $B\subset S$ of type β which is $(\mathcal{J}', k)$-complete. The set B is also $(\mathcal{J}, k)$-complete since $\mathcal{J}'\subset\mathcal{J}$.

$(*)$ THEOREM 4. 4. *If* $m\geqq\aleph_0 2^{|\alpha|}$ *and* $\alpha\to[\beta, \gamma]^k_{m'}$, *then* $\alpha\to[\beta, \gamma]^k_m$.

REMARK. Similarly, *if* $m\geqq\aleph_0 2^{|\alpha|}$ *and* $\alpha\Rightarrow[\beta, \gamma]_{m'}$ *then* $\alpha\Rightarrow[\beta, \gamma]_m$.

PROOF OF THEOREM 4. 4. We can assume that $m>m'$. Then $(*)$ implies that $m>2^{|\alpha|}$. Let $\mathcal{J}=(F, M, S)$ be any (m, α)-system and suppose there is no $(\mathcal{J}, k)$-complete subset of S of type β.

Let $\lambda=\omega(m')$. Then $m=m_0+\ldots+\hat{m}_\lambda$, where $m_\nu<m(\nu<\lambda)$. For $\nu<\lambda$ there is $\mu(\nu)\in M$ such that

$$|\{\mu\in M: F_\mu=F_{\mu(\nu)}\}|>m_\nu. \tag{4. 1}$$

Otherwise, we should obtain the contradiction $m=|M|\leqq 2^{|\alpha|}m_\nu<m$. Put $N=[0, \lambda)$ and let $F'_\nu=F_{\mu(\nu)}$ $(\nu\in N)$. Then $\mathcal{J}'=(F', N, S)$ is an (m', α)-system. Moreover, since the k-graph $\mathcal{G}_k(\mathcal{J}')$ is a subgraph of $\mathcal{G}_k(\mathcal{J})$, there is no $(\mathcal{J}', k)$-complete subset of S of type β. The hypothesis of the theorem now implies that there is a set $C\subset S$ of type γ which is $(\mathcal{J}', m')$-free. Hence, there is $N'\in[N]^{m'}$ such that $C\cap F'_\nu=0$ if $\nu\in N'$. Put $M'=\bigcup(\nu\in N')\{\mu: F_\mu=F'_\nu\}$. Then (4. 1) implies that $|M'|>m_\nu$ for all $\nu<\lambda$, i. e. $|M'|=m$. Since $C\cap F_\mu=0$ $(\mu\in M')$, it follows that C is $(\mathcal{J}, m)$-free.

We have also the following

$(*)$ THEOREM 4. 5. *If* $m>n\geqq\aleph_0$ *and* $m'\to[m', m']^k_n$, *then* $m\to[m, m]^k_n$.

REMARK. We have also, *if* $m>n>\aleph_0$ *and* $m'\Rightarrow[m', m']_n$, *then* $m\Rightarrow[m, m]_n$.

PROOF OF THEOREM 4. 5. We may assume that $m>m'$. Let $\lambda=\omega(m')$. Then we may write $m=m_0+m_1+\ldots+\hat{m}_\lambda$, where $n^+<m_0<m_1<\ldots<\hat{m}_\lambda$ and $m'_\nu=m_\nu$ $(\nu<\lambda)$.

Let $\mathcal{J}=(F, N, S)$ be any (n, m)-system. Since N contains only n^+ subsets and $n^+<m'_\nu$, it follows that for $\nu<\lambda$ there are $N_\nu\subset N$ and $S_\nu\in[S]^{m_\nu}$ such that

$$\{\mu\in N: x\in F_\mu\}=N_\nu \qquad (x\in S_\nu).$$

Choose $x_\nu \in S_\nu \sim \cup (\varrho < \nu) S_\varrho$ $(\nu < \lambda)$. Put $S' = \{x_\nu : \nu < \lambda\}$ and let $F'_\mu = F_\mu \cap S'$ $(\mu \in N)$. Then $\mathscr{J}' = (F', N, S')$ is an (n, m')-system. The k-graph $\mathscr{G}_k(\mathscr{J})$ restricted to the set $\cup (\nu < \lambda) S_\nu$ is completely characterized by the k-graph $\mathscr{G}_k(\mathscr{J}')$. More precisely, if $T' \subset S'$ is a $(\mathscr{J}', k)$-complete set, then the corresponding set

$$T = \cup (x_\nu \in T', \nu < \lambda) S_\nu \tag{4. 2}$$

is $(\mathscr{J}, k)$-complete. Similarly, if T' is an $(\mathscr{J}', n)$-free subset of S', then the set T given by (4. 2) is $(\mathscr{J}, n)$-free. The result now follows from the hypothesis $m' \to [m', m']^k_n$, for if $|T'| = m'$, the set T in (4. 2) has cardinal m.

Later on we require

THEOREM 4. 6. *If* $\delta \geqq 0$; $|\varepsilon| < m'$; $\alpha \Rightarrow [\beta, \gamma]_m$, *then*

$$\delta\alpha\varepsilon \Rightarrow [\beta\varepsilon, \delta\gamma]_m.$$

PROOF. Let $\mathscr{J} = (F, M, S)$ be a $(m, \delta\alpha\varepsilon)$-system such that $\text{tp}\, F_\mu < \beta\varepsilon$ $(\mu \in M)$. We may write $S = \cup (\varrho < \varepsilon) S_\varrho$ (tp), where $\text{tp}\, S_\varrho = \delta\alpha$ $(\varrho < \varepsilon)$. If $\mu \in M$, then there is $\varrho(\mu) < \varepsilon$ such that $\text{tp}\, F_\mu \cap S_{\varrho(\mu)} < \beta$. Since $|\varepsilon| < m'$ it follows that there is $M' \in [M]^m$ such that $\varrho(\mu) = \varrho$ $(\mu \in M')$. We have that $S_\varrho = \cup (\lambda < \alpha) T_\lambda$ (tp), where $\text{tp}\, T_\lambda = \delta$ $(\lambda < \alpha)$. For $\mu \in M'$ let $F'_\mu = \{\lambda < \alpha : T_\lambda \cap F_\mu \neq 0\}$. Then $\text{tp}\, F'_\mu < \beta$ $(\mu \in M')$. The hypothesis $\alpha \Rightarrow [\beta, \gamma]_m$ implies that there is $A \subset [0, \alpha)$ and $M'' \in [M']^m$ such that $F'_\mu \cap A = 0$ $(\mu \in M'')$ and $\text{tp}\, A = \gamma$. The set $\cup (\lambda \in A) T_\lambda$ is disjoint from the sets F_μ $(\mu \in M'')$ and has type $\delta\gamma$.

§ 5. Two special cases. In this section we analyse the symbol $\alpha \to [\beta, \gamma]^{<k}_m$ in the following two simple cases (i) $k = 2$, m arbitrary, (ii) k arbitrary and m finite. Our results are expressed in terms of the partition symbol (2. 13) in the special case when $r = 1$; these 'unitary' partition relations have been completely discussed in [7].

THEOREM 5. 1. *For any* $m \geqq 1$, *the relations* $\alpha \to [\beta, \gamma]^1_m$ *and* $\alpha \to (\beta, \gamma)^1$ *are equivalent.*

PROOF. Suppose that $\alpha \to (\beta, \gamma)^1$. Let $\mathscr{J} = (F, M, S)$ be any (m, α)-system. Then $B = \cup (\mu \in M) F_\mu$ is $(\mathscr{J}, 1)$-complete and $C = S \sim B$ is $(\mathscr{J}, m)$-free. Since either $\text{tp}\, B \geqq \beta$ or $\text{tp}\, C \geqq \gamma$, it follows that $\alpha \to [\beta, \gamma]^1_m$.

Now suppose that $\alpha \not\to (\beta, \gamma)^1$. Let $\text{tp}\, S = \alpha$. Then $S = X \cup Y$, where $\text{tp}\, X \ngeqq \beta$, $\text{tp}\, Y \ngeqq \gamma$. Let $M = (0, \omega(m))$ and let $F_\mu = X$ $(\mu \in M)$. Consider the (m, α)-system $\mathscr{J} = (F, M, S)$. If B' is $(\mathscr{J}, 1)$-complete, then $B' \subset X$ and $\text{tp}\, B' \ngeqq \beta$. If C' is $(\mathscr{J}, m)$-free then (since $m \geqq 1$) $C' \subset Y$ and $\text{tp}\, C' \ngeqq \gamma$. This shows that $\alpha \not\to [\beta, \gamma]^1_m$.

The condition that m is finite is only required for the second part of

THEOREM 5. 2. (i) *Let* $m \geqq 1$, $\mu = \omega(m)$, $\alpha \to (\beta_0, \dots, \hat{\beta}_\mu, \gamma)^1$, *where* $\beta_\lambda = \beta$ $(\lambda < \mu)$. *Then* $\alpha \Rightarrow [\beta, \gamma]_m$.

(ii) *Let* $1 \leqq m < \aleph_0$, $k \geqq 2$. *If* $\alpha \not\to (\beta_0, \dots, \hat{\beta}_m, \gamma)^1$, *where* $\beta_\lambda = \beta$ $(\lambda < m)$, *then* $\alpha \not\to [\beta, \gamma]^k_m$.

PROOF. Let $\mathscr{J} = (F, M, S)$ be any (m, α)-system of sets such that $\text{tp}\, F_\mu < \beta$ $(\mu \in M)$. The hypothesis of (i) implies that $\text{tp}\, C \geqq \gamma$, where $C = S \sim \cup (\mu \in M) F_\mu$. Since C is $(\mathscr{J}, m)$-free, this proves that $\alpha \Rightarrow [\beta, \gamma]_m$.

Now assume the hypothesis of (ii) holds. If S is an ordered set of type α, then there are disjoint sets $F_\mu \subset S$ $(\mu \leqq m)$ such that $\text{tp}\, F_\mu < \beta$ $(\mu < m)$ and $\text{tp}\, F_m < \gamma$.

Let $M=[0, m)$ and let $\mathscr{J}=(F, M, S)$. If B is any $(\mathscr{J}, k)$-complete subset of S then, since $k\geqq 2$ and the sets F_μ are disjoint, it follows that there is some $\mu<m$ such that $B\subset F_\mu$. Hence there is no $(\mathscr{J}, k)$-complete subset of S of type β. On the other hand, since m is finite, a set $C\subset S$ is $(\mathscr{J}, m)$-free only if $C\subset F_m$. Hence S contains no $(\mathscr{J}, m)$-free subset of type γ. This proves (ii).

We mentioned in § 3 that the relations

$$\alpha\to[\beta, \gamma]^2_m \quad\text{and}\quad \alpha\to[\beta, \gamma]^{<\aleph_0}_m$$

are equivalent for denumerable α, β, γ. It is easy to see that a similar remark does not apply to the symbol $\alpha\to[\beta, \gamma]^1_m$. For example, since

$$\omega 2\to(\omega 2, \omega)^1 \quad\text{and}\quad \omega 2\nrightarrow(\omega 2, \omega 2, \omega)^1,$$

it follows from the two theorems that

$$\omega 2\to[\omega 2, \omega]^1_m \quad\text{and}\quad \omega 2\nrightarrow[\omega 2, \omega]^2_m \quad\text{if}\quad m\geqq 2.$$

§ 6. Cardinal numbers. The main result of this section is

(∗) THEOREM 6. 1. *Let* $m, n\geqq\aleph_0$. *Then*

$$m\to[m, m]^{<\aleph_0}_n. \tag{6. 1}$$

Also, if $m'\neq n'$, *then*

$$m\Rightarrow[m, m]_n. \tag{6.2}$$

In order to prove this theorem, we prove first a lemma. This is a special case of a more general result (Canonization Lemma) proved in [2].

(∗) LEMMA 6. 1. *Let* $p<\aleph_0$; $m>m'\geqq\aleph_0$; $\lambda=\omega(m')$; $m=m_0+\ldots+\hat{m}_\lambda$; $m_\mu=m'_\mu>(m'+m_0+\ldots+\hat{m}_\mu)^+$ $(\mu<\lambda)$. *Let* $S=S_0\cup\ldots\cup\hat{S}_\lambda$; $|S_\mu|=m_\mu$ $(\mu<\lambda)$; $[S]^2=L_0\cup\ldots\cup L_p$. *Then there are disjoint sets* $S'_\mu\subset S_\mu$ $(\mu<\lambda)$ *such that* $|S'_\mu|=m_\mu$ $(\mu<\lambda)$ *and*

$$[S'_\mu, S'_\nu]^{1,1}\subset L_{\pi(\mu,\nu)} \qquad (\mu<\nu<\lambda),$$

where $\pi(\mu,\nu)\leqq p$ $(\mu<\nu<\lambda)$.

PROOF. Let $P=\{0, 1, \ldots, p\}$. Since $[S]^2=\bigcup(\nu\in P)L_\nu$, there is a function $\varphi\in P^{[S]^2}$ such that $\{x, y\}\in L_{\varphi(\{x,y\})}$ $(\{x, y\}_{\neq}\subset S)$. Let $T_\mu=\bigcup(\nu<\mu)S_\nu$ for $\mu<\lambda$. The hypothesis implies that $|P^{T_\mu}|<m'_\mu$. Hence there is a function $\psi_\mu\in P^{T_\mu}$ and a set $S''_\mu\in[S_\mu\sim T_\mu]^{m_\mu}$ $(\mu<\lambda)$ such that

$$\varphi(\{x, y\})=\psi_\mu(y) \qquad (x\in S''_\mu;\ y\in T_\mu;\ \mu<\lambda). \tag{6. 3}$$

Choose $x_\mu\in S''_\mu$ for $\mu<\lambda$. Let $M=[0, \lambda)$. Since $|P^M|<m'_\mu$ $(\mu<\lambda)$, it follows that there is a function $\theta_\mu\in P^M$ and a set $S'_\mu\in[S''_\mu\sim\{x_\mu\}]^{m_\mu}$ $(\mu<\lambda)$ such that

$$\varphi(\{x, x_\nu\})=\theta_\mu(\nu) \qquad (x\in S'_\mu;\ \mu, \nu<\lambda). \tag{6. 4}$$

The sets S'_μ $(\mu<\lambda)$ are clearly disjoint since the S''_μ are disjoint. Let $x\in S'_\mu$, $y\in S'_\nu$, where $\mu<\nu<\lambda$. Then, by (6. 3) and (6. 4)

$$\varphi(\{x, y\})=\psi_\nu(x)=\varphi(\{x, x_\nu\})=\theta_\mu(\nu).$$

This proves that

$$[S'_\mu, S'_\nu]^{1,1} \subset L_{\pi(\mu,\nu)} \qquad (\mu < \nu < \lambda),$$

where $\pi(\mu, \nu) = \theta_\mu(\nu)$.

PROOF OF THEOREM 6. 1. First we prove (6. 2). Let $\mathscr{J} = (F, N, S)$ be a (n, m)-system of subsets of S such that $|F_\nu| < m$ $(\nu \in N)$. We have to show that, if $m' \neq n'$, then S contains an $(\mathscr{J}, n)$-free subset of cardinal m.

Case 1. $n' > m$. There are disjoint sets $S_\lambda \in [S]^m$ for $\lambda < \omega(m)$ such that $S = U(\lambda < \omega(m)) S_\lambda$. Each of the sets $F_\nu (\nu \in N)$ is disjoint from at least one of the sets S_λ $(\lambda < \omega(n))$. Hence, there is a set $N_1 \in [N]^n$ and $\varkappa < \omega(n)$ such that $S_\varkappa \cap F_\nu = 0$ $(\nu \in N_1)$. Thus $S_\varkappa$ is a $(\mathscr{J}, n)$-free subset of S of cardinal m.

Case 2. $m' > n$. Then $S \sim \bigcup(\nu \in N) F_\nu$ is a $(\mathscr{J}, n)$-free subset of S of cardinal m.

Cases 1 and 2 above show that (6. 2) holds if $n' > m$ or if $m' > n$. In particular, since $m' \neq n'$, we have proved

$$(6.5) \qquad m' \Rightarrow [m', m']_{n'}.$$

The remaining cases which have to be considered follow from (6. 5) and the remarks which follow Theorems 4. 4 and 4. 5.

Case 3. $m > n > m'$. Since $m > n^+$, it follows from Theorem 4. 4 and (6. 5) that $m' \Rightarrow [m', m']_n$. Therefore, by Theorem 4.5, $m \Rightarrow [m, m]_n$.

Case 4. $n > m > n'$. From (6.5) and Theorem 4.5, it follows that $m \Rightarrow [m, m]_{n'}$. Therefore, since $n > m^+$, it follows by Theorem 4.4 that $m \Rightarrow [m, m]_n$.

These four cases exhaust all possibilities and complete the proof of (6. 2).

Since (6. 2) is stronger than (6. 1) when $m' \neq n'$, it is only necessary to prove (6. 1) under the added assumption

$$(6.6) \qquad m' = n'.$$

In fact, it is enough to prove

$$(6.7) \qquad m \to [m, m]_m^{<\aleph_0} \qquad (m \geqq \aleph_0).$$

To see this we must show that (6. 6) and (6. 7) imply (6. 1) when $m \neq n$. From (6. 6) and (6. 7) we have

$$(6.8) \qquad m' \to [m', m']_{n'}^{<\aleph_0}.$$

Suppose first that $m > n$. By Theorem 4. 4 and (6. 8), it follows that $m' \to [m', m']_n^{<\aleph_0}$. Now Theorem 4. 5 implies that $m \to [m, m]_n^{<\aleph_0}$. Now assume that $m < n$. From (6. 8) and Theorem 4. 5 we have $m \to [m, m]_{n'}^{<\aleph_0}$. Applying Theorem 4. 4 to this last relation, we deduce that $m \to [m, m]_n^{<\aleph_0}$. Therefore, in order to prove the Theorem it suffices to prove (6. 7).

Let $\mathscr{J} = (F, M, S)$ be any (m, m)-system. We will assume that S contains no $(\mathscr{J}, m)$-free subset of cardinal m and deduce that there is a $(\mathscr{J}, <\aleph_0)$-complete subset of cardinal m. Since the sets F_μ $(\mu \in M)$ are $(\mathscr{J}, <\aleph_0)$-complete, there is no loss of generality if we assume also that

$$(6.9) \qquad |F_\mu| < m \qquad (\mu \in M).$$

We shall consider separately the cases (i) m regular, (ii) m singular.

Case (i). $m=m'$. If $x \in S$ and $M' \subset M$, we put $M'(x)=\{\mu \in M': x \in F_\mu\}$. Suppose that there is $M' \in [M]^m$ such that the set

$$T=\{x \in S: |M'(x)|<m\}$$

has cardinal m. Then, in view of (6.9), we can define inductively x_ν and μ_ν for $\nu<\omega(m)$ such that

$$x_\nu \in T \sim \bigcup (\varrho<\nu) F_{\mu_\varrho} \cup \{x_\varrho\},$$

$$\mu_\nu \in M' \sim \bigcup (\varrho \leqq \nu) M'(x_\varrho) \cup \{\mu_0, \dots, \hat{\mu}_\nu\}.$$

Put $S^*=\{x_\nu: \nu<\omega(m)\}$, $M^*=\{\mu_\nu: \nu<\omega(m)\}$. From the way the x_ν and μ_ν are chosen, it follows that $|S^*|=|M^*|=m$ and

$$x \notin F_\mu \qquad (x \in S^*, \mu \in M^*).$$

This contradicts the initial assumption that S does not contain any $(\mathscr{J}, m)$-free subset of cardinal m. Hence,

$$|\{x \in S: |M'(x)|<m\}|<m \qquad (M' \in [M]^m). \tag{6. 10}$$

Let $\lambda<\omega(m)$ and suppose we have already chosen $y_0, \dots, \hat{y}_\lambda \in S$ in such a way that each finite subset of $\{y_0, \dots, \hat{y}_\lambda\}$ is contained in m of the sets F_μ $(\mu \in M)$, i. e.

$$|\bigcap (x \in X) M(x)|=m \qquad (X \in [\{y_0, \dots, \hat{y}_\lambda\}]^{<\aleph_0}). \tag{6. 11}$$

We write $M(X)=\bigcap (x \in X) M(x)$. Since $Y=\{y_0, \dots, \hat{y}_\lambda\}$ contains fewer than $m(=m')$ finite subsets, it follows from (6. 10) and (6. 11) that the set

$$A=\bigcup (X \in [Y]^{<\aleph_0}) \{x \in S: |M(X+\{x\})|<m\}$$

has cardinal $a<m$. Therefore, we can choose

$$y_\lambda \in S \sim A \cup Y.$$

Since $y_\lambda \notin A$, it follows that if X is any finite subset of $Y \cup \{y_\lambda\}$, then X is contained in m of the sets F_μ $(\mu \in M)$. Therefore, by induction, there is a set $Y^*=\{y_\lambda: \lambda<\omega(m)\}$ such that each finite subset of Y^* is contained in m of the sets F_μ $(\mu \in M)$. Hence Y^* is $(\mathscr{J}, <\aleph_0)$-complete. Since $|Y^*|=m$, this proves (6. 7) for regular cardinals m.

Case (ii). $m>m'$. Then we can assume that $m=m_0+\dots+\hat{m}_\lambda$, where $\lambda=\omega(m)$ and $m_\mu=m'_\mu>(m'+m_0+\dots+\hat{m}_\mu)^+$ $(\mu<\lambda)$.

There are sets K_ν for $\nu<\lambda$ such that $|K_\nu|=m_\nu$ and so that $K_\nu \subset S$ (if ν is even), $K_\nu \subset M$ (if ν is odd). Let $K=\bigcup (\nu<\lambda) K_\nu$. Consider the set $E \subset [K]^2$, where $\{x, y\}_{\neq} \in E$ if and only if $x \in S$, $y \in M$ and $x \in F_y$. By lemma 6. 1, there are disjoint sets $K'_\nu \subset K_\nu$ $(\nu<\lambda)$ such that $|K'_\nu|=m_\nu$ and, for $\mu<\nu<\lambda$, either

$$[K'_\mu, K'_\nu]^{1,1} \cap E=0, \tag{6. 12}$$

or

$$[K'_\mu, K'_\nu]^{1,1} \subset E. \tag{6. 13}$$

Choose $z_\nu \in K'_\nu$ $(\nu<\lambda)$ and put $S'=\{z_\nu: \nu<\lambda, \nu \text{ even}\}$, $M'=\{z_\nu: \nu<\lambda, \nu \text{ odd}\}$. Then, since the sets K'_ν are disjoint, $S' \in [S]^{m'}$, $M' \in [M]^{m'}$. Consider the (m', m')-system $\mathscr{J}'=(F', M', S')$, where $F'_\nu=F_\nu \cap S'$ $(\nu \in M')$.

Suppose that there is a $(\mathcal{J}', m')$-free subset of S' of cardinal m'. Then there is $N_1 \subset \{\nu: \nu < \lambda, \nu \textit{ even}\}$ and $N_2 \subset \{\nu: \nu < \lambda, \nu \textit{ odd}\}$ such that $|N_1| = |N_2| = m'$ and

$$\{z_\mu, z_\nu\} \notin E \qquad (\mu \in N_1,\ \nu \in N_2).$$

Therefore, (6. 13) is false and (6. 12) holds whenever $\mu \in N_1$ and $\nu \in N_2$ (or vice-versa). Hence,

$$x \notin F_y$$

if $x \in S_1 = \bigcup(\mu \in N_1) K_\mu$ and $y \in M_1 = \bigcup(\nu \in N_2) K_\nu$. Since $|S_1| = |M_1| = m$, this contradicts our initial assumption that S does not contain any $(\mathcal{J}, m)$-free subset of power m. Hence, S' does not contain any $(\mathcal{J}', m')$-free subset of power m'.

Since $m' \to [m', m']_{m'}^{<\aleph_0}$ by case (i), it follows that S' contains a subset of cardinal m' which is $(\mathcal{J}', <\aleph_0)$-complete. This means that there is a set $N_3 \subset \{\nu: \nu < \lambda, \nu \textit{ even}\}$ of cardinal m' such that, whenever $N \in [N_3]^{<\aleph_0}$, then there is $\nu = \nu(N) \in M'$ such that

$$\{z_\mu: \mu \in N\} \subset F_\nu \cap S'. \tag{6. 14}$$

Let $S_3 = \bigcup(\nu \in N_3) K_\nu$. Then $S_3 \in [S]^m$. If Y is any finite subset of S_3, then $N = N(Y) = \{\mu: \mu \in N_3,\ Y \cap K_\mu \neq 0\}$ is a finite subset of N_3. Hence, there is $\nu = \nu(N) \in M'$ such that (6. 14) holds. This implies that (6. 12) is false if $\nu = \nu(N(Y))$ and $\mu \in N(Y)$. Hence, for these values of μ, ν (6. 13) holds. This implies that

$$Y \subset \bigcup(\mu \in N(Y)) K_\mu \subset F_{z_\nu}.$$

Therefore, S_3 is $(\mathcal{J}, <\aleph_0)$-complete. This completes the proof of (6. 7) and concludes the proof of Theorem 6. 1.

The condition $m' \neq n'$ for (6. 2) is a necessary one since

$$m \not\Rightarrow [m, 1]_n \quad \textit{if} \quad m' = n' \geqq \aleph_0. \tag{6. 15}$$

To see this let $\lambda = \omega(m') = \omega(n')$; $S = \bigcup(\varrho < \lambda) S_\varrho$; $M = \bigcup(\varrho < \lambda) M_\varrho$; $|S| = m > |S_\varrho| (\varrho < \lambda)$; $M_\varrho \cap M_\sigma = 0$ $(\varrho < \sigma < \lambda)$; $|M| = n > |M_\varrho|$ $(\varrho < \lambda)$. Now consider the (n, m)-system $\mathcal{J} = (F, M, S)$, where

$$F_\mu = S_0 \cup \ldots \cup \hat{S}_\mu \qquad (\mu \in M_\varrho;\ \varrho < \lambda).$$

Clearly, $|F_\mu| < m$ $(\mu \in M)$ and S does not contain any element which is disjoint from n of the sets F_μ $(\mu \in M)$.

In contrast with (6. 15), however, we do have the following

(*) THEOREM 6. 2. *If $m, n \geqq \aleph_0$ and $\alpha < \omega(m)$, then*

$$\omega(m) \Rightarrow [\alpha, \omega(m)]_n. \tag{6. 16}$$

In order to prove Theorem 6. 2 we make use of the following result established in [4].

LEMMA 6. 2. *Let $m \geqq \aleph_0$ and let S be an ordered set of type $\omega(m)$. If ψ is any mapping of S into $[S]$ such that $x \notin \psi(x)$ and* $\operatorname{tp} \psi(x) < \alpha < \omega(m)$ $(x \in S)$, *then there is a set $S' \in [S]^m$ such that $x \notin \psi(y)$ $(x, y \in S')$.*

PROOF OF THEOREM 6. 2. We will first prove (6. 16) in the case $m=n$.

Let $\lambda=\omega(m)$ and let $S=\{x_0, x_1, \ldots, \hat{x}_\lambda\}_<$. Let $M=[0, \lambda)$ and let $\mathcal{J}=(F, M, S)$ be any (m, m)-system of subsets of S such that $\operatorname{tp} F_\mu<\alpha$ $(\mu<\lambda)$. By induction, there are ordinals $\varrho_\mu<\lambda$ such that $\varrho_\mu<\varrho_\nu$ $(\mu<\nu<\lambda)$ and

$$y_\mu=x_{\varrho_\mu}\in S\sim F_\mu \qquad (\mu<\lambda).$$

Let $S_1=\{y_0, \ldots, \hat{y}_\lambda\}_<$ and consider the set-mapping $\psi\in[S_1]^{S_1}$ where $\psi(y_\mu)=F_\mu\cap S_1$. Then $x\notin\psi(x)$ and $\operatorname{tp}\psi(x)<\alpha$ $(x\in S_1)$. Therefore, by Lemma 6. 2, there is $M'\in[M]^m$ such that

$$y_\mu\notin F_\nu \qquad (\mu, \nu\in M').$$

The set $S'=\{y_\mu:\mu\in M'\}$ is $(\mathcal{J}, m)$-free and has order type $\lambda=\omega(m)$. This proves

$$\lambda\Rightarrow[\alpha, \lambda]_m. \tag{6.17}$$

Now suppose $m\neq n$. If $m'\neq n'$, then (6. 2) implies (6. 16). Therefore, we may assume that $m'=n'$.

Case 1. Suppose $n>m$. Then (6. 17) and the remark which follows Theorem 4.2 imply that $\lambda\Rightarrow[\alpha, \lambda]_{n'}$. Therefore by Theorem 4. 4, $\lambda\Rightarrow[\alpha, \lambda]_n$.

Case 2. Suppose $m>n$. Let $\mathcal{J}=(F, N, S)$ be any (n, λ)-system such that $\operatorname{tp} F_\mu<\alpha$ $(\mu\in N)$. Then $S'=S\sim\bigcup(\mu\in N)F_\mu$ is an $(\mathcal{J}, n)$-free subset of type λ. Hence, $\lambda\Rightarrow[\alpha, \lambda]_n$. This completes the proof of (6. 16) in the case $m\neq n$ and concludes the proof of Theorem 6. 2.

The formula (6. 1) provides a complete discussion of the symbol (2, 5)′ in the case $k<\aleph_0$. In contrast, the only results we have of this kind when $k\geqq\aleph_0$ are given by Theorems 6. 3 and 6. 4. We conclude this section by stating the three simplest problems not covered by our results.

THEOREM 6. 3. *If* $m\geqq\aleph_0$ *and* $k^n<m'$, *then*

$$m\rightarrow[m, n]^k_m. \tag{6.18}$$

PROOF. Let $\mathcal{J}=(F, M, S)$ be a (m, m)-system and suppose that S does not contain a $(\mathcal{J}, k)$-complete subset of cardinal m.

Let $\nu=\omega(n)$, $\varkappa=\omega(k)$. Put $N=[0, \nu)$, $K=[0, \varkappa)$. Let $\varrho<\nu$ and suppose we have already chosen disjoint sets $S_\sigma\in[S]^k$ for $\sigma<\varrho$. Since $T=S\sim\bigcup(\sigma<\varrho)S_\sigma$ has cardinal m, it is not $(\mathcal{J}, k)$-complete. Hence, there is a set $S_\varrho\in[T]^k$ such that $S_\varrho\nsubseteq F_\mu$ $(\mu\in M)$. Therefore, by induction, there are disjoint sets $S_\varrho\in[S]^k$ $(\varrho<\nu)$ such that $S_\varrho\nsubseteq F_\mu$ $(\varrho<\nu;\ \mu\in M)$. Let $S_\varrho=\{x_{\varrho,\tau}:\tau\in K\}_{\neq}$ $(\varrho\in N)$.

For any function $\varphi\in K^N$ we put

$$M(\varphi)=\{\mu\in M:\ \bigcup(\varrho\in N)\{x_{\varrho,\varphi(\varrho)}\}\cap F_\mu=0\}.$$

If $\mu\in M$, then by the definition of the sets S_ϱ, it follows that there is some function $\varphi\in K^N$ such that

$$x_{\varrho,\varphi(\varrho)}\notin F_\mu \qquad (\varrho\in N).$$

Therefore,

$$M\subset\bigcup(\varphi\in K^N)M(\varphi).$$

Since $k^n < m'$, it follows that there is $\psi \in K^N$ such that $|M(\psi)| = m$. Put $S' = \{x_{\varrho, \psi(\varrho)}: \varrho \in N\}$. Then $|S'| = n$ and $x \notin F_\mu$ if $x \in S'$ and $\mu \in M(\psi)$, i. e., S' is a $(\mathcal{J}, m)$-free subset of S of cardinal n. This proves (6. 18).

We do not know if the condition $k^n < m'$ in Theorem 6. 3 is necessary. The negative results that we have suggest that this might be the case. For example, it is easy to see that

$$m \nrightarrow [m, 1]^{m'}_{m} \quad \textit{if} \quad m \geqq \aleph_0.$$

We will prove the slightly more general result that

$$\text{(6. 19)} \qquad m \nrightarrow [m, 1]^{m'}_{n} \quad \textit{if} \quad m' = n'.$$

To see this let $\lambda = \omega(m')$ and let $S = \bigcup(\mu < \lambda) S_\mu(\text{tp})$, $M = \bigcup(\mu < \lambda) M_\mu$, where $\text{tp } S = \omega(m)$, $|S_\mu| < m \ (\mu < \lambda)$ and $M_\mu \cap M_\varrho = 0 \ (\mu < \varrho < \lambda)$, $|M| = n > |M_\mu| \ (\mu < \lambda)$. Now consider the set-system $\mathcal{J} = (F, M, S)$, where

$$F_\nu = \bigcup(\varrho < \mu) S_\varrho \qquad (\nu \in M_\mu;\ \mu < \lambda).$$

Every subset of S of cardinal m has a subset of cardinal m' which is cofinal with S and which, therefore, is not contained in any of the sets F_ν $(\nu \in M)$. Hence, there is no $(\mathcal{J}, m')$-complete subset of S of cardinal m. Also, if $x \in S$, then there is $\mu < \lambda$ such that $x \in S_\mu$. Hence, $x \in F_\nu$ if $\nu \in \bigcup(\mu < \varrho < \lambda) M_\varrho$, i. e. $\{x\}$ is not $(\mathcal{J}, n)$-free. This proves (6. 19).

Less obvious than (6. 19) is

(∗) THEOREM 6. 4. *If* $m \geqq \aleph_0$, *then* $m^+ \nrightarrow [m^+, m]^{m}_{m^+}$.

PROOF. Let $\mu = \omega(m)$, $\pi = \omega(m^+)$. Let S be a set of cardinal m^+ and let $[S]^m = \{B_0, \ldots, \hat{B}_\pi\}_{\neq}$. Let $M = [\mu, \pi)$. We will define a (m^+, m^+)-system $\mathcal{J} = (F, M, S)$ in the following way. Let $\varrho \in M$. Then we may write $\{B_0, \ldots, \hat{B}_\varrho\} = \{C_0, \ldots, \hat{C}_\mu\}_{\neq}$. Since each of the sets C_λ $(\lambda < \mu)$ has cardinal m, there are x_λ, y_λ for $\lambda < \mu$ such that

$$\{x_\lambda, y_\lambda\}_{\neq} \subset C_\lambda \sim \bigcup(\sigma < \lambda)\{x_\sigma, y_\sigma\}.$$

Now put $F_\varrho = \{x_\lambda: \lambda < \mu\}$. This defines the set system $\mathcal{J}$. Since $x_\lambda \in F_\varrho \cap C_\lambda$ and $y_\lambda \in C_\lambda \sim F_\varrho$, it follows that

$$\text{(6. 20)} \qquad F_\varrho \cap B_\sigma \neq 0, \quad B_\sigma \sim F_\varrho \neq 0 \qquad (\sigma < \varrho \in M).$$

Let B be any subset of S of cardinal m. Then there is $\sigma < \pi$ such that $B = B_\sigma$. Hence, by (6. 20), $F_\varrho \cap B \neq 0$ $(\sigma < \varrho \in M)$. Therefore, S does not contain a $(\mathcal{J}, m^+)$-free subset of cardinal m. Let T be a subset of S of cardinal m^+. Then there are $\tau < \pi$ and $x \in T$ such that $B_\tau \subset T$ and

$$\text{(6. 21)} \qquad x \in T \sim \bigcup(\tau \geqq \varrho \in M) F_\varrho.$$

Put $X = B_\tau \cup \{x\}$. Then $|X| = m$. Also, by (6. 20) and (6. 21), $X \not\subset F_\varrho$ $(\varrho \in M)$. Hence, T is not a $(\mathcal{J}, m)$-complete subset of S. This proves the theorem.

We do not know the answers to the following questions.

PROBLEM 1. (?) $\aleph_2 \rightarrow [\aleph_2, a]^{\aleph_0}_{\aleph_2}$ $\quad (a = \aleph_1$ or $\aleph_2)$.

PROBLEM 2. (?) $\aleph_{\omega+1} \rightarrow [\aleph_{\omega+1}, \aleph_0]^{\aleph_\omega}_{\aleph_{\omega+1}}$.

PROBLEM 3. (?) $\aleph_{\omega_1} \rightarrow [\aleph_{\omega_1}, \aleph_0]^{\aleph_0}_{\aleph_1}$.

§ 7. Preliminary results for denumerable ordinals. The results of this section are required in § 8, where we give a complete analysis of the symbol $\alpha \to [\beta, \gamma]^{<\aleph_0}_{\aleph_0}$ for $\alpha, \beta, \gamma < \omega_1$. Most of the theorems in this section are special cases of Theorem 8. 2, but we do not restrict ourselves entirely to the denumerable case.

We remind the reader that if $\mathcal{J} = (F, M, S)$ is an *infinite* system of sets, then $P(\mathcal{J}) = \{x \in S : |\{\mu \in M : x \notin F_\mu\}| < |M|\}$.

LEMMA 7. 1. *Let* $\mathcal{J} = (F, M, S), \mathcal{J}' = (F, M', S)$, *where* $F \in [S]^{M \cup M'}$, $|M| = |M'| = m \geq \aleph_0$. *If* $|M' \sim M| < m$, *then* $P(\mathcal{J}) \subset P(\mathcal{J}')$.

PROOF. Let $x \in P(\mathcal{J})$. Then $|M \sim M(x)| < m$, where $M(x) = \{\mu \in M : x \in F_\mu\}$. Put $N = (M' \sim M) \cup (M \sim M(x))$. Then $|N| < m$ and $M' \sim N \subset M(x)$. Hence, $x \in F_\mu$ if $\mu \in M' \sim N$, i. e. $x \in P(\mathcal{J}')$. This proves the lemma.

LEMMA 7. 2. *Let S be an ordered set and let* $S_\lambda \subset S\, (\lambda \in L)$, *where* $|L| \leq \aleph_0$, $S_\lambda \cap S_\mu = 0$ *if* $\{\lambda, \mu\}_{\neq} \subset L$, $\operatorname{tp} S_\lambda = \omega^{\varrho_\lambda}$ *and* $\varrho_\lambda > 0$ $(\lambda \in L)$. *Let* $\mathcal{J} = (F, M, S)$ *be a* $(\aleph_0, \operatorname{tp} S)$*-system such that* (i) $\operatorname{tp} F_\mu \cap S_\lambda < \omega^{\varrho_\lambda}$ $(\mu \in M, \lambda \in L)$, (ii) $\operatorname{tp} P(\mathcal{J}') \cap S_\lambda < \omega^{\varrho_\lambda}$ $(\lambda \in L; \mathcal{J}' \subset \mathcal{J}; |\mathcal{J}'| = \aleph_0)$. *Then there is a* $(\mathcal{J}, \aleph_0)$*-free set* $X \subset S$ *such that* $|X \cap S_\lambda| = \aleph_0$ $(\lambda \in L)$.

PROOF. If $L = 0$ there is nothing to prove. Therefore, we assume $L \neq 0$. We can assume that $M = [0, \omega)$. There is a function $\varphi \in L^M$ such that $|\{\mu \in M : \varphi(\mu) = \lambda\}| = \aleph_0$ $(\lambda \in L)$, i. e. the sequence $\varphi(0), \varphi(1), \ldots, \hat{\varphi}(\omega)$ repeats each element of L infinitely often.

Let $\alpha < \omega$ and suppose that we have already chosen $x_\nu \in S$ and $\mu_\nu \in M$ for $\nu < \alpha$ in such a way that

$$M_\nu = M \sim \bigcup (\sigma < \nu)\, M(x_\sigma)$$

is infinite for $\nu \leq \alpha$, where $M(x) = \{\mu : \mu < \omega, x \in F_\mu\}$. Let $\mathcal{J}_\alpha$ denote the sub-system (F, M_α, S) of $\mathcal{J}$. From the hypothesis (i), (ii) of the Lemma, it follows that there is

$$x_\alpha \in S_{\varphi(\alpha)} \sim \bigcup (\nu < \alpha)\, F_{\mu_\nu} \cup \{x_\nu\} \cup P(\mathcal{J}_\alpha)$$

(because $\operatorname{tp} S_{\varphi(\alpha)}$ is indecomposable). With this choice for x_α we notice that $M_{\alpha+1} = M \sim \bigcup (\sigma \leq \alpha)\, M(x_\sigma) = M_\alpha \sim M(x_\alpha)$ is also infinite since $x_\alpha \notin P(\mathcal{J}_\alpha)$. Hence, we can choose $\mu_\alpha \in M_{\alpha+1} \sim \{\mu_0, \ldots, \hat{\mu}_\alpha\}$. Therefore, by induction, there is $X = \{x_0, \ldots, \hat{x}_\omega\}_{\neq} \subset S$ and $M' = \{\mu_0, \ldots, \hat{x}_\omega\}_{\neq} \subset M$ such that [2]

$$x \notin F_\mu \qquad (x \in X, \mu \in M').$$

Thus the set X is $(\mathcal{J}, \aleph_0)$-free. Moreover, since $x_\alpha \in S_{\varphi(\alpha)}$ and $|\{\alpha : \alpha < \omega; \varphi(\alpha) = \lambda\}| = \aleph_0$ for each $\lambda \in L$, it follows that $|X \cap S_\lambda| = \aleph_0\, (\lambda \in L)$. This proves the lemma.

As an immediate deduction from Lemma 7. 2 we have

THEOREM 7. 1. *If* $\varrho > 0$ *and* $\gamma < \omega_1$, *then* $\omega^\varrho \gamma \not\to [\omega^\varrho, \omega\gamma]_{\aleph_0}$.

PROOF. Let $L = [0, \gamma)$; $S = \bigcup (\lambda \in L)\, S_\lambda$ (tp); $\operatorname{tp} S_\lambda = \omega^\varrho$ $(\lambda \in L)$. Let $\mathcal{J} = (F, M, S)$ be an $(\aleph_0, \omega^\varrho \gamma)$-system such that $\operatorname{tp} F_\mu < \omega^\varrho$ $(\mu \in M)$ and $\operatorname{tp} P(\mathcal{J}') < \omega^\varrho$ $(\mathcal{J}' \subset \mathcal{J};$ $|\mathcal{J}'| = \aleph_0)$. Then, by the last Lemma, there is a $(\mathcal{J}, \aleph_0)$-free subset $X \subset S$ such that $|X \cap S_\lambda| = \aleph_0$ $(\lambda \in L)$. Hence, $\operatorname{tp} X \geq \omega\gamma$. This proves the result.

[2] Suppose $\alpha, \beta < \omega$. If $\beta < \alpha$, then $x_\alpha \notin F_{\mu_\beta}$ by the definition of x_α. Also, if $\alpha \leq \beta$, then $\mu_\beta \in M_{\beta+1} \subset M_{\alpha+1}$. Hence, $\mu_\beta \notin M(x_\alpha)$, i. e. $x_\alpha \notin F_{\mu_\beta}$.

THEOREM 7. 2. *If* $r<\omega$, $\gamma<\omega_1$, *then* $\omega\gamma\Rightarrow[r,\omega\gamma]_{\aleph_0}$.

PROOF. The conclusion is obvious if $r\leqq 1$. Therefore, we assume that $r>1$ and use induction on r, i. e. we assume that

$$\omega\gamma\Rightarrow[r-1,\omega\gamma]_{\aleph_0}. \tag{7. 1}$$

Let $\mathscr{J}=(F,M,S)$ be a $(\aleph_0,\omega\gamma)$-system such that

$$\operatorname{tp} F_\mu<r \qquad (\mu\in M). \tag{7. 2}$$

We want to show that S contains a $(\mathscr{J},\aleph_0)$-free subset of type $\omega\gamma$.

Suppose that there is an element $x\in S$ such that $M(x)=\{\mu\in M\colon x\in F_\mu\}$ is infinite. Put $F'_\mu=F_\mu-\{x\}$ for $\mu\in M(x)$. Then $\mathscr{J}'=(F',M(x),S\sim\{x\})$ is a $(\aleph_0,\omega\gamma)$-system such that $\operatorname{tp} F'_\mu<r-1$ $(\mu\in M(x))$. It follows from (7. 1) that $S\sim\{x\}$ contains a $(\mathscr{J}',\aleph_0)$-free subset of type $\omega\gamma$. Hence there is $S'\subset S\sim\{x\}$ and $M'\subset M(x)$ such that $\operatorname{tp} S'=\omega\gamma$, $|M'|=\aleph_0$ and $S'\cap F'_\mu=0$ $(\mu\in M')$. Since $x\notin S'$, it follows that $S'\cap F_\mu=0$ $(\mu\in M')$. Therefore, S' is also $(\mathscr{J},\aleph_0)$-free. Consequently, we may suppose that

$$|M(x)|<\aleph_0 \qquad (x\in S). \tag{7. 3}$$

By (7. 2) and (7. 3) it follows that each element $x\in S$ is joined to only a finite number of elements of S by edges of the graph $\mathscr{G}_2(\mathscr{J})$. Since $P(\mathscr{J}^*)$ is a complete subgraph of $\mathscr{G}_2(\mathscr{J})$ if $|\mathscr{J}^*|=\aleph_0$ and $\mathscr{J}^*\subset\mathscr{J}$ (see (2. 4)), it follows that

$$\operatorname{tp} P(\mathscr{J}^*)<\omega \qquad (|\mathscr{J}^*|=\aleph_0;\ \mathscr{J}^*\subset\mathscr{J}). \tag{7. 4}$$

By Theorem 7. 1, we have that

$$\omega\gamma\rightleftarrows[\omega,\omega\gamma]_{\aleph_0}.$$

This relation, together with (7. 2) and (7. 4), implies that there is a $(\mathscr{J},\aleph_0)$-free subset of S of type $\omega\gamma$. The Theorem now follows by induction on r.

THEOREM 7. 3. *Let* $r<\omega$, $\gamma<\omega_1$, $\beta=\omega^\varrho$, $\varrho>0$. *Then*

$$\beta\omega\gamma\rightleftarrows[\beta r,\omega^2\gamma]_{\aleph_0}.$$

PROOF. We can assume that $r,\gamma>0$. Let $\mathscr{J}=(F,M,S)$ be a $(\aleph_0,\beta\omega\gamma)$-system such that

$$\operatorname{tp} F_\mu<\beta r \qquad (\mu\in M), \tag{7. 5}$$

$$\operatorname{tp} P(\mathscr{J}')<\beta r \qquad (\mathscr{J}'\subset\mathscr{J};\ |\mathscr{J}'|=\aleph_0). \tag{7. 6}$$

Since $\operatorname{tp} S=\beta\omega\gamma$, we may write $S=\bigcup(\lambda<\omega\gamma)\,S_\lambda$ (tp), where $\operatorname{tp} S_\lambda=\beta$ $(\lambda<\omega\gamma)$. Consider the $(\aleph_0,\omega\gamma)$-system $\mathscr{J}^*=(F^*,M,L)$, where $L=[0,\omega\gamma)$ and

$$F^*_\mu=\{\lambda\in L\colon \operatorname{tp} F_\mu\cap S_\lambda=\beta\} \qquad (\mu\in M).$$

By (7. 5), $\operatorname{tp} F^*_\mu<r$ $(\mu\in M)$. Therefore, by Theorem 7. 2, there is $M_0\in[M]^{\aleph_0}$ and $L_0\subset L$ such that $\operatorname{tp} L_0=\omega\gamma$ and

$$\operatorname{tp} F_\mu\cap S_\lambda<\beta \qquad (\mu\in M_0;\ \lambda\in L_0). \tag{7. 7}$$

Put $\mathscr{J}_0=(F,M_0,S)$.

Suppose that $\mathscr{S}$: *whenever* $\mathscr{J}' \subset \mathscr{J}_0$, $L' \subset L_0$, $|\mathscr{J}'| = \aleph_0$, $\operatorname{tp} L' = \omega\gamma$, *then there is* $\lambda \in L'$ *and an infinite subsystem* $\mathscr{J}'' \subset \mathscr{J}'$ *such that* $\operatorname{tp} S_\lambda \cap P(\mathscr{J}'') = \beta$.

Since the removal of a finite number of elements from L_0 does not change the order type of L_0, it follows from the assumption $\mathscr{S}$ that there are $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}_{\neq} \subset \subset L_0$ and infinite set-systems $\mathscr{J}_1, \mathscr{J}_2, \ldots, \mathscr{J}_r$ such that $\mathscr{J}_0 \supset \mathscr{J}_1 \supset \ldots \supset \mathscr{J}_r$ and

$$\operatorname{tp} S_{\lambda_\varrho} \cap P(\mathscr{J}_\varrho) = \beta \qquad (1 \leqq \varrho \leqq r).$$

By Lemma 7. 1, $P(\mathscr{J}_\varrho) \subset P(\mathscr{J}_r)$ $(1 \leqq \varrho \leqq r)$. Therefore, $\operatorname{tp} P(\mathscr{J}_r) \geqq \beta r$. This contradicts (7. 6) and proves that the statement $\mathscr{S}$ is false.

Hence, there is $\mathscr{J}' \subset \mathscr{J}_0$ and $L' \subset L_0$ such that $|\mathscr{J}'| = \aleph_0$, $\operatorname{tp} L' = \omega\gamma$ and

$$(7.\,8) \qquad \operatorname{tp} S_\lambda \cap P(\mathscr{J}'') < \beta \qquad (\lambda \in L';\ \mathscr{J}'' \subset \mathscr{J}';\ |\mathscr{J}''| = \aleph_0).$$

Applying Lemma 7. 2 to the set $S' = \bigcup(\lambda \in L') S_\lambda$, it follows from (7. 7) and (7. 8) that there is a $(\mathscr{J}', \aleph_0)$-free set $X \subset S'$ such that $|X \cap S_\lambda| = \aleph_0$ $(\lambda \in L')$. The set X is also $(\mathscr{J}, \aleph_0)$-free and $\operatorname{tp} X \geqq \omega^2\gamma$. This completes the proof of Theorem 7. 3.

The next theorem shows that, in a certain sense, the results given by Theorems 7. 1 and 7. 3 are best possible.

THEOREM 7. 4. *Let* $\gamma < \omega_1$, $\beta = \omega^\varrho$, $\varrho < \omega_1$. *Then*

$$\beta\gamma \nrightarrow [\beta, \omega\gamma + 1]^2_{\aleph_0}.$$

PROOF. If $\beta\gamma \leqq \omega\gamma$ the result is obvious. Therefore, we assume that $\gamma > 0$ and $\beta = \beta_0 + \ldots + \hat{\beta}_\omega$, where $\omega \leqq \beta_0 \leqq \beta_1 \leqq \ldots \leqq \hat{\beta}_\omega < \beta < \omega_1$.

Let $S = \bigcup(\lambda < \gamma) S_\lambda$ (tp); $\operatorname{tp} S_\lambda = \beta$ $(\lambda < \gamma)$. Then $\operatorname{tp} S = \beta\gamma$. Since $0 < \gamma < \omega_1$, we may write $[0, \gamma) = \{\lambda_0, \ldots, \hat{\lambda}_\pi\}_{\neq}$, where $0 < \pi \leqq \omega$. Put $T_\varkappa = S_{\lambda_\varkappa}$ $(\varkappa < \pi)$. Then $S = \bigcup(\varkappa < \pi) T_\varkappa$ and $\operatorname{tp} T_\varkappa = \beta$ $(\varkappa < \pi)$. Therefore, $T_\varkappa = \bigcup(\mu < \omega) T_{\varkappa\mu}$ (tp), where $\operatorname{tp} T_{\varkappa\mu} = \beta_\mu$ $(\varkappa < \pi;\ \mu < \omega)$. Since $\omega \leqq \beta_\mu < \omega_1$, we can assume that

$$T_{\varkappa\mu} = \{x_{\varkappa\mu 0}, \ldots, \hat{x}_{\varkappa\mu\omega}\}_{\neq} \qquad (\varkappa < \pi;\ \mu < \omega).$$

Now consider the $(\aleph_0, \beta\gamma)$-system $\mathscr{J} = (F, M, S)$, where $M = [0, \omega)$ and

$$F_n = \{x_{\varkappa\mu\nu} : \varkappa, \mu < n < \nu < \omega\} \qquad (n < \omega).$$

Let S' be any subset of S of type β. Let $x_1 = x_{\varkappa_1\mu_1\nu_1} \in S'$, where $\varkappa_1 < \pi$; $\mu_1, \nu_1 < \omega$. Then there is $\varkappa_2 < \pi$ and $\mu_2, \nu_2 < \omega$ such that $x_2 = x_{\varkappa_2\mu_2\nu_2} \in S'$ and either $\varkappa_2 \geqq \nu_1$ or $\mu_2 \geqq \nu_1$. Otherwise, we have $S' \subset \bigcup(\varkappa, \mu < \nu_1) T_{\varkappa\mu}$ and this leads to the contradiction $\beta = \operatorname{tp} S' \leqq (\beta_0 + \ldots + \hat{\beta}_{\nu_1})\nu_1 < \beta$. Since $\{x_1, x_2\}_{\neq} \nsubseteq F_n$ $(n < \omega)$, it follows that the graph $\mathscr{G}(\mathscr{J})$ does not contain a complete subgraph of type β.

Now let S'' be a subset of S of type $\omega\gamma + 1$. Then there is $\varkappa < \pi$ and $\mu < \omega$ such that $\operatorname{tp} S'' \cap T_{\varkappa\mu} \geqq \omega$. Let $n_0 = \max\{\varkappa, \mu\}$. If $n > n_0$, then there is $\nu > n$ such that $x_{\varkappa\mu\nu} \in S''$. Therefore, $S'' \cap F_n \neq 0$ $(n_0 < n < \omega)$. This proves that S does not contain a $(\mathscr{J}, \aleph_0)$-free subset of type $\omega\gamma + 1$.

The next theorem gives two ways of obtaining new relations of the form (2. 8) from known relations of this kind.

THEOREM 7. 5. (i) *Let* $0 < i + j < \omega$; $\alpha \rightleftarrows [\beta, j]_{\aleph_0}$. *Then* $\alpha + (i + j - 1) \rightleftarrows [\beta + i, j]_{\aleph_0}$.

(ii) *If* $\beta' < \beta$; $\alpha \rightleftarrows [\beta, \gamma]_{\aleph_0}$; $\alpha' \rightleftarrows [\beta', \gamma']_{\aleph_0}$, *then* $\alpha + \alpha' \rightleftarrows [\beta', \gamma + \gamma']_{\aleph_0}$.

PROOF. (i) Let $S=S_1\cup S_2$ (tp); tp $S_1=\alpha$; tp $S_2=i+j-1$. Suppose $\mathscr{J}=(F, M, S)$ is a set-system such that $|M|=\aleph_0$ and

$$\text{tp } F_\mu<\beta+i \qquad (\mu\in M). \tag{7. 9}$$

Suppose also that S does not contain a subset of j elements which is $(\mathscr{J}, \aleph_0)$-free. Since S_2 is finite, there is $X\subset S_2$ and $M'\in[M]^{\aleph_0}$ such that $F_\mu\cap S_2=X$ $(\mu\in M')$. Since $S_2\sim X$ is $(\mathscr{J}, \aleph_0)$-free, it follows that $|X|\geqq i$. Therefore, by (7. 9),

$$\text{tp } F_\mu\cap S_1<\beta \qquad (\mu\in M').$$

It now follows from the hypothesis $\alpha\rightarrow[\beta, j]_{\aleph_0}$ that there is $M_1\in[M']^{\aleph_0}$ such that tp $S_1\cap P(\mathscr{J}_1)\geqq\beta$, where $\mathscr{J}_1=(F, M_1, S)$. Since $X\subset P(\mathscr{J}_1)$, it follows that tp $P(\mathscr{J}_1)\geqq\beta+i$. This proves (i).

(ii) Let $S=T\cup T'$ (tp); tp $T=\alpha$; tp $T'=\alpha'$. Then tp $S=\alpha+\alpha'$. Let $\mathscr{J}=(F, M, S)$ be a system of subsets of S such that $|M|=\aleph_0$ and

$$\text{tp } F_\mu<\beta' \qquad (\mu\in M), \tag{7. 10}$$

$$\text{tp } P(\mathscr{J}')<\beta' \qquad (\mathscr{J}'\subset\mathscr{J};\ |\mathscr{J}'|=\aleph_0). \tag{7. 11}$$

Since $\alpha\rightarrow[\beta, \gamma]_{\aleph_0}$, it follows from (7. 10) and (7. 11) that there is $M'\in[M]^{\aleph_0}$ and $C\subset T$ such that tp $C=\gamma$ and $C\cap F_\mu=0$ $(\mu\in M')$. Similarly, since $\alpha'\rightarrow[\beta', \gamma']_{\aleph_0}$, it follows that there is $M''\in[M']^{\aleph_0}$ and a set $C'\subset T'$ such that tp $C'=\gamma'$ and $C'\cap F_\mu=0$ $(\mu\in M'')$. The set $C\cup C'$ is $(\mathscr{J}, \aleph_0)$-free and has type $\gamma+\gamma'$. This proves (ii).

For negative relations of the form (2. 6) we have

THEOREM 7. 6. *If* $\alpha_\varrho\nrightarrow[\beta_\varrho, \gamma_\varrho]_m^{<k}$ $(\varrho<\lambda)$ *and if* β, γ *are such that* $\beta>\Sigma(\varrho<\lambda)\beta'_\varrho$ *and* $\gamma>\Sigma(\varrho<\lambda)\gamma'_\varrho$ *whenever* $\beta'_\varrho<\beta_\varrho$ *and* $\gamma'_\varrho<\gamma_\varrho$ $(\varrho<\lambda)$, *then*

$$\Sigma(\varrho<\lambda)\alpha_\varrho\nrightarrow[\beta, \gamma]_m^{<k}. \tag{7. 12}$$

REMARK. The theorem remains true if $<k$ is replaced throughout by k.

PROOF OF THEOREM 7. 6. Let $S=\cup(\varrho<\lambda)S_\varrho$ (tp), where tp $S_\varrho=\alpha_\varrho$ $(\varrho<\lambda)$. Let $|M|=m$. By the hypothesis, there are set-systems $\mathscr{J}_\varrho=(F^{(\varrho)}, M, S_\varrho)$ $(\varrho<\lambda)$ such that S_ϱ does not contain a $(\mathscr{J}_\varrho, <k)$-complete subset of type β_ϱ or a $(\mathscr{J}_\varrho, m)$-free subset of type γ_ϱ. Consider the set-system $\mathscr{J}=(F, M, S)$, where $F_\mu=\cup(\varrho<\lambda)F_\mu^{(\varrho)}$ $(\mu\in M)$.

If X is $(\mathscr{J}, <k)$-complete, the $X\cap S_\varrho$ is $(\mathscr{J}_\varrho, <k)$-complete $(\varrho<\lambda)$. Therefore, tp $X=\Sigma(\varrho<\lambda)$ tp $(X\cap S_\varrho)<\beta$. Similarly, if Y is $(\mathscr{J}, m)$-free, then $Y\cap S_\varrho$ is $(\mathscr{J}_\varrho, m)$-free $(\varrho<\lambda)$ and hence, tp $Y=\Sigma(\varrho<\lambda)$ tp $(Y\cap S_\varrho)<\gamma$. This proves (7. 12).

LEMMA 7. 3. *Let* $r<\omega$; tp $S\geqq\omega\alpha$. *If* $\mathscr{J}_0$ *is a countable system of subsets of* S *such that*

$$\text{tp } P(\mathscr{J}')<\omega\alpha \qquad (\mathscr{J}'\subset\mathscr{J}_0;\ |\mathscr{J}'|=\aleph_0)$$

then S *contains a subset of* r *elements which is* $(\mathscr{J}_0, \aleph_0)$*-free.*

REMARK. The lemma clearly implies

$$\omega\alpha\rightarrow[\omega\alpha, r]_{\aleph_0} \qquad (r<\omega;\ \alpha\geqq 0). \tag{7. 13}$$

PROOF OF LEMMA 7. 3. The hypothesis implies that $\alpha > 0$ and that $S \sim P(\mathscr{J}')$ is infinite if $\mathscr{J}' \subset \mathscr{J}_0, |\mathscr{J}'| = \aleph_0$. Let $\varrho < r$ and suppose we have already defined $x_1, \dots, x_\varrho \in S$ and $\mathscr{J}_\varrho = (F, M_\varrho, S) \subset \mathscr{J}_0$ such that $|\mathscr{J}_\varrho| = \aleph_0$. Then there is $x_{\varrho+1} \in S \sim \sim P(\mathscr{J}_\varrho) \cup \{x_1, \dots, x_\varrho\}$. Let $M_{\varrho+1} = \{\mu \in M_\varrho : x_{\varrho+1} \notin F_\mu\}$. Then $|M_{\varrho+1}| = \aleph_0$ since $x_{\varrho+1} \notin P(\mathscr{J}_\varrho)$. Now put $\mathscr{J}_{\varrho+1} = (F, M_{\varrho+1}, S)$. By induction, it follows that there is $X = \{x_1, \dots, x_r\}_{\neq} \subset S$ and infinite set-systems $\mathscr{J}_\varrho = (F, M_\varrho, S)$ such that $M_0 \supset \dots \supset M_r$ and $x_\varrho \notin F_\mu$ ($\mu \in M_\varrho$; $1 \leqq \varrho \leqq r$). Hence $X \cap F_\mu = 0$ ($\mu \in M_r$), i. e. X is a $(\mathscr{J}_0, \aleph_0)$-free subset of type r.

LEMMA 7. 4. *Let* $r < \omega$; $\beta' < \beta = \omega^\varrho$; $\varrho > 1$. *Let* $\mathscr{J} = (F, M, S)$ *be an* $(\aleph_0, \beta)$-*system such that* (i) $\text{tp } F_\mu < \beta'$ ($\mu \in M$), (ii) $\text{tp } P(\mathscr{J}') < \beta$ ($\mathscr{J}' \subset \mathscr{J}$; $|\mathscr{J}'| = \aleph_0$). *Then there is a* $(\mathscr{J}, \aleph_0)$-*free subset of S of type* $\omega + r$.

PROOF. The hypothesis implies that there are indecomposable ordinals β_ν ($\nu < \omega$) and $s < \omega$ such that $\beta' < \beta_0 s$ and

$$\omega \leqq \beta_0 \leqq \beta_1 \leqq \dots \leqq \hat{\beta}_\omega < \beta = \Sigma(\nu < \omega)\beta_\nu.$$

Therefore, $S = \bigcup(\nu < \omega) S_\nu$ (tp), where $\text{tp } S_\nu = \beta_\nu$ ($\nu < \lambda$). Put $F^*_\mu = \{\nu : \nu < \omega;$ $\text{tp } F_\mu \cap S_\nu = \beta_\nu\}$ ($\mu \in M$). Then, by (i), $\text{tp } F^*_\mu < s$ ($\mu \in M$). Since $\omega \to [s, \omega]_{\aleph_0}$ by Theorem 7. 2, it follows that there are infinite sets $L \subset [0, \omega)$ and $M_1 \subset M$ such that $L \cap F^*_\mu = 0$ ($\mu \in M_1$), i. e.

(7. 14) $$\text{tp } S_\lambda \cap F_\mu < \beta_\lambda \qquad (\lambda \in L;\ \mu \in M_1).$$

Put $\mathscr{J}_1 = (F, M_1, S)$, $L = \{\lambda_0, \dots, \hat{\lambda}_\omega\}_<$.

Suppose that $\mathscr{S}$: *whenever* $\lambda \in L$, $\mathscr{J}' \subset \mathscr{J}_1$, $|\mathscr{J}'| = \aleph_0$, *then there is* $\mathscr{J}'' \subset \mathscr{J}'$ *such that* $|\mathscr{J}''| = \aleph_0$ *and* $\text{tp } S_\lambda \cap P(\mathscr{J}'') = \beta_\lambda$. Then there are infinite set-systems $\mathscr{J}'_\varrho = (F, M'_\varrho, S)$ ($\varrho < \omega$) such that $\mathscr{J}_1 \supset \mathscr{J}'_0 \supset \mathscr{J}'_1 \supset \dots \supset \hat{\mathscr{J}}'_\omega$ and

(7. 15) $$\text{tp } S_{\lambda_\varrho} \cap P(\mathscr{J}'_\varrho) = \beta_{\lambda_\varrho} \qquad (\varrho < \omega).$$

Choose μ_ϱ for $\varrho < \omega$ so that $\mu_\varrho \in M'_\varrho \sim \{\mu_0, \dots, \hat{\mu}_\varrho\}$. Put $\mathscr{J}' = (F, M', S)$, where $M' = \{\mu_0, \dots, \hat{\mu}_\omega\}_{\neq}$. Since $M' \sim M'_\varrho \subset \{\mu_0, \dots, \hat{\mu}_\varrho\}$, it is a finite set and hence, by Lemma 7. 1, $P(\mathscr{J}'_\varrho) \subset P(\mathscr{J}')$ for $\varrho < \omega$. Therefore, by (7. 15), $\text{tp } P(\mathscr{J}') \geqq \geqq \Sigma(\varrho < \omega)\beta_{\lambda_\varrho} = \beta$. This contradicts the hypothesis (ii). Hence, $\mathscr{S}$ is false.

It follows that there is some $\lambda \in L$ and an infinite set-system $\mathscr{J}^*_1 \subset \mathscr{J}_1$ such that

(7. 16) $$\text{tp } S_\lambda \cap P(\mathscr{J}') < \beta_\lambda \qquad (\mathscr{J}' \subset \mathscr{J}^*_1;\ |\mathscr{J}'| = \aleph_0).$$

By Theorem 7. 1, $\beta_\lambda \rightarrow [\beta_\lambda, \omega]_{\aleph_0}$. Therefore, in view of (7. 14) and (7. 16), there is a set $X \subset S_\lambda$ of type ω and an infinite set-system $\mathscr{J}^*_2 = (F, M^*_2, S) \subset \mathscr{J}^*_1$ such that

$$X \cap F_\mu = 0 \qquad (\mu \in M^*_2).$$

Put $T = \bigcup(\lambda < \nu < \omega) S_\nu$. Then $\text{tp } T = \beta$. By (ii) of the hypothesis and Lemma 7. 3 we deduce that there is $Y \in [T]^r$ and $M^*_3 \in [M^*_2]^{\aleph_0}$ such that $Y \cap F_\mu = 0$ ($\mu \in M^*_3$). It follows that the set $X \cup Y$ is $(\mathscr{J}, \aleph_0)$-free and has type $\omega + r$. This proves the lemma.

LEMMA 7. 5. *Let* $r, s, t < \omega$; $\alpha \rightarrow [\beta, \gamma]_{\aleph_0}$. *Let* $\text{tp } S = \alpha(r + s + t)$ *and let* $\mathscr{J} = = (F, M, S)$ *be an infinite set-system such that* (i) $\text{tp } F_\mu < \beta(r+1)$ ($\mu \in M$); (ii) $\text{tp } P(\mathscr{J}') < < \beta(s+1)$ ($\mathscr{J}' \subset \mathscr{J}$; $|\mathscr{J}'| = \aleph_0$); (iii) *there is no* $(\mathscr{J}, \aleph_0)$-*free subset of S of type*

$\gamma(t+1)$. *Then there is an infinite sub-system* $\mathscr{J}^*=(F, M^*, S)\subset\mathscr{J}$ *such that* (i)′ $\operatorname{tp} F_\mu \geqq \beta r$ $(\mu\in M^*)$; (ii)′ $\operatorname{tp} P(\mathscr{J}^*)\geqq\beta s$; (iii)′ $\operatorname{tp}\bigl(S\sim\bigcup(\mu\in M^*)F_\mu\bigr)\geqq\gamma t$.

PROOF. Put $n=r+s+t$. Then we may write $S=\bigcup(\nu<n)S_\nu$ (tp), where $\operatorname{tp} S_\nu=\alpha$ $(\nu<n)$. We shall define infinite set-systems $\mathscr{J}_\nu=(F, M_\nu, S)\subset\mathscr{J}$ $(0\leqq\nu\leqq n)$ and a partition of $[0, n)$ into disjoint sets A, B, C in the following way.

Put $M_0=M$. Let $\nu<n$ and suppose we have already defined $M_\nu\in[M]^{\aleph_0}$. Put $N_\nu=\{\mu\in M_\nu\colon \operatorname{tp} S_\nu\cap F_\mu\geqq\beta\}$.

Case 1. $|N_\nu|=\aleph_0$. Then we put $M_{\nu+1}=N_\nu$ and put ν in A.

Case 2. $|N_\nu|<\aleph_0$. Then $M'_\nu=M_\nu\sim N_\nu$ is infinite and

$$\operatorname{tp} S_\nu\cap F_\mu<\beta \qquad (\mu\in M'_\mu). \tag{7.17}$$

Let $\mathscr{J}'_\nu=(F, M'_\nu, S)$.

Case 2a. There is an infinite set-system $\mathscr{J}''_\nu\subset\mathscr{J}'_\nu$ such that $\operatorname{tp} S_\nu\cap P(\mathscr{J}''_\nu)\geqq\beta$. Then we put $\mathscr{J}_{\nu+1}=\mathscr{J}''_\nu$ and put ν in B.

Case 2b. $\operatorname{tp} S_\nu\cap P(\mathscr{J}'')<\beta$ $(\mathscr{J}''\subset\mathscr{J}'_\nu;\ |\mathscr{J}''|=\aleph_0)$. By (7. 17) and the hypothesis $\alpha\rightleftarrows[\beta, \gamma]_{\aleph_0}$, it follows in this case that there is a $(\mathscr{J}'_\nu, \aleph_0)$-free subset of S_ν of type γ. Thus, there is a set $M'''_\nu\in[M'_\nu]^{\aleph_0}$ such that $\operatorname{tp}\bigl(S_\nu\sim\bigcup(\mu\in M'''_\nu)F_\mu\bigr)\geqq\gamma$. In this case we set $M_{\nu+1}=M'''_\nu$ and put ν in C.

This procedure defines the partition $[0, n)=A\cup B\cup C$ and the set systems $\mathscr{J}_\nu=(F, M_\nu, S)$ $(\nu\leqq n)$ so that $M=M_0\supset M_1\supset\ldots\supset M_n$. Moreover, we have

$$\textit{if}\quad \nu\in A,\quad \textit{then}\quad \operatorname{tp} S_\nu\cap F_\mu\geqq\beta \qquad (\mu\in M_{\nu+1}), \tag{7.18}$$

$$\textit{if}\quad \nu\in B,\quad \textit{then}\quad \operatorname{tp} S_\nu\cap P(\mathscr{J}_{\nu+1})\geqq\beta, \tag{7.19}$$

$$\textit{if}\quad \nu\in C,\quad \textit{then}\quad \operatorname{tp}\bigl(S_\nu\sim\bigcup(\mu\in M_{\nu+1})F_\mu\bigr)\geqq\gamma. \tag{7.20}$$

Let $|A|=a$, $|B|=b$, $|C|=c$. Since $M_n\subset M_\nu$ $(\nu\leqq n)$, it follows from (7. 18) that

$$\operatorname{tp} F_\mu\geqq\beta a \qquad (\mu\in M_n). \tag{7.21}$$

Also, by Lemma 7. 1 and (7. 19), we have

$$\operatorname{tp} P(\mathscr{J}_n)\geqq\beta b. \tag{7.22}$$

Similarly, (7. 20) implies

$$\operatorname{tp}\bigl(S\sim\bigcup(\mu\in M_n)F_\mu\bigr)\geqq\gamma c. \tag{7.23}$$

By (i), (ii) and (iii) of the hypothesis, it follows from the above inequalities that $a\leqq r$; $b\leqq s$; $c\leqq t$. Moreover, since the sets A, B, C are disjoint, $a+b+c=n=$ $=r+s+t$. Hence, $a=r$; $b=s$; $c=t$. If we put $N=M_n$, then (7. 21), (7. 22), (7. 23) coincide with the conclusions of the lemma.

A consequence of the last lemma is

THEOREM 7. 7. *If* $\alpha\rightleftarrows[\beta, \gamma]_{\aleph_0}$ *and* $s, t<\omega$, *then*

$$\alpha(2s+t+1)\rightleftarrows[\beta(s+1), \gamma(t+1)]_{\aleph_0}. \tag{7.24}$$

PROOF. Let $S=\cup(\lambda\leqq 2s+t)S_\lambda$ (tp), tp $S_\lambda=\alpha$ $(\lambda\leqq 2s+t)$. Then tp $S=\alpha(2s+t+1)$. Suppose (7. 24) is false. Then there is a countable set-system $\mathscr{J}=(F, M, S)$ such that (i) tp $F_\mu<\beta(s+1)$ $(\mu\in M)$; (ii) tp $P(\mathscr{J}')<\beta(s+1)$ $(\mathscr{J}'\subset\mathscr{J}; |\mathscr{J}'|=\aleph_0)$; (iii) S does not contain a $(\mathscr{J}, \aleph_0)$-free subset of type $\gamma(t+1)$. Now consider the set-system $\mathscr{J}^0=(F^0, M, S^0)$, where $S^0=S\sim S_{2s+t}$ and $F^0_\mu=S^0\cap F_\mu$ $(\mu\in M)$. By (i), (ii), (iii) and Lemma 7. 5, there is an infinite set-system $\mathscr{J}^*=(F, M^*, S)\subset\mathscr{J}$ such that (i)′ tp $S^0\cap F_\mu\geqq\beta s$ $(\mu\in M^*)$; (ii)′ tp $S^0\cap P(\mathscr{J}^*)\geqq\beta s$; (iii)′ tp $(S^0\sim\cup(\mu\in M^*)F_\mu)\geqq\gamma t$. From (i)—(iii)′ it follows that

$$\text{tp } S_{2s+t}\cap F_\mu<\beta \qquad (\mu\in M^*);$$

$$\text{tp } S_{2s+t}\cap P(\mathscr{J}')<\beta \quad \text{if} \quad \mathscr{J}'\subset\mathscr{J}^* \quad \text{and} \quad |\mathscr{J}'|=\aleph_0;$$

$$\text{tp}\left(S_{2s+t}\sim\cup(\mu\in M')F_\mu\right)<\gamma \quad \text{if} \quad M'\in[M^*]^{\aleph_0}.$$

These last three statements contradict the hypothesis $\alpha\rightarrow[\beta, \gamma]_{\aleph_0}$. This proves (7. 24).

We need also the following lemma which resembles Lemma 7. 5.

LEMMA 7. 6. *Let* $q, r, s, t<\omega$; $\beta=\omega^\varrho>\omega\theta$; $\theta>0$. *Let* $S=T_0\cup T_1$ (tp); tp $T_0=\beta(r+s+t)$; tp $T_1\geqq\omega\theta$. *Suppose that* $\mathscr{J}=(F, M, S)$ *is a countable set-system such that* (i) tp $F_\mu<\beta r+\omega\theta$ $(\mu\in M)$; (ii) tp $P(\mathscr{J}')<\beta s+\omega\theta$ $(\mathscr{J}'\subset\mathscr{J}; |\mathscr{J}'|=\aleph_0)$; (iii) *$S$ does not contain a $(\mathscr{J}, \aleph_0)$-free subset of type $\omega(t+1)+q$. Then there is* $\mathscr{J}^*=(F, M^*, S)\subset\mathscr{J}$ *such that* $|M^*|=\aleph_0$ *and* (i)′ tp $T_0\cap F_\mu\geqq\beta r$ $(\mu\in M^*)$; (ii)′ tp $T_0\cap P(\mathscr{J}^*)\geqq\beta s$; (iii)′ tp $(T_0\sim\cup(\mu\in M^*)F_\mu)\geqq\omega t$.

PROOF. We proceed as in the proof of Lemma 7. 5. Put $n=r+s+t$. Then $T_0=\cup(\lambda<n)S_\lambda$ (tp), where tp $S_\lambda=\beta$ $(\lambda<n)$. We define infinite set-systems $\mathscr{J}_\lambda=(F, M_\lambda, S)\subset\mathscr{J}$ $(\lambda\leqq n)$ and a partition of $[0, n)$ into disjoint sets A, B, C in the following way.

Put $M_0=M$. Let $\lambda<n$ and suppose we have already defined $M_\lambda\in[M]^{\aleph_0}$. Put $N_\lambda=\{\mu\in M_\lambda: \text{tp } S_\lambda\cap F_\mu=\beta\}$.

Case 1. $|N_\lambda|=\aleph_0$. Then put $M_{\lambda+1}=N_\lambda$ and put λ in A.

Case 2. $|N_\lambda|<\aleph_0$. Then $M'_\lambda=M_\lambda\sim N_\lambda$ is infinite and

(7. 25) $$\text{tp } S_\lambda\cap F_\mu<\beta \qquad (\mu\in M'_\lambda).$$

Put $\mathscr{J}'_\lambda=(F, M'_\lambda, S)$.

Case 2a. There is an infinite set-system $\mathscr{J}''_\lambda\subset\mathscr{J}'_\lambda$ such that tp $S_\lambda\cap P(\mathscr{J}''_\lambda)=\beta$. Then we define $\mathscr{J}_{\lambda+1}=\mathscr{J}''_\lambda$ and put λ in B.

Case 2b. tp $S_\lambda\cap P(\mathscr{J}')<\beta$ $(\mathscr{J}'\subset\mathscr{J}'_\lambda; |\mathscr{J}'|=\aleph_0)$. Since $\beta\rightarrow[\beta, \omega]_{\aleph_0}$ by Theorem 7. 1, it follows from (7. 25) that, in this case, there is $M'''_\lambda\in[M'_\lambda]_{\aleph_0}$ such that tp $(S_\lambda\sim\cup(\mu\in M'''_\lambda)F_\mu)\geqq\omega$. In this case we define $M_{\lambda+1}=M'''_\lambda$ and put λ in C.

This defines the set-systems $\mathscr{J}_\lambda=(F, M_\lambda, S)$ $(\lambda\leqq n)$ and disjoint sets A, B, C such that $[0, n)=A\cup B\cup C$; $M=M_0\supset M_1\supset\ldots\supset M_n\in[M]^{\aleph_0}$. Also, we have

(7. 26) *if* $\lambda\in A$, *then* tp $S_\lambda\cap F_\mu=\beta$ $\quad(\mu\in M_{\lambda+1})$;

(7. 27) *if* $\lambda\in B$, *then* tp $S_\lambda\cap P(\mathscr{J}_{\lambda+1})=\beta$;

(7. 28) *if* $\lambda\in C$, *then* tp$(S_\lambda\sim\cup(\mu\in M_{\lambda+1})F_\mu)\geqq\omega$;

(7. 29) *if* $\lambda\in C$, *then* tp $S_\lambda\cap P(\mathscr{J}')<\beta$ $(\mathscr{J}'\subset\mathscr{J}_n; |\mathscr{J}'|=\aleph_0)$.

Let $|A|=a$, $|B|=b$, $|C|=c$. By Lemma 7. 1, $P(\mathscr{J}_\lambda)\subset P(\mathscr{J}_n)$ $(\lambda\leqq n)$. Therefore, from (7. 26), (7. 27) and (7. 28) respectively, we deduce that

(7. 30) $$\text{tp } T_0\cap F_\mu\geqq\beta a \qquad (\mu\in M_n);$$

(7. 31) $$\text{tp } T_0\cap P(\mathscr{J}_n)\geqq\beta b;$$

(7. 32) $$\text{tp}\left(T_0\sim\bigcup(\mu\in M_n)F_\mu\right)\geqq\omega c.$$

These last three statements coincide with the conclusion of the lemma with $M^*=M_n$, provided that

(7. 33) $$a=r;\quad b=s;\quad c=t.$$

To complete the proof we will assume that (7. 33) is false and deduce a contradiction.

Since the sets A, B, C are disjoint, we have that $a+b+c = r+s+t = n$. Also, (7. 30)—(7. 32) and the hypothesis (i)—(iii) imply that $a\leqq r$, $b\leqq s$; $c\leqq t+1$. Therefore, since (7. 33) is false by assumption, we must have

(7. 34) $$c = t+1;\quad a\in\{r, r-1\};\quad a+b = r+s-1.$$

Suppose that $a = r-1$ and $b=s$. Then, by Lemma 7. 1, (7. 31) and the hypothesis (ii),

$$\text{tp } T_1\cap P(\mathscr{J}')<\omega\theta \qquad (\mathscr{J}'\subset\mathscr{J}_n;\ |\mathscr{J}'|=\aleph_0).$$

Therefore, by Lemma 7. 3, there is a $(\mathscr{J}_n, \aleph_0)$-free set $Y\subset T_1$ such that $|Y|=q$. Put $X=T_0\sim\bigcup(\mu\in M_n)F_\mu$. Then $X\cup Y$ is $(\mathscr{J}_n, \aleph_0)$-free and, by (7. 32) and (7. 34), $\text{tp } X\cup Y\geqq\omega(t+1)+q$. This contradicts (iii) of the hypothesis.

Therefore $a=r$ and $b = s-1$. By (7. 34), $|C|=c\geqq 1$. Let π denote the *largest* element of C. Suppose that

(7. 35) $$A\subset[0, \pi).$$

Then, since $D=\bigcup(\mu\in A)S_\lambda$ precedes S_π in the ordering of S and $\text{tp } D\cap F_\mu\geqq\beta a=\beta r$, it follows from (i) that

(7. 36) $$\text{tp } S_\pi\cap F_\mu<\omega\theta \qquad (\mu\in M_n).$$

Since $\pi\in C$, it follows from (7. 36), (7. 29) and Lemma 7. 4 that S_π contains a $(\mathscr{J}_n, \aleph_0)$-free subset of type $\omega+q$. Thus, there is $N\in[M_n]^{\aleph_0}$ such that $\text{tp}\left(S_\pi\sim\right.$ $\left.\sim\bigcup(\mu\in N)F_\mu\right)\geqq\omega+q$. Since π is the largest element of C, it follows from (7. 28) (7. 34) that

$$\text{tp}\left(\bigcup(\lambda\in C)S_\lambda\sim\bigcup(\mu\in N)F_\mu\right)\geqq\omega(t+1)+q.$$

This contradicts (iii). Hence, (7. 35) is false and there is $\alpha\in A$ such that $\pi<\alpha$. This means that the set

$$Z=\bigcup(\lambda\in C)S_\lambda\sim\bigcup(\mu\in M_n)F_\mu$$

precedes the set S_α in the ordering of S. By (7. 28) and (7. 34), $\text{tp } Z\geqq\omega\ (t+1)$. Therefore, by (iii), $S_\alpha\cup T_1$ does not contain a $(\mathscr{J}_n, \aleph_0)$-free set of q elements. Therefore, by Lemma 7. 3, there is an infinite sub-system $\mathscr{J}'\subset\mathscr{J}_n$ such that

$$\text{tp}\,(S_\alpha\cup T_1)\cap P(\mathscr{J}')\geqq\beta+\omega\theta.$$

Since $P(\mathcal{J}_n) \subset P(\mathcal{J}')$, by Lemma 7. 1, it follows from (7. 28) and the fact that $\alpha \notin B$ that

$$\mathrm{tp}\,(T_0 \cup T_1) \cap P(\mathcal{J}') \geqq \beta b + \beta + \omega\theta = \beta s + \omega\theta.$$

This contradicts the hypothesis (ii). Hence, (7.33) holds and the proof of Lemma 7. 6 is complete.

§ 8. The case of denumerable ordinals continued. In this section we show that, for given $\beta, \gamma < \omega_1$, there is $\alpha_0 < \omega_1$ such that

$$\alpha_0 \rightleftarrows [\beta, \gamma]_{\aleph_0}, \tag{8. 1}$$

$$\alpha \nrightarrow [\beta, \gamma]^2_{\aleph_0} \qquad (\alpha < \alpha_0). \tag{8. 2}$$

Our results also show how to evaluate α_0 in terms of the given β, γ.

If $\beta = 0$, the above relations hold trivially with $\alpha_0 = 0$. If $0 < \beta < \omega$, we have the stronger statement of Theorem 8. 1. The more general case when $\omega \leqq \beta < \omega_1$ is dealt with in Theorem 8. 2.

THEOREM 8. 1. *If $0 < \beta < \omega$ and $\gamma < \omega_1$, then there is α_0 such that*

$$\alpha_0 \rightarrow [\beta, \gamma]_{\aleph_0}, \tag{8. 3}$$

$$\alpha \nrightarrow [\beta, \gamma]^1_m \qquad (\alpha < \alpha_0;\ m \geqq 1). \tag{8. 4}$$

The value of α_0 is given by

$$\alpha_0 = \begin{cases} \gamma & \text{if} \quad \gamma = \gamma^-, \\ \gamma_1 + \beta & \text{if} \quad \gamma = \gamma_1 + 1. \end{cases} \tag{8. 5}$$

PROOF. If α_0 is given by (8. 5) and $\alpha < \alpha_0$, then $\alpha \nrightarrow (\beta, \gamma)^1$. Now (8. 4) follows from Theorem 5. 1.

Let $\gamma = \omega\gamma_0 + j$, where $j < \omega$. We define δ as follows (i) $\delta = 0$ if $j = 0$, (ii) $\delta = (j-1) + \beta$ if $j > 0$. Then (8. 5) gives $\alpha_0 = \omega\gamma_0 + \delta$. Let $\mathcal{J} = (F, M, S)$ be a $(\aleph_0, \alpha_0)$-system such that $\mathrm{tp}\, F_\mu < \beta$ $(\mu \in M)$. Then $S = S_0 \cup S_1$ (tp), where $\mathrm{tp}\, S_0 = \omega\gamma_0$, $\mathrm{tp}\, S_1 = \delta$. Since S_1 is finite, there is $M' \in [M]^{\aleph_0}$ such that $S_1 \cap F_\mu = X$ $(\mu \in M')$. By Theorem 7. 2, $\omega\gamma_0 \rightarrow [\beta, \omega\gamma_0]_{\aleph_0}$. Hence, there is $M'' \in [M']^{\aleph_0}$ such that $Y = S_0 \sim \bigcup (\mu \in M'') F_\mu$ has type $\geqq \omega\gamma_0$. The set $Y \cup (S_1 \sim X)$ is $(\mathcal{J}, \aleph_0)$-free and has type $\geqq \omega\gamma_0 + j = \gamma$. This proves (8. 3).

Lemmas 8. 1 and 8. 2 give special cases of the positive and negative parts of Theorem 8. 2.

LEMMA 8. 1. *Let $b, c, i, j < \omega \leqq \beta < \omega_1$; $\beta = \omega^\varrho$. Then*

$$\alpha_0 \rightleftarrows [\beta(b+1) + i,\ \omega(c+1) + j]_{\aleph_0}, \tag{8. 6}$$

where

$$\alpha_0 = \begin{cases} \beta(2b+c+1) & \text{if} \quad i = j = 0, \\ \beta(2b+c+2) & \text{if} \quad i = 0 < j, \\ \beta(2b+c+2) + \omega & \text{if} \quad i > 0 = j, \\ \beta(2b+c+2) + \omega + (i+j-1) & \text{if} \quad i, j > 0. \end{cases} \tag{8. 7}$$

PROOF. By Theorem 7. 1, $\beta \rightarrow [\beta, \omega]_{\aleph_0}$. Therefore, by Theorem 7. 7,

(8. 8) $$\beta(2b+c+1) \rightarrow [\beta(b+1), \omega(c+1)]_{\aleph_0},$$

(8. 9) $$\beta(2b+c+2) \rightarrow [\beta(b+1), \omega(c+2)]_{\aleph_0}.$$

The relations (8. 8) and (8. 9) respectively imply (8. 6) in the cases (i) $i=j=0$ and (ii) $i=0<j$.

For the remainder of the proof we assume that $i>0$. If $j=0$ we put $\delta=0$; if $j>0$ we put $\delta = i+j-1$. Then, for the remaining two cases of (8. 7), we have $\alpha_0=\beta(2b+c+2)+\omega+\delta$. Suppose that (8. 6) is false. Then there is a $(\aleph_0, \alpha_0)$-system $\mathscr{J}=(F, M, S)$ such that

(8. 10) $$\text{tp } F_\mu < \beta(b+1)+i \qquad (\mu \in M);$$

(8. 11) $$\text{tp } P(\mathscr{J}') < \beta(b+1)+i \qquad (\mathscr{J}' \subset \mathscr{J};\ |\mathscr{J}'| = \aleph_0);$$

(8. 12) *S contains no* $(\mathscr{J}, \aleph_0)$*-free subset of type* $\omega(c+1)+j$.

We may write $S=S_0 \cup U \cup V$ (tp), where tp $S_0=\beta(2b+c+2)$, tp $U=\omega$ and tp $V=\delta$. We consider separately the cases $\beta=\omega$ and $\beta>\omega$.

Case 1. $\beta=\omega$. Then tp $(S_0 \cup U)=\beta(2b+c+3)$. It follows from (8. 10)—(8. 12) and[3] Lemma 7. 5 that there is an infinite system $\mathscr{J}^*=(F, M^*, S) \subset \mathscr{J}$ such that

(8. 13) $$\text{tp } (S_0 \cup U) \cap F_\mu \geqq \beta(b+1) \qquad (\mu \in M^*);$$

(8. 14) $$\text{tp } (S_0 \cup U) \cap P(\mathscr{J}^*) \geqq \beta(b+1);$$

(8. 15) $$\text{tp } (S_0 \cup U \sim \cup(\mu \in M^*) F_\mu) \geqq \omega(c+1).$$

From (8. 10) and (8. 13), it follows that tp $V \cap F_\mu < i$ $(\mu \in M^*)$. Also, (8. 12) and (8. 15) together imply that V does not contain a $(\mathscr{J}^*, \aleph_0)$-free subset of type j. This is a contradiction, since Theorem 8. 1 implies that tp $V=\delta \rightarrow [i, j]_{\aleph_0}$.

Case 2. $\beta>\omega$. In this case (8. 10)—(8. 12) and[4] Lemma 7. 6 imply that there is $\mathscr{J}^*=(F, M^*, S) \subset \mathscr{J}$ such that $|\mathscr{J}^*|=\aleph_0$ and

(8. 16) $$\text{tp } S_0 \cap F_\mu \geqq \beta(b+1) \qquad (\mu \in M^*);$$

(8. 17) $$\text{tp } S_0 \cap P(\mathscr{J}^*) \geqq \beta(b+1);$$

(8. 18) $$\text{tp } \big(S_0 \sim \cup(\mu \in M^*) F_\mu\big) \geqq \omega c.$$

From (8. 10)—(8. 12) and (8. 16)—(8. 18), we deduce that

$$\text{tp } (U \cup V) \cap F_\mu < i \qquad (\mu \in M^*);$$

$$\text{tp } (U \cup V) \cap P(\mathscr{J}') < i \qquad (\mathscr{J}' \subset \mathscr{J}^*;\ |\mathscr{J}'| = \aleph_0);$$

and $U \cup V$ does not contain a $(\mathscr{J}^*, \aleph_0)$-free subset of type $\omega+j$. This is a contradiction since

$$\omega+\delta \rightarrow [i, \omega+j]_{\aleph_0}$$

by Theorem 7. 5 (ii) (for $\omega \rightarrow [\omega, \omega]_{\aleph_0}$ and $\delta \rightarrow [i, j]_{\aleph_0}$). This completes the proof of Lemma 8. 1.

[3] Apply Lemma 7. 5 to the set $S_0 \cup U$ with $r = s = b+1$, $t=c$ and $\alpha=\beta=\gamma=\omega$.

[4] Apply Lemma 7. 6 with $T_0=S_0$, $T_1=U \cup V$, $r = s = b+1$, $t=c$, $q=j$ and $\theta=1$.

LEMMA 8. 2. *Let $i, j<\omega\leqq\beta<\omega_1$; $\beta=\omega^{\varrho}$. Then*

$$\alpha\nrightarrow[\beta+i, \omega+j]^2_{\aleph_0} \qquad (\alpha<\alpha_0), \tag{8. 19}$$

where α_0 has the value given by

$$\alpha_0=\begin{cases}\beta & \text{if } i=j=0,\\ \beta2 & \text{if } i=0<j,\\ \beta2+\omega & \text{if } i>0=j,\\ \beta2+\omega+(i+j-1) & \text{if } i,j>0.\end{cases} \tag{8. 20}$$

PROOF. *Case* 1. $i=j=0$. Clearly (8. 19) holds with $\alpha_0=\beta$.

Case 2. $i=0<j$. By Theorem 7. 4, $\beta\nrightarrow[\beta, \omega+1]^2_{\aleph_0}$, and, if $\beta'<\beta$, clearly $\beta'\nrightarrow[\beta, 1]^2_{\aleph_0}$. Therefore, by the remark after Theorem 7. 6, $\beta+\beta'\nrightarrow[\beta, \omega+1]^2_{\aleph_0}$. This implies (8. 19) with $\alpha_0=\beta2$.

Case 3. $i>0=j$. Let $\alpha<\beta2+\omega$. Then there is $q<\omega$ such that $\alpha\leqq\beta2+q$. Let $S=B\cup B'\cup Q$ (tp), where $\text{tp } B=\text{tp } B'=\beta$, $\text{tp } Q=q$. Let $B=\{x_0, \dots, \hat{x}_\omega\}_{\neq}$, $B'=\{x'_0, \dots, \hat{x}'_\omega\}_{\neq}$. Let $M=[0, \omega)$ and consider the set-system $\mathscr{J}=(F, M, S)$, where $F_\mu=\{x_\lambda: \lambda<\mu\}\cup\{x'_\lambda: \lambda>\mu\}$ $(\mu\in M)$.

Suppose $S'\subset S$ is $(\mathscr{J}, 2)$-complete and $|S'|>1$. Then $S'\cap Q=0$. Suppose there is $\varrho<\omega$ such that $x'_\varrho\in S'$. Then $S'\cap B\subset\{x_0, \dots, \hat{x}_\varrho\}$. Hence, $\text{tp } S'\leqq\varrho+\beta=\beta$. Thus, there is no $(\mathscr{J}, 2)$-complete subset of S of type $\beta+1$.

Suppose that S'' is $(\mathscr{J}, \aleph_0)$-free. Then there is $\{\mu_0, \dots, \hat{\mu}_\omega\}_<\subset[0, \omega)$ such that $S''\cap F_{\mu_\nu}=0$ $(\nu<\omega)$. This implies that $S''\subset\{x'_\lambda: \lambda<\mu_0\}\cup Q$, i. e. S'' is finite. We have proved that

$$\beta2+q\nrightarrow[\beta+1, \omega]^2_{\aleph_0} \quad \text{if} \quad q<\omega. \tag{8. 21}$$

Hence, (8. 19) holds in this case with $\alpha_0=\beta2+\omega$.

Case 4. $i, j>0$. By Theorems 8. 1 and 4. 1 we have that

$$\omega+i+j-2\nrightarrow[i, \omega+j]^2_{\aleph_0}.$$

Therefore, by (8. 21) and Theorem 7. 6,

$$\beta2+\omega+i+j-2\nrightarrow[\beta+i, \omega+j]^2_{\aleph_0}.$$

This proves that (8. 19) holds in this case with $\alpha_0=\beta2+\omega+i+j-1$ and concludes the proof of Lemma 8. 2.

THEOREM 8. 2. *Let $i, j, k<\omega\leqq\beta=\beta_0+\beta_1+\dots+\beta_k+i<\omega_1$; $\beta_\lambda=\omega^{\varrho_\lambda}$ $(\lambda\leqq k)$; $\varrho_0\geqq\varrho_1\geqq\dots\geqq\varrho_k>0$; $\gamma=\omega\gamma_0+j<\omega_1$. Then* (8. 1) *and* (8. 2) *both hold if α_0 has the following value:*

Case 1. $\gamma_0=\gamma_0^-$. *Then*

$$\alpha_0=\begin{cases}\beta_0\gamma_0 & \text{if } j=0,\\ \beta_0\gamma_0+\beta & \text{if } i=0<j,\\ \beta_0\gamma_0+\beta+j-1 & \text{if } i,j>0.\end{cases}$$

Case 2. $\gamma_0=\gamma_1+1$. *Then* $\alpha_0=\beta_0\gamma_1+\Sigma(\lambda<k)\beta_\lambda 2+\delta$, *where*

$$\delta=\begin{cases}\beta_k & \text{if } i=j=0,\\ \beta_k 2 & \text{if } i=0<j,\\ \beta_k 2+\omega & \text{if } i>0=j,\\ \beta_k 2+\omega+i+j-1 & \text{if } i,j>0.\end{cases}$$

PROOF. *Case* 1. $\gamma_0=\gamma_0^-$. Then $\gamma_0=\omega\gamma'$ and, by Theorem 7. 3,

(8. 22) $$\beta_0\gamma_0 \rightleftarrows [\beta_0(k+2), \omega\gamma_0]_{\aleph_0}.$$

Case 1a. $j=0$. Then $\gamma=\omega\gamma_0$ and since $\beta_0(k+2)>\beta$, it follows from (8. 22) that (8. 1) holds with $\alpha_0=\beta_0\gamma_0$.

If $\alpha<\beta_0\gamma_0$, then there is $\gamma''<\gamma'$ and $q<\omega$ such that $\alpha\leqq\beta_0(\omega\gamma''+q)$. By Theorem 7. 4,

$$\beta_0\omega\gamma'' \nrightarrow [\beta_0, \omega^2\gamma''+1]^2_{\aleph_0}$$
$$\beta_0 \nrightarrow [\beta_0, \omega+1]^2_{\aleph_0}.$$

Using Theorem 7. 6 and the fact that β_0 is indecomposable, we deduce that

$$\beta_0\omega\gamma''+\beta_0 q \nrightarrow [\beta_0, \omega^2\gamma''+\omega q+1]^2_{\aleph_0}.$$

Therefore, by Theorem 4. 1, $\alpha \nrightarrow [\beta, \gamma]^2_{\aleph_0}$. This proves that (8. 2) also holds in this case with $\alpha_0=\beta_0\gamma_0$.

Case 1b. $i=0<j$. In this case β is a multiple of ω and, by (7. 13), $\beta \rightleftarrows [\beta, j]_{\aleph_0}$. Therefore, by (8. 22) and Theorem 7. 5 (ii),

$$\beta_0\gamma_0+\beta \rightleftarrows [\beta, \omega\gamma_0+j]_{\aleph_0}.$$

Therefore, (8. 1) holds with $\alpha_0=\beta_0\gamma_0+\beta$.

By Theorem 7. 4,

(8. 23) $$\beta_0\gamma_0 \nrightarrow [\beta_0, \omega\gamma_0+1]^2_{\aleph_0}.$$

Therefore, by Theorem 7. 6 and the fact that $\beta' \nrightarrow [\beta, 1]^2_{\aleph_0}$ if $\beta'<\beta$, we deduce that

$$\beta_0\gamma_0+\beta' \nrightarrow [\beta, \omega\gamma_0+1]^2_{\aleph_0} \qquad (\beta'<\beta_0).$$

Hence, (8. 2) also holds in this case if $\alpha_0=\beta_0\gamma_0+\beta$.

Case 1c. $i,j>0$. Put $\beta=\beta^*+i$. Then β^* is a multiple of ω and, by (7. 13), $\beta^* \rightleftarrows [\beta^*, j]_{\aleph_0}$. Therefore, by Theorem 7. 5 (i),

$$\beta+(j-1)=\beta^*+(i+j-1) \rightleftarrows [\beta, j]_{\aleph_0}.$$

Therefore, by Theorem 7. 5 (ii), this last relation and (8. 22), we have

$$\beta_0\gamma_0+\beta+(j-1) \rightleftarrows [\beta, \gamma]_{\aleph_0}.$$

Therefore, (8. 1) holds with $\alpha_0=\beta_0\gamma_0+\beta+(j-1)$.

Clearly, $\beta^*+(i+j-2) \nrightarrow [\beta, j]^2_{\aleph_0}$. Therefore, by (8. 23) and Theorem 7. 6,

$$\beta_0\gamma_0+\beta^*+(i+j-2) \nrightarrow [\beta, \omega\gamma_0+j]^2_{\aleph_0}.$$

Hence, (8. 2) also holds in this case with $\alpha_0=\beta_0\gamma_0+\beta+(j-1)$.

Case 2. $\gamma_0=\gamma_1+1$. We must prove that both (8. 1) and (8. 2) hold if $\alpha_0=\beta_0\gamma_1+\Sigma(\lambda<k)\beta_\lambda 2+\delta$, where δ has the value given in the Theorem.

We consider first the negative relation (8. 2). By Theorem 7. 4,

$$\beta_0\gamma_1 \nrightarrow [\beta_0, \omega\gamma_1+1]^2_{\aleph_0}.$$

Also, by Lemma 8. 2, we have

$$\beta_\lambda 2 \nrightarrow [\beta_\lambda+1, \omega]^2_{\aleph_0},$$

$$\delta' \nrightarrow [\beta_k+i, \omega+j]^2_{\aleph_0} \quad \text{if} \quad \delta'<\delta.$$

Therefore, by Theorem 7. 6,

$$\beta_0\gamma_1+\Sigma(\lambda<k)\beta_\lambda 2+\delta' \nrightarrow [\Sigma(\lambda\leqq k)\beta_\lambda+i, \omega\gamma_1+\omega+j]^2_{\aleph_0}$$

if $\delta'<\delta$. This proves that (8. 2) holds if α_0 has the value given by the Theorem.

It remains to prove (8. 1) in case 2. It is more convenient to re-write the standard representation for β in the form

$$\beta = \beta'_0 b_0+\beta'_1 b_1+\dots+\beta'_l b_l+i,$$

where $l<\omega$; $0<b_\lambda<\omega$ $(\lambda\leqq l)$; $\{\beta'_0, \beta'_1, \dots, \beta'_l\}_> = \{\beta_0, \beta_1, \dots, \beta_k\}_\geqq$. There is $c_0<\omega$ such that

$$\gamma_1=\omega\gamma_2+c_0.$$

We put $c_\lambda=0$ $(0<\lambda\leqq l)$. Then we may write

$$\gamma=\omega^2\gamma_2+\omega(c_0+c_1+\dots+c_l+1)+j$$

and

$$\alpha_0=\beta_0\omega\gamma_2+\Sigma(\lambda<l)\beta'_\lambda(2b_\lambda+c_\lambda)+\beta'_l(2b_l+c_l-2)+\delta.$$

Let $\mathcal{J}=(F, M, S)$ be a $(\aleph_0, \alpha_0)$-system and suppose that

(8. 24) $$\text{tp } F_\mu<\beta \qquad (\mu\in M);$$

(8. 25) $$\text{tp } P(\mathcal{J}')<\beta \qquad (\mathcal{J}'\subset\mathcal{J};\ |\mathcal{J}'|=\aleph_0);$$

(8. 26) *S contains no $(\mathcal{J}, \aleph_0)$-free subset of type γ.*

We will deduce a contradiction.

Since tp $S=\alpha_0$, we have $S=S_0\cup T_0\cup T_1\cup\dots\cup T_l$ (tp), where tp $S_0=\beta_0\omega\gamma_2$, tp $T_\lambda=\beta'_\lambda(2b_\lambda+c_\lambda)$ $(\lambda<l)$, tp $T_l=\beta'_l(2b_l+c_l-2)+\delta$.

By Theorem 7. 3,

$$\text{tp } S_0=\beta_0\omega\gamma_2 \rightarrow [\beta_0(k+2), \omega^2\gamma_2]_{\aleph_0}.$$

Therefore, since $\beta_0(k+2)>\beta$, it follows from (8. 24) and (8. 25) that there is $M_0\in[M]^{\aleph_0}$ such that

(8. 27) $$\text{tp }(S_0\sim\cup(\mu\in M_0)F_\mu)\geqq\omega^2\gamma_2.$$

We will prove by induction that there are infinite set-systems $\mathcal{J}_\lambda=(F, M_\lambda, S)$ such that $M_0\supset M_1\supset\dots\supset M_l$ and

(8. 28) $$\text{tp }(\cup(\nu<\lambda)T_\nu\cap F_\mu)\geqq\Sigma(\nu<\lambda)\beta'_\nu b_\nu \qquad (\mu\in M_\lambda),$$

(8. 29) $$\text{tp }(\cup(\nu<\lambda)T_\nu\cap P(\mathcal{J}_\lambda))\geqq\Sigma(\nu<\lambda)\beta'_\nu b_\nu,$$

(8. 30) $$\text{tp }(\cup(\nu<\lambda)T_\nu\sim\cup(\mu\in M_\lambda)F_\mu)\geqq\Sigma(\nu<\lambda)\omega c_\nu$$

all hold for $\lambda\leqq l$.

If $l=0$ there is nothing to prove since (8. 28)—(8. 30) merely assert $0 \geqq 0$. Now suppose $l>0$. Let $\lambda < l$ and suppose that we havre already defined $\mathscr{J}_\lambda = (F, M_\lambda, S)$ so that the above statements hold. Put

$$Z_0 = T_\lambda, \quad Z_1 = \cup(\lambda < \nu \leqq l) T_\nu.$$

Also, let θ be the *least* ordinal satisfying

$$\omega\theta \geqq \bar{\beta}_\lambda = \Sigma(\lambda < \nu \leqq l)\beta'_\nu b_\nu + i.$$

Since $\beta'_{\lambda+1}(b_{\lambda+1}+1)$ is a multiple of ω which is strictly greater than $\bar{\beta}_\lambda$, it follows from the definition of θ that

$$(8.\ 31) \qquad \beta'_\lambda > \beta'_{\lambda+1}(b_{\lambda+1}+1) \geqq \omega\theta.$$

We note also, that

$$(8.\ 32) \qquad \operatorname{tp} Z_1 \geqq \omega\theta.$$

To see this consider separately the cases (i) $\lambda = l-1$ and (ii) $\lambda < l-1$. If $\lambda = l-1$, then $\bar{\beta}_\lambda = \beta'_l b_l + i$ and $\operatorname{tp} Z_1 = \operatorname{tp} T_l \geqq \beta'_l(2b_l - 2) + \delta$. If $i=0$, then $\delta \geqq \beta_k = \beta'_l$ and $\operatorname{tp} Z_1 \geqq \beta'_l b_l \geqq \omega\theta$ by the minimal property of θ. If $i>0$, then $\delta \geqq \beta'_l 2$ and again $\operatorname{tp} Z_1 \geqq \beta'_l(2b_l) \geqq \omega\theta$. Thus (8. 32) holds if $\lambda = l-1$. If $\lambda < l-1$, then by (8. 31),

$$\operatorname{tp} Z_1 \geqq \operatorname{tp} T_{\lambda+1} = \beta'_{\lambda+1}(2b_{\lambda+1}) \geqq \beta'_{\lambda+1}(b_{\lambda+1}+1) \geqq \omega\theta.$$

This proves (8. 32).

By (8. 24) and (8. 28) we have

$$(8.\ 33) \qquad \operatorname{tp}(Z_0 \cup Z_1) \cap F_\mu < \Sigma(\lambda \leqq \nu \leqq l)\beta'_\nu b_\nu + i \leqq \beta'_\lambda b_\lambda + \omega\theta \qquad (\mu \in M_\lambda).$$

Also, by (8. 25), (8. 29) and Lemma 7. 1,

$$(8.\ 34) \qquad \operatorname{tp}(Z_0 \cup Z_1) \cap P(\mathscr{J}') < \beta'_\lambda b_\lambda + \omega\theta \qquad (\mathscr{J}' \subset \mathscr{J}_\lambda;\ |\mathscr{J}'| = \aleph_0).$$

Similarly, by (8. 26), (8. 27) and (8. 30),

(8. 35) $\qquad Z_0 \cup Z_1$ *contains no* $(\mathscr{J}_\lambda, \aleph_0)$*-free subset of type* $\omega(c_\lambda+1)+j$.

By (8. 31), $\operatorname{tp} Z_0 = \beta'_\lambda(2b_\lambda + c_\lambda) > \omega\theta$. Therefore, by[5] (8. 33)—(8. 35) and Lemma 7. 6, there is an infinite set-system $\mathscr{J}_{\lambda+1} = (F, M_{\lambda+1}, S) \subset \mathscr{J}_\lambda$, such that

$$\operatorname{tp} Z_0 \cap F_\mu \geqq \beta'_\lambda b_\lambda \quad (\mu \in M_{\lambda+1}),$$

$$\operatorname{tp} Z_0 \cap P(\mathscr{J}_{\lambda+1}) \geqq \beta'_\lambda b_\lambda,$$

$$\operatorname{tp}\big(Z_0 \sim \cup(\mu \in M_{\lambda+1}) F_\mu\big) \geqq \omega c_\lambda.$$

The last three inequalities, together with (8. 28)—(8. 30), imply that (8. 28)—(8. 30) remain valid when we replace λ by $\lambda+1$. Therefore, by induction, there is $\mathscr{J}_l = (F, M_l, S) \subset \mathscr{J}$ such that $|M_l| = \aleph_0$ and such that (8. 28)—(8. 30) hold with $\lambda = l$. These three inequalities (with $\lambda = l$) together with (8. 24)—(8. 27) imply that

$$\operatorname{tp} T_l \cap F_\mu < \beta'_l b_l + i \qquad (\mu \in M_l),$$

$$\operatorname{tp} T_l \cap P(\mathscr{J}') < \beta'_l b_l + i \qquad (\mathscr{J}' \subset \mathscr{J}_l;\ |\mathscr{J}'| = \aleph_0)$$

[5] We apply Lemma 7. 6 with $\beta = \beta'_\lambda$; $q=j$; $r=s=b_\lambda$; $t=c_\lambda$.

and T_l does not contain a $(\mathscr{I}_l, \aleph_0)$-free subset of type $\omega(c_l+1)+j$. This is a contradiction, for, by Lemma 8.1 and the definition of δ,

$$\text{tp } T_l = \beta_l(2b_l+c_l-2)+\delta \not\rightrightarrows [\beta_l b_l + i, \omega(c_l+1)+j]_{\aleph_0}.$$

This concludes the proof of Theorem 8.2.

§ 9. The case of denumerable ordinals concluded. By Theorems 4.2 and 4.4 (and the continuum hypothesis $(*)$), the relations

$$\alpha \rightarrow [\beta, \gamma]^k_m \quad \text{and} \quad \alpha \rightarrow [\beta, \gamma]^k_{m'}$$

are equivalent if $\alpha < \omega_1$ and $m \geqq \aleph_0$. Therefore, if $m' = \aleph_0$, $\beta, \gamma < \omega_1$ and α_0 has the value prescribed by Theorems 8.1, 8.2, then $\alpha_0 \rightarrow [\beta, \gamma]^{<\aleph_0}_m$ and $\alpha \not\rightarrow [\beta, \gamma]^2_m$ if $\alpha < \alpha_0$.

In this section we examine relations of the form

$$\alpha \rightarrow [\beta, \gamma]^{<k}_m \tag{9.1}$$

for $m' > \aleph_0$ and denumerable α, β, γ. Assuming $(*)$ it is only necessary to consider the case $m = \aleph_1$. For, if $m' \geqq 2^{|\alpha|} \cdot \aleph_0$ (α not necessarily denumerable), then a simple argument shows that (9.1) is equivalent to

$$\alpha \rightarrow (\beta, \gamma)^1.$$

Thus, for denumerable ordinals, it suffices to discuss (9.1) in the special case $m = \aleph_1$. We show (Theorems 9.2, 9.3) that for given $\beta, \gamma < \omega_1$, there is $\alpha_1 < \omega_1$ such that

$$\alpha_1 \Rightarrow [\beta, \gamma]_{\aleph_1}, \tag{9.2}$$

$$\alpha \not\rightarrow [\beta, \gamma]^2_{\aleph_1} \qquad (\alpha < \alpha_1). \tag{9.3}$$

LEMMA 9.1. *If $\beta, \gamma < \omega_1$, then $\omega\gamma\beta \Rightarrow [\omega\beta, \omega\gamma]_{\aleph_1}$.*

PROOF. Let $\mathscr{I} = (F, M, S)$ be a $(\aleph_1, \omega\gamma\beta)$-system such that tp $F_\mu < \omega\beta$ $(\mu \in M)$. Since tp $S = \omega\gamma\beta$, we may write $S = \bigcup(\lambda < \beta) S_\lambda$ (tp), where tp $S_\lambda = \omega\gamma$ $(\lambda < \beta)$. If $\mu \in M$, then there is $\lambda(\mu) < \beta$ such that $F_\mu \cap S_{\lambda(\mu)}$ is finite. Hence, there is $M' \in [M]^{\aleph_1}$ such that $\lambda(\mu) = \lambda$ $(\mu \in M')$. Since S_λ contains only countably many finite sets, there is $M'' \in [M']^{\aleph_1}$ such that $F_\mu \cap S_\lambda = A$ $(\mu \in M'')$. The set $S_\lambda \sim A$ is $(\mathscr{I}, \aleph_1)$-free and has order type $\omega\gamma$.

LEMMA 9.2. (i) *If $0 < i < \omega$ and $\gamma < \omega_1$, then $\gamma + (i-1) \Rightarrow [i, \gamma]_{\aleph_1}$.* (ii) *If $r, s < \omega$ and $\gamma = \omega^\varrho < \omega_1$, then $\omega\gamma(r+s) \Rightarrow [\omega r, \omega\gamma(s+1)]_{\aleph_1}$.*

PROOF. (i) Let $\mathscr{I} = (F, M, S)$ be a $(\aleph_1, \gamma + i - 1)$-system such that tp $F_\mu < i$ $(\mu \in M)$. Then there is $M' \in [M]^{\aleph_1}$ such that $F_\mu = A$ $(\mu \in M')$. Then tp $(S \sim A) \geqq \gamma$ and $S \sim A$ is $(\mathscr{I}, \aleph_1)$-free. (ii) Let $S = \bigcup(\lambda < r+s) S_\lambda$ (tp); tp $S_\lambda = \omega\gamma$ $(\lambda < r+s)$. Then tp $S = \omega\gamma(r+s)$. Let $\mathscr{I} = (F, M, S)$ be a set system such that $|M| = \aleph_1$ and tp $F_\mu < \omega r$ $(\mu \in M)$. Then there are sets $M' \in [M]^{\aleph_1}$ and $N \subset [0, r+s)$ such that $|N| > s$ and $F_\mu \cap S_\lambda$ is finite for $\mu \in M', \lambda \in N$. Since $T = \bigcup(\lambda \in N) S_\lambda$ has only countably many finite sets, there is $M'' \in [M']^{\aleph_1}$ such that $F_\mu \cap T = B$ $(\mu \in M'')$. The set $T \sim B$ is $(\mathscr{I}, \aleph_1)$-free and has type $\geqq \omega\gamma(s+1)$.

The negative result Theorem 9.1 is given in more general terms than we immediately require in this section.

($*$) THEOREM 9. 1. *Let* $\alpha \geqq 0$; $0 < \beta < \omega_{\alpha+2}$; $0 < \gamma_0 \leqq \gamma_1 \leqq \ldots \leqq \hat{\gamma}_{\omega_\alpha} < \omega_{\alpha+2}$; $\gamma = \gamma_0 + \gamma_1 + \ldots + \hat{\gamma}_{\omega_\alpha}$. *Then* $\gamma\beta \nrightarrow [\omega_\alpha\beta + 1, \gamma]^2_{\aleph_{\alpha+1}}$.

PROOF. Let $S = \bigcup(\lambda < \omega_\alpha\beta) S_\lambda$ (tp); tp $S_\lambda = \gamma_{\pi(\lambda)}$ where $\pi(\lambda) < \omega_\alpha$ and $\lambda = \omega_\alpha \varrho(\lambda) + \pi(\lambda)$ $(\lambda < \omega_\alpha\beta)$. Then tp $S = \gamma\beta$. Now consider all sets $B \subset S$ such that tp $B = \omega_\alpha$ and $|B \cap S_\lambda| \leqq 1$ $(\lambda < \omega_\alpha\beta)$. Since $1 \leqq |S_\lambda| \leqq \aleph_{\alpha+1}$ $(\lambda < \omega_\alpha\beta)$ and $\aleph_{\alpha+1}^{\aleph_\alpha} = \aleph_{\alpha+1}$ by ($*$), it follows that there are $\aleph_{\alpha+1}$ such sets B, say $B_0, B_1, \ldots, \hat{B}_{\omega_{\alpha+1}}$. Let $M = [0, \omega_{\alpha+1})$. We will construct a set-system $\mathscr{J} = (F, M, S)$ such that

$$F_\mu \cap B_\nu \neq 0 \qquad (\nu < \mu < \omega_{\alpha+1}), \tag{9. 4}$$

$$|F_\mu \cap S_\lambda| \leqq 1 \qquad (\mu < \omega_{\alpha+1};\ \lambda < \omega_\alpha\beta). \tag{9. 5}$$

Let $\mu < \omega_{\alpha+1}$. Then $\{B_0, \ldots, \hat{B}_\mu\} = \{C_0, \ldots, \hat{C}_\theta\}_{\neq}$, where $\theta \leqq \omega_\alpha$. Since each C is a B, there is $\{\lambda_0, \ldots, \hat{\lambda}_\theta\}_{\neq} \subset [0, \omega_\alpha\beta)$ such that $C_\varrho \cap S_{\lambda_\varrho} \neq 0$ $(\varrho < \theta)$. Put $F_\mu = \bigcup(\varrho < \theta) C_\varrho \cap S_{\lambda_\varrho}$. Clearly, F_μ has a non-empty intersection with each set C_ϱ $(\varrho < \theta)$. Therefore, (9. 4) holds. Also, (9. 5) holds since $\lambda_0, \ldots, \hat{\lambda}_\theta$ are distinct.

If S' is a complete subgraph of $\mathscr{G}(\mathscr{J})$, then $|S' \cap S_\lambda| \leqq 1$ $(\lambda < \omega_\alpha\beta)$. Therefore, tp $S' < \omega_\alpha\beta + 1$. If $S'' \subset S$ has type $\geqq \gamma$, then there is $\pi < \omega_{\alpha+1}$ such that $B_\pi \subset S''$. Therefore, $F_\mu \cap S'' \neq 0$ $(\pi < \mu < \omega_{\alpha+1})$, i. e. S'' is not $(\mathscr{J}, \aleph_{\alpha+1})$-free. This proves the Theorem.

Theorems 9. 2, 9. 3 show how to find the ordinal α_1 such that (9. 2) and (9. 3) hold for given $\beta, \gamma < \omega_1$. Theorem 9. 2 deals with the trivial case of finite γ. In this case (9. 3) can be replaced by the stronger relation (9. 6) — we omit the proof.

THEOREM 9. 2. *If* $\beta < \omega_1$ *and* $0 < \gamma < \omega$, *then*

$$\alpha_1 \rightarrow [\beta, \gamma]_{\aleph_1},$$

$$\alpha \nrightarrow [\beta, \gamma]^1_{\aleph_1} \qquad (\alpha < \alpha_1), \tag{9. 6}$$

where

$$\alpha_1 = \begin{cases} \beta & \text{if } \beta = \beta^-, \\ \beta_1 + \gamma & \text{if } \beta = \beta_1 + 1. \end{cases}$$

THEOREM 9. 3. *Let* $b, i, j, k < \omega$; $\beta = \omega^2\beta_0 + \omega b + i < \omega_1$; $\gamma = \gamma_0 + \ldots + \gamma_k + j < \omega_1$; $\gamma_\lambda = \omega^{\varrho_\lambda}$ $(\lambda \leqq k)$; $\varrho_0 \geqq \ldots \geqq \varrho_k > 0$. *Then* (9. 2) *and* (9. 3) *both hold if* α_1 *is given by:*

$$\alpha_1 = \begin{cases} \gamma_0\omega\beta_0 & \text{if } i = b = 0, \\ \gamma_0(\omega\beta_0 + b - 1) + \gamma & \text{if } i = j = 0,\ b > 0, \\ \gamma_0(\omega\beta_0 + b - 1) + \gamma + \omega & \text{if } i = 0,\ j, b > 0; \\ \gamma_0(\omega\beta_0 + b) + \gamma + (i - 1) & \text{if } i \neq 0. \end{cases}$$

PROOF. *Case* 1. $i = b = 0$. By Lemma 9. 1,

$$\gamma_0\omega\beta_0 \rightarrow [\omega^2\beta_0, \gamma_0(k+1)]_{\aleph_1}. \tag{9. 7}$$

This implies that (9. 2) holds with $\alpha_1 = \gamma_0\omega\beta_0$.

Let $\alpha < \gamma_0\omega\beta_0$. Then there is $\beta_1 < \beta_0$ and $r < \omega$ such that $\alpha \leqq \gamma_0(\omega\beta_1 + r)$. By Theorem 9. 1,

$$\gamma_0\omega\beta_1 \nrightarrow [\omega^2\beta_1 + 1, \gamma_0]^2_{\aleph_1},$$

$$\gamma_0 r \nrightarrow [\omega r + 1, \gamma_0]^2_{\aleph_1}.$$

Therefore, by Theorem 7. 6, $\alpha \nrightarrow [\omega^2\beta_1+\omega r+1, \gamma_0]^2_{\aleph_1}$. This implies that (9. 3) also holds with $\alpha_1=\gamma_0\omega\beta_0$.

Case 2. $i+b\neq 0$. There is $c\leqq k$ such that $\gamma_0=\ldots=\gamma_c>\gamma_{c+1}$. Put $\gamma'=\gamma_{c+1}+ +\ldots+\gamma_k$

$$\delta=\begin{cases}\gamma' & \text{if} \quad i=j=0;\\ \gamma'+\omega & \text{if} \quad i=0<j;\\ \gamma'+(i+j-1) & \text{if} \quad i\neq 0.\end{cases}$$

Also, we put

$$b'=\begin{cases} b & \text{if} \quad i\neq 0\\ b-1 & \text{if} \quad i=0\end{cases}; \qquad \pi=\begin{cases} i & \text{if} \quad i\neq 0\\ \omega & \text{if} \quad i=0.\end{cases}$$

With these definitions for γ', δ and π we have, by Lemma 9. 1 and Lemma 9. 2 (i), that

$$(9.\,8) \qquad \delta\Rightarrow[\pi, \gamma'+j]_{\aleph_1}.$$

Let $S=S_0\cup S_1\cup S_2$ (tp); tp $S_0=\gamma_0\omega\beta_0$; tp $S_1=\gamma_0(b'+c)$; tp $S_2=\delta$. Then, by definition of b', c, δ we have tp $S=\alpha_1$, where α_1 has the value given in the statement of the Theorem for the case $i+b\neq 0$. Let $\mathscr{J}=(F, M, S)$ be a $(\aleph_1, \alpha_1)$-system such that tp $F_\mu<\beta$ $(\mu\in M)$. We will assume that S does not contain a $(\mathscr{J}, \aleph_1)$-free subset of type γ and deduce a contradiction.

Since $\gamma_0(k+1)>\gamma$, it follows from (9. 7) and our assumption that there is $M'\in[M]^{\aleph_1}$ such that tp $F_\mu\cap S_0\geqq\omega^2\beta$ $(\mu\in M')$. Therefore,

$$(9.\,9) \qquad \text{tp } F_\mu\cap(S_1\cup S_2)<\omega b+i \qquad (\mu\in M').$$

By Lemma (9. 2) (ii), we have

$$\gamma_0(b'+c)\Rightarrow[\omega b', \gamma_0(c+1)]_{\aleph_1}; \quad \gamma_0(b'+c)\Rightarrow[\omega(b'+1), \gamma_0 c]_{\aleph_1}.$$

Therefore, since $\omega(b'+1)\geqq\omega b+i$ and $\gamma_0(c+1)>\gamma$, there is $M''\in[M']^{\aleph_1}$ such that

$$(9.\,10) \qquad \text{tp } F_\mu\cap S_1\geqq\omega b' \qquad (\mu\in M''),$$

$$(9.\,11) \qquad \text{tp}\left(S_1\sim\bigcup(\mu\in M'')F_\mu\right)\geqq\gamma_0 c.$$

(9. 9) and (9. 10) imply that tp $F_\mu\cap S_2<\pi$ $(\mu\in M'')$. Also, (9. 11) implies that there is no $(\mathscr{J}'', \aleph_1)$-free subset of S_2 of type $\gamma'+j$, where $\mathscr{J}''=(F, M'', S)$. This contradicts (9. 8). Therefore, (9. 2) holds if α_1 has the value stated in the Theorem.

It remains to show that (9. 3) also holds in the case $i+b\neq 0$. We consider the various sub-cases separately.

Case 2a. $i=j=0<b$. If $\alpha<\gamma_0(\omega\beta_0+b-1)+\gamma$, then there is $\gamma^*<\gamma$ such that $\alpha\leqq\gamma_0(\omega\beta_0+b-1)+\gamma^*$. By Theorem 9. 1,

$$(9.\,12) \qquad \gamma_0(\omega\beta_0+b-1)\nrightarrow[\omega^2\beta_0+\omega(b-1)+1, \gamma_0]^2_{\aleph_1}.$$

Since $\gamma^*\nrightarrow[1, \gamma]^2_{\aleph_1}$, it follows from Theorem 7. 6 that

$$\gamma_0(\omega\beta_0+b-1)+\gamma^*\nrightarrow[\omega^2\beta_0+\omega(b-1)+1, \gamma]^2_{\aleph_1}.$$

Hence (9. 3) holds with $\alpha_1=\gamma_0(\omega\beta_0+b-1)+\gamma$.

Case 2b. $i=0<j, b$. Let $\alpha<\alpha_1=\gamma_0(\omega\beta_0+b-1)+\gamma+\omega$. Then there is $\theta<\omega$ such that $\alpha\leqq\gamma_0(\omega\beta_0+b-1)+\gamma+\theta$. From Theorem 7. 6, (9. 12) and the trivial relations

$$\gamma_0+\ldots+\gamma_k\nrightarrow[1, \gamma_0+\ldots+\gamma_k+1]^2_{\aleph_1}$$

and $j+\theta\nrightarrow[\omega, 1]^2_{\aleph_1}$, we deduce that

$$\gamma_0(\omega\beta_0+b-1)+\gamma+\theta\nrightarrow[\omega^2\beta_0+\omega b, \gamma_0+\ldots+\gamma_k+1]^2_{\aleph_1}.$$

This implies (9. 3).

Case 2c. $i\neq0$. Let $\alpha<\alpha_1=\gamma_0(\omega\beta_0+b)+\gamma+(i-1)$. Then there is $\varphi<\gamma+(i-1)$ such that $\alpha\leqq\gamma_0(\omega\beta_0+b)+\varphi$. By Theorem 9. 1, $\gamma_0(\omega\beta_0+b)\nrightarrow[\omega^2\beta_0+\omega b+1, \gamma_0]^2_{\aleph_1}$. Therefore, by Theorem 7.6 and the trivial relation $\varphi\nrightarrow[i, \gamma]^2_{\aleph_1}$, $\alpha\nrightarrow[\beta, \gamma]^2_{\aleph_1}$. This completes the proof of Theorem 9. 3.

§ 10. Non-denumerable ordinals. Although some of our results are expressed in more general terms, the discussion in this section is mainly directed towards the relation

$$\alpha\rightarrow[\beta, \gamma]^2_{\aleph_1} \tag{10. 1}$$

in the case when α is an *indecomposable ordinal* of cardinal $\aleph_1$. Even in this restricted form, our discussion is incomplete. We are able to decide the truth or otherwise of (10. 1) if α is indecomposable and $\alpha\leqq\omega_1^{\omega+1}$. The first relation of this kind which we cannot decide is

PROBLEM 4. (?) $\omega_1^{\omega+1}\omega\rightarrow[\omega_1^{\omega+1}, \omega_1\omega]^2_{\aleph_1}$.

The section is concluded with a discussion of relations of the form

$$\alpha\rightarrow[\beta, \gamma]^2_{\aleph_0}$$

when $|\alpha|=\aleph_1$. Essentially we consider only the case when α is a power of ω_1 and, even for this case, our analysis is incomplete. In Theorem 10. 9 we show that

$$\omega_1^\lambda\rightarrow[\omega_1^\lambda, \omega_1^\omega]^2_{\aleph_0} \quad \textit{if} \quad \omega\leqq\lambda<\omega_2$$

and this result is best possible in the case cf $(\lambda)=\omega$ by Theorem 10. 10. In Theorem 10. 15 we establish the stronger relation

$$\omega_1^\lambda\rightarrow[\omega_1^\lambda, \gamma]^2_{\aleph_0} \qquad (\gamma<\omega_1^{\omega+1})$$

provided cf $(\lambda)\neq\omega$ and $\omega<\lambda<\omega_2$. Also, in Theorem 10. 14, we prove the negative result that

$$\alpha\nrightarrow[\omega_1^\omega, \omega_1^{\omega+2}]^2_{\aleph_0}.$$

However, we do not know if

$$(?) \quad \omega_1^\lambda\rightarrow[\omega_1^\omega, \omega_1^{\omega+1}]^2_{\aleph_0}$$

for any $\lambda<\omega_2$ (Problem 6).

All the negative results of the form (10. 1) in this section derive from Theorem 9. 1 and the next three theorems.

(✳) THEOREM 10. 1. *Let* $\nu\geqq0$; $0<\alpha\beta<\omega_{\nu+2}$; $\beta<\omega_\nu(\gamma+1)$. *Then* $\alpha\beta\nrightarrow[\omega_\nu\gamma+1, \alpha\omega_\nu]^2_{\aleph_{\nu+1}}$.

PROOF. There is $\delta < \omega_\nu$ such that $\beta \leqq \omega_\nu \gamma + \delta$. By Theorem 9.1, $\alpha\omega_\nu\gamma \nrightarrow [\omega_\nu\gamma + 1, \alpha\omega_\nu]^2_{\aleph_{\nu+1}}$; also, $\alpha\delta \nrightarrow [1, \alpha\omega_\nu]^2_{\aleph_{\nu+1}}$. Therefore, by Theorem 7.6, $\alpha(\omega_\nu\gamma + \delta) \nrightarrow [\omega_\nu\gamma + 1, \alpha\omega_\nu]^2_{\aleph_{\nu+1}}$. This implies the Theorem.

THEOREM 10.2. *Let* $\nu \geqq 0$; $0 < \alpha\beta < \omega_{\nu+2}$. *If* $\gamma > \Sigma(\varrho < \beta)\lambda_\varrho$ *whenever* $\lambda_\varrho < \omega_{\nu+1}$ $(\varrho < \beta)$, then $\alpha\beta \nrightarrow [\alpha\omega_{\nu+1}, \gamma]^2_{\aleph_{\nu+1}}$.

PROOF. If $\beta < \omega_{\nu+1}$ the result is obvious. The result is also obvious if $\alpha < \omega_{\nu+1}$ since, in this case, $\alpha\beta < \gamma$. We shall therefore assume that $\alpha, \beta \geqq \omega_{\nu+1}$.

Let tp $S = \alpha\beta$. Then $S = \cup(\varrho < \beta) S_\varrho$ (tp), where tp $S_\varrho = \alpha$ $(\varrho < \beta)$. Let $M = [0, \omega_{\nu+1})$ and let φ be a $(1-1)$-mapping of M onto $[0, \beta)$. Put $T_\sigma = S_{\varphi(\sigma)} = \{x_{\sigma\tau} : \tau < \omega_{\nu+1}\}_{\neq}$ $(\sigma < \omega_{\nu+1})$. Consider the set-system $\mathscr{J} = (F, M, S)$, where

$$F_\mu = \{x_{\sigma\tau} : \sigma < \mu < \tau < \omega_{\nu+1}\} \qquad (\mu \in M).$$

If $S' \subset S$ and tp $S' = \alpha\omega_{\nu+1}$, then there is $M' \in [M]^{\aleph_{\nu+1}}$ such that $S' \cap T_\mu \neq 0$ $(\mu \in M')$. Let $x_{\sigma\tau} \in S'$. Then there is $\varrho \in M'$ and $\pi < \omega_{\nu+1}$ such that $\tau < \varrho$ and $x_{\varrho\pi} \in S'$. The elements $x_{\sigma\tau}$ and $x_{\varrho\pi}$ are not joined by an edge of the graph $\mathscr{G}(\mathscr{J})$. Hence $\mathscr{G}(\mathscr{J})$ contains no complete sub-graph of type $\alpha\omega_{\nu+1}$.

If $S'' \subset S$ and tp $S'' = \gamma$, then there is $\varrho < \beta$ such that $|S'' \cap S_\varrho| = \aleph_{\nu+1}$. If $\mu \in M$ and $\mu > \sigma = \varphi^{-1}(\varrho)$, then there is τ such that $\mu < \tau < \omega_{\nu+1}$ and $x_{\sigma\tau} \in S''$. Hence, $S'' \cap F_\mu \neq 0$ if $\mu > \sigma$. Thus, there is no $(\mathscr{J}, \aleph_{\nu+1})$-free set of type γ. This proves the Theorem.

(∗) THEOREM 10.3. *If* $\lambda < \omega_2$, *then* $\lambda \nrightarrow [\omega_1, \omega_1^\omega]^2_{\aleph_1}$.

PROOF. It has been proved in[6] [7] that

$$\lambda \nrightarrow (\omega_1, \omega_1^2, \ldots, \omega_1^{\hat{\omega}})^1.$$

Therefore, if tp $S = \lambda$, there are disjoint sets $S_\nu \subset S$ $(\nu < \omega)$ such that $S = \cup(\nu < \omega) S_\nu$ and tp $S_\nu < \omega_1^{\nu+1}$ $(\nu < \omega)$.

We assume that $\lambda \geqq \omega_1^\omega$. Then there are $\aleph_1$ sets, say $B_0, \ldots, \hat{B}_{\omega_1}$ such that $|B_\varrho| = \aleph_0$ and $|B_\varrho \cap S_\nu| \leqq 1$ $(\varrho < \omega_1;\ \nu < \omega)$. Let $M = [0, \omega_1)$. Using the same construction as in the proof of Theorem 9.1, it is easy to see that there is a set-system $\mathscr{J} = (F, M, S)$ such that $|F_\mu \cap S_\nu| \leqq 1$ $(\mu \in M;\ \nu < \omega)$ and $F_\mu \cap B_\varrho \neq 0$ $(\varrho < \mu < \omega_1)$.

If S' is a complete sub-graph of $\mathscr{G}(\mathscr{J})$, then $|S' \cap S_\nu| \leqq 1$ $(\nu < \omega)$. Therefore, S is countable and tp $S' < \omega_1$. If $S'' \subset S$ and tp $S'' = \omega_1^\omega$, then S'' intersects infinitely many of the sets S_ν $(\nu < \omega)$. Hence $B_\varrho \subset S''$ for some $\varrho < \omega_1$ and $F_\mu \cap S'' \neq 0$ $(\varrho < \mu < \omega_1)$. Thus, S'' is not $(\mathscr{J}, \aleph_1)$-free. This proves the Theorem.

The next Theorem generalizes Theorem 6.2 in one direction.

(∗) THEOREM 10.4. *Let* $m \geqq \aleph_0$; $\alpha < \omega(m) = \nu$; $\beta = \Sigma(\varrho < \nu)\beta_\varrho$; $0 < \beta_0 \leqq \beta_1 \leqq \ldots \leqq \hat{\beta}_\nu$. *Then* $\beta \rightarrow [\alpha, \beta]_m$.

PROOF. Let $\mathscr{J} = (F, M, S)$ be a (m, β)-system such that tp $F_\mu < \alpha$ $(\mu \in M)$. Then $S = \cup(\varrho < \nu) S_\varrho$ (tp), where tp $S_\varrho = \beta_\varrho$ $(\varrho < \nu)$. Let $F'_\mu = \{\varrho : F_\mu \cap S_\varrho \neq 0\}$ $(\mu \in M)$. Then tp $F'_\mu < \alpha$ $(\mu \in M)$. By Theorem 6.2, $\nu \rightarrow [\alpha, \nu]_m$. Hence, there are sets M', N' such that $|M'| = |N'| = m$, $M' \subset M$, $N' \subset [0, \nu)$ and $\varrho \notin F'_\mu$ $(\varrho \in N';\ \mu \in M')$. The set $\cup(\varrho \in N') S_\varrho$ is $(\mathscr{J}, m)$-free and has type β.

[6] [7], Theorem 5.

(*) THEOREM 10. 5. *If* $\aleph'_\mu > \aleph_\nu$; $\alpha, \theta \geqq 0$; $|\beta| < \aleph'_\mu$, *then*

$$\theta\omega_\nu\alpha\beta \Rightarrow [\omega_\nu\beta, \theta\omega_\nu\alpha]_{\aleph_\mu}.$$

PROOF. By Theorem 4. 6, it is enough to prove that

$$\omega_\nu\alpha \Rightarrow [\omega_\nu, \omega_\nu\alpha]_{\aleph_\mu}. \tag{10. 3}$$

Let $\mathscr{J} = (F, M, S)$ be a $(\aleph_\mu, \omega_\nu\alpha)$-system such that tp $F_\lambda < \omega_\nu$ $(\lambda \in M)$. We may assume that S is the set $\{(\varrho, \sigma) : \varrho < \omega_\nu;\ \sigma < \alpha\}$ ordered anti-lexicographically. For $\lambda \in M$, put

$$F'_\lambda = \{\varrho : \varrho < \omega_\nu, (\varrho, \sigma) \in F_\lambda \quad \textit{for some} \quad \sigma < \alpha\}.$$

By (6. 2), we have $\omega_\nu \Rightarrow [\omega_\nu, \omega_\nu]_{\aleph_\mu}$. Therefore, since tp $F'_\lambda < \omega_\nu$ $(\lambda \in M)$, there are $M' \in [M]^{\aleph_0}$ and $A \subset [0, \omega_\nu)$ such that tp $A = \omega_\nu$ and $F'_\mu \cap A = 0$ $(\mu \in M')$. The set $\{(\varrho, \sigma) : \varrho \in A;\ \sigma < \alpha\}$ is $(\mathscr{J}, \aleph_\mu)$-free and has type $\omega_\nu\alpha$. This proves (10. 3) and hence Theorem 10. 5.

Although not required for the discussion of (10. 1), we include here the following result.

THEOREM 10. 6. *If* $\nu \geqq 0$ *and* $\alpha < \omega_{\nu+1}$, *then*

$$\omega_{\nu+2}\alpha \Rightarrow [\omega^\omega_{\nu+1}, \omega_{\nu+2}\alpha]_{\aleph_{\nu+2}}.$$

PROOF. Let $|M| = m = \aleph_{\nu+2}$ and let $S = \{(\lambda, \varrho) : \lambda < \omega_{\nu+2};\ \varrho < \alpha\}$ be ordered anti-lexicographically. Let $\mathscr{J} = (F, M, S)$ be a $(m, \omega_{\nu+2}\alpha)$-system such that tp $F_\mu < \omega^\omega_{\nu+1}$ $(\mu \in M)$. Since $m = m' > \aleph_0$, it follows that there is $r < \omega$ and $M' \in [M]^m$ such that tp $F_\mu < \omega^r_{\nu+1}$ $(\mu \in M')$.

Suppose that $\mathscr{S}$: *whenever* $S' \subset S$ *and* tp $S' = \omega_{\nu+2}\alpha$, *then there is* $X \in [S']^{<m}$ *such that* $|\{\mu \in M' : F_\mu \cap X = 0\}| < m$. Then we define ordinals $\sigma_\theta, \tau_\theta < \omega_{\nu+2}$ and sets $X_\theta \subset S$ for $\theta < \omega^r_{\nu+1}$ by induction in the following way. Let $\theta < \omega^r_{\nu+1}$ and suppose we have already defined $\sigma_\varphi, \tau_\varphi < \omega_{\nu+2}$ and $X_\varphi \subset S$ for $\varphi < \theta$. Then we can choose $\sigma_\theta < \omega_{\nu+2}$ such that $\sigma_\varphi, \tau_\varphi < \sigma_\theta$ $(\varphi < \theta)$. Put $T_\theta = \{(\lambda, \varrho) : \sigma_\theta < \lambda < \omega_{\nu+2};\ \varrho < \alpha\}$. Then tp $T_\theta = \omega_{\nu+2}\alpha$. Therefore, by $\mathscr{S}$, there is $X_\theta \in [T_\theta]^{<m}$ such that

$$|\{\mu \in M' : F_\mu \cap X_\theta = 0\}| < m. \tag{10. 4}$$

Since $|X_\theta| < m'$, there is $\tau_\theta < \omega_{\nu+2}$ such that

$$X_\theta \subset \{(\lambda, \varrho) : \sigma_\theta < \lambda < \tau_\theta;\ \varrho < \alpha\}. \tag{10. 5}$$

It follows by induction that there are $\sigma_\theta, \tau_\theta < \omega_{\nu+2}$ and $X_\theta \subset S$ for $\theta < \omega^r_{\nu+1}$ such that (10. 4) and (10. 5) hold. Moreover, by the construction we have

$$\sigma_\varphi < \tau_\varphi < \sigma_\theta < \tau_\theta \qquad (\varphi < \theta < \omega^r_{\nu+1}). \tag{10. 6}$$

By (10. 4), there is some $\mu \in M'$ such that $F_\mu \cap X_\theta \neq 0$ for all $\theta < \omega^r_{\nu+1}$. Therefore, by (10. 5), there is λ_θ and ϱ_θ such that $\sigma_\theta < \lambda_\theta < \tau_\theta$; $\varrho_\theta < \alpha$ and $(\lambda_\theta, \varrho_\theta) \in F_\mu$ $(\theta < \omega^r_{\nu+1})$. By (10. 6), we see that tp $\{\lambda_\theta : \theta < \omega^r_{\nu+1}\} = \omega^r_{\nu+1}$.

It has been proved in [7] that, for $\alpha < \omega_{\nu+1}$,

$$\omega^r_{\nu+1} \to (\omega^r_{\nu+1})^1_\alpha.$$

It follows from this partition relation that there is $\delta < \alpha$ such that tp $\{\lambda_\theta : \theta < \omega^r_{\nu+1};\ \varrho_\theta = \delta\} = \omega^r_{\nu+1}$. Since $\{(\lambda_\theta, \delta) : \theta < \omega_{\nu+1};\ \varrho_\theta = \delta\} \subset F_\mu$, it follows that tp $F_\mu \geqq \omega^r_{\nu+1}$. This is a contradiction.

Hence $\mathscr{S}$ is false and there is a set $S'\subset S$ such that tp $S'=\omega_{\nu+2}\alpha$ and

$$(10.7)\qquad |\{\mu\in M':\ F_\mu\cap X=0\}|=m \qquad (X\in[S']^{<m}).$$

Let $A=\{\varrho<\alpha: |\{(\lambda,\varrho):\lambda<\omega_{\nu+2}\}\cap S'|=m\}$. Then tp $A=\alpha$. Consider any sequence[7] $\varrho_0,\dots,\hat{\varrho}_{\omega_{\nu+2}}$ which repeats each ordinal, $\varrho\in A$ $\aleph_{\nu+2}$ times, i. e. such that

$$(10.8)\qquad |\{\varrho_\gamma:\ \gamma<\omega_{\nu+2};\ \varrho_\gamma=\varrho\}|=m \qquad (\varrho\in A).$$

Let $\varepsilon<\omega_{\nu+2}$ and suppose we have already defined $x_\gamma\in S'$ and $\mu_\gamma\in M'$ for $\gamma<\varepsilon$. Then we can choose

$$(10.9)\qquad x_\varepsilon\in S'\cap\{(\lambda,\varrho_\varepsilon):\lambda<\omega_{\nu+2}\}\sim\cup(\gamma<\varepsilon)F_{\mu_\gamma}\cup\{x_\gamma\}.$$

Also, by (10. 7), there is $\mu_\varepsilon\in M'\sim\{\mu_\gamma:\ \gamma<\varepsilon\}$ such that

$$F_{\mu_\varepsilon}\cap\{x_0,\dots,x_\varepsilon\}=0.$$

This defines by induction $X=\{x_\varepsilon:\varepsilon<\omega_{\nu+2}\}_{\neq}\subset S$ and $M''=\{\mu_\varepsilon:\varepsilon<\omega_{\nu+2}\}_{\neq}\subset M'$ such that $F_\mu\cap X=0$ $(\mu\in M'')$. Hence, X is $(\mathscr{J},m)$-free. In addition, if $\varrho\in A$, then by (10. 8) and (10. 9),

$$|X\cap\{(\lambda,\varrho):\lambda<\omega_{\nu+2}\}|=m.$$

Therefore, tp $X=\omega_{\nu+2}\alpha$. This proves Theorem 10. 6.

By the last Theorem we have, in particular,

$$(10.10)\qquad \omega_2\alpha\Rightarrow[\omega_1^\omega,\omega_2\alpha]_{\aleph_2} \qquad (\alpha<\omega_1).$$

This is not the best possible relation when $\alpha=1$ since, by Theorem 10. 4, $\omega_2\Rightarrow[\beta,\omega_2]_{\aleph_2}$ $(\beta<\omega_2)$. We do not know if (10. 10) is best possible in the case $\alpha=\omega$. Thus, we have

PROBLEM 5. (?) $\omega_2\omega\Rightarrow[\omega_1^\omega+1,\omega_2\omega]_{\aleph_2}$.

Incidentally, the condition $\alpha<\omega_{\nu+1}$ in Theorem 10. 6 cannot be relaxed. For, by Theorem 10. 1, we have $\omega_2\omega_1\nrightarrow[\omega_1+1,\omega_2\omega_1]^2_{\aleph_2}$.

We require the following two lemmas.

LEMMA 10. 1. *Let* $s,n<\omega$; $\beta=\omega^\varrho$; $n\leqq\beta<\omega_2$. *Let* $S\supset B=\cup(\lambda<\beta)B_\lambda$ (tp); tp $B_\lambda=\omega_1^{1+s}$ $(\lambda<\beta)$. *If* $\mathscr{J}=(F,M,S)$ *is a set system such that* $|M|=\aleph_1$ *and*[8]

$$(10.11)\qquad B\subset Q(\mathscr{J}),$$

then either (i) *there is a set* $X\subset S$ *which is a complete subgraph of* $\mathscr{G}(\mathscr{J})$ *and* $|X\cap B_\lambda|=\aleph_1$ $(\lambda<\beta)$, *or* (ii) *there is a* $(\mathscr{J},\aleph_1)$*-free set* $Y\subset B$ *such that* tp $Y=\omega_1^{1+s}n$.

PROOF. We shall use induction with respect to n. For $n=0$ (ii) holds trivially. We therefore assume that $n>0$ and that the result is true with $n-1$ in place of n. We will assume that (i) is false and deduce (ii).

We will show that there is some $\lambda<\beta$ and a set $M'\in[M]^{\aleph_1}$ such that

$$(10.12)\qquad \mathrm{tp}\,(B_\lambda\sim\cup(\mu\in M')F_\mu)=\omega_1^{1+s}.$$

[7] We may assume $\alpha\neq0$.

[8] We remind the reader that $Q(\mathscr{J})=\{x\in S:|\{\mu\in M: x\in F_\mu\}|=|M|\}$.

Suppose this is not the case. Then, by the hypothesis (10. 11),

$$\text{tp}\,(B_\lambda \sim V(x)) < \omega_1^{1+s} \qquad (\lambda < \beta;\, x \in B),$$

where $V(x) = \bigcup(\mu \in M;\, x \in F_\mu) F_\mu \sim \{x\}$ is the set of points of S joined to x by edges of the graph $\mathscr{G}(\mathscr{J})$. Let $\lambda_0, \ldots, \hat{\lambda}_{\omega_1}$ be a sequence of ordinals such that $\lambda_\theta < \beta$ $(\theta < \omega_1)$ and

$$|\{\theta < \omega_1 : \lambda_\theta = \lambda\}| = \aleph_1 \qquad (\lambda < \beta).$$

Let $\theta < \omega_1$, and suppose we have already defined $x_\varrho \in B$ for $\varrho < \theta$. As we have already remarked,

(10. 13) $$\omega_1^{1+s} \to (\omega_1^{1+s})^1_\theta.$$

Therefore,

$$\text{tp} \bigcup(\varrho < \theta)(B_\lambda \sim V(x_\varrho)) < \omega_1^{1+s} \qquad (\lambda < \beta).$$

Hence, there is $x_\theta \in B_{\lambda_\theta} \cap (\varrho < \theta) V(x_\varrho)$. The set $X = \{x_\theta : \theta < \omega_1\}$ is a complete sub-graph of $\mathscr{G}(\mathscr{J})$ and $|X \cap B_\lambda| = \aleph_1$ $(\lambda < \beta)$. This contradicts our assumption that (i) is false. Hence, there is some $\lambda < \beta$ and $M' \in [M]^{\aleph_1}$ such that (10. 12) holds.

If $n = 1$, then (ii) holds with $Y = B_\lambda \sim \bigcup(\mu \in M') F_\mu = Y_0$. Suppose $n > 1$. Then the hypothesis $\beta \geq n$ implies that $\beta \geq \omega$. Hence, tp $[\lambda, \beta) = \beta$, since β is indecomposable. Let $\mathscr{J}' = (F, M', S)$ and let $B' = \bigcup(\lambda < \varrho < \beta) B_\varrho$. Then it follows from the induction hypothesis and the assumption that (i) is false, that there is a $(\mathscr{J}', \aleph_1)$-free set $Y' \subset B'$ such that tp $Y' = \omega_1^{1+s}(n-1)$. Then the set $Y = Y_0 \cup Y'$ is $(\mathscr{J}, \aleph_1)$-free and has type $\omega_1^{1+s} n$. This proves the Lemma.

LEMMA 10. 2. *Let* $r < \omega;\, 0 < \beta < \omega_2;\, B = \bigcup(\lambda < \beta) B_\lambda$ (tp); $B \subset S$; tp $B_\lambda = \omega_1^{1+r}$ $(\lambda < \beta)$. *If* $|M| = \aleph_1$ *and* $\mathscr{J} = (F, M, S)$ *is a set-system such that* $B \cap Q(\mathscr{J}) = 0$ *and* tp $B_\lambda \cap F_\mu < \omega_1^{1+r}$ $(\lambda < \beta;\, \mu \in M)$, *then there is a* $(\mathscr{J}, \aleph_1)$*-free set* $X \subset S$ *such that* $|X \cap B_\lambda| = \aleph_1$ $(\lambda < \beta)$.

PROOF. Let $\lambda_0, \ldots, \hat{\lambda}_{\omega_1}$ be a sequence of ordinals such that $\lambda_\theta < \beta$ $(\theta < \omega_1)$ and such that $|\{\theta < \omega_1 : \lambda_\theta = \lambda\}| = \aleph_1$ $(\lambda < \beta)$. Let $\theta < \omega_1$ and suppose we have already chosen $x_\varrho \in B$ and $\mu_\varrho \in M$ for $\varrho < \theta$. By (10. 13) and the hypothesis that tp $B_\lambda \cap F_\mu < \omega_1^{1+r}$ $(\lambda < \beta;\, \mu \in M)$, it follows that there is

$$x_\theta \in B_{\lambda_\theta} \sim \bigcup(\varrho < \theta) F_{\mu_\varrho} \cup \{x_\varrho\}.$$

Also, since each $x \in B$ is a member of only countably many of the sets F_μ, there is $\mu_\theta \in M \sim \{\mu_0, \ldots, \hat{\mu}_\theta\}$ such that

$$F_{\mu_\theta} \cap \{x_0, \ldots, x_\theta\} = 0.$$

Put $X = \{x_0, \ldots, \hat{x}_{\omega_1}\}$ and $M' = \{\mu_0, \ldots, \hat{\mu}_{\omega_1}\}$. Then $X \cap F_\mu = 0$ $(\mu \in M')$ and $|X \cap B_\lambda| = \aleph_1$ $(\lambda < \beta)$. This proves Lemma 10. 2.

THEOREM 10. 7. *Let* $n, r, s < \omega;\, \beta = \omega^\varrho;\, n \leq \beta < \omega_2$. *If* $\beta \Rightarrow [\alpha, 1]_{\aleph_1}$, *then*

$$\omega_1^{r+s+1}\beta \to [\omega_1^{r+1}\alpha,\, \omega_1^{s+1} n]^2_{\aleph_1}.$$

PROOF. Let $\mathscr{J} = (F, M, S)$ be a $(\aleph_1, \omega_1^{r+S+1}\beta)$-system. We will assume that there is no complete sub-graph of $\mathscr{G}(\mathscr{J})$ of type $\omega_1^{r+1}\alpha$ and deduce that S contains a $(\mathscr{J}, \aleph_1)$-free subset of type $\omega_1^{s+1} n$.

Suppose that tp $Q(\mathscr{J})=\omega_1^{r+s+1}\beta$. Then $Q(\mathscr{J})=\cup(\lambda<\beta')B_\lambda(\text{tp})$, where $\beta'=\omega_1^r\beta$ and tp $B_\lambda=\omega_1^{s+1}$ $(\lambda<\beta')$. By the hypothesis β' is indecomposable and $n\leqq\beta'<\omega_2$. By the assumption that $\mathscr{G}(\mathscr{J})$ does not contain a complete sub-graph of type $\omega_1^{r+1}\alpha$ $(\leqq\omega_1\beta')$, it follows from Lemma 10. 1 that there is a $(\mathscr{J}, \aleph_1)$-free subset of $Q(\mathscr{J})$ of order type $\omega_1^{s+1}n$.

Hence, we may assume that tp $Q(\mathscr{J})<$ tp S. Since tp S is indecomposable we have (e. g. see [7]) that

$$\text{tp } S\rightarrow(\text{tp } S, \text{tp } S)^1.$$

Therefore, $B=S\sim Q(\mathscr{J})$ has type $\omega_1^{r+s+1}\beta$. Therefore, $B=\cup(\lambda<\beta)B'_\lambda$ (tp), where tp $B'_\lambda=\omega_1^{r+s+1}$ $(\lambda<\beta)$. For $\mu\in M$, put $F'_\mu=\{\lambda<\beta\colon \text{tp } B'_\lambda\cap F_\mu\geqq\omega_1^{r+1}\}$. Since F_μ $(\mu\in M)$ is a complete sub-graph of $\mathscr{G}(\mathscr{J})$, it follows that tp $F'_\mu<\alpha$ $(\mu\in M)$. The hypothesis $\beta\Rightarrow[\alpha, 1]_{\aleph_1}$ implies that there is $M'\in[M]^{\aleph_1}$ and $\pi<\beta$ such that $\pi\notin F'_\mu$ $(\mu\in M')$, i. e.

$$\text{tp } F_\mu\cap B'_\pi<\omega_1^{r+1}; \qquad (\mu\in M'). \tag{10. 14}$$

We may write $B'_\pi=\cup(\varrho<\omega_1^s)T_\varrho$ (tp), where tp $T_\varrho=\omega_1^{r+1}$ $(\varrho<\omega_1^s)$. By (10. 14), tp $F_\mu\cap T_\varrho<\omega_1^{r+1}$ $(\varrho<\omega_1^s;\ \mu\in M')$. Therefore, by Lemma 10. 2, there is a set $Y\subset B'_\pi$ such that Y is $(\mathscr{J}, \aleph_1)$-free and $|Y\cap T_\varrho|=\aleph_1$ $(\varrho<\omega_1^s)$. Hence tp $Y\geqq\omega_1^{s+1}$. If $n=1$, there is nothing more to prove. If $n>1$, then $\beta\geqq\omega$ and the set $\cup(\pi<\lambda<\beta)B'_\lambda$ also has type $\omega_1^{r+s+1}\beta$. Now a simple induction argument on n achieves the result. We omit the details.

The results we have proved so far are sufficient to analyse (10. 1) for indecomposable ordinals $\alpha<\omega_1^\omega$. The case $\alpha<\omega_1$ has already been dealt with in § 9. The case $\omega_1\leqq\alpha<\omega_1^\omega$ is summarized by the formulae (10. 15)—(10. 24). In these formulae, $r+s=m<\omega$; $n<\omega$; $\lambda, \mu, \nu<\omega_1$.

$$\omega_1^{m+1}\Rightarrow[\lambda, \omega_1^{m+1}]_{\aleph_1}. \tag{10. 15}$$

$$\omega_1^{m+1}\Rightarrow[\omega_1, 1]_{\aleph_1}. \tag{10. 16}$$

$$\omega_1^{m+1}\rightarrow[\omega_1^{r+1}, \omega_1^{s+1}]^2_{\aleph_1}. \tag{10. 17}$$

$$\omega_1^{m+1}\nrightarrow[\omega_1^{r+1}, \omega_1^{s+1}+1]^2_{\aleph_1}. \tag{10. 18}$$

$$\omega_1^{m+1}\nrightarrow[\omega_1^{r+1}+1, \omega_1^{s}\omega]^2_{\aleph_1}. \tag{10. 19}$$

$$\omega_1^{m+1}\rightarrow[\omega_1^{r+1}\lambda, \omega_1^{s}n]^2_{\aleph_1}. \tag{10. 20}$$

$$\omega_1^{m+1}\omega^{1+\mu+\nu}\Rightarrow[\omega^{1+\nu}, \omega_1^{m+1}\omega^{1+\mu}]_{\aleph_1}. \tag{10. 21}$$

$$\omega_1^{m+1}\omega^{1+\mu+\nu}\nrightarrow[\omega^{1+\nu}+1, \omega_1^{m+1}\omega^{1+\mu}]^2_{\aleph_1}. \tag{10. 22}$$

$$\omega_1^{m+1}\omega^{1+\mu+\nu}\rightarrow[\omega_1^{r+1}\omega^{1+\mu+\nu}, \omega_1^{s+1}n]^2_{\aleph_1}. \tag{10. 23}$$

$$\omega_1^{m+1}\omega^{1+\mu+\nu}\nrightarrow[\omega_1^{r+1}\omega^{1+\mu+\nu}+1, \omega_1^{s}\omega]^2_{\aleph_1}. \tag{10. 24}$$

(10. 15) follows from Theorem 10. 4 and (10. 16) follows from (6. 15). (10. 17) follows from Theorem 10. 7 and the fact that $1\Rightarrow[1, 1]_{\aleph_1}$. (10. 18) is a consequence of Theorem 10. 2 since $\omega_1^{s+1}+1>\Sigma(\varrho<\omega_1^{s+1})\lambda_\varrho$ if $\lambda_\varrho<\omega_1$ $(\varrho<\omega_1^{s+1})$. (10. 19) is an immediate deduction from Theorem 10. 1. Since $\omega_1\Rightarrow[\lambda, 1]$ by (10. 15), Theorem 10. 7 implies (10. 20). (10. 21) follows from Theorem 10. 5. The negative relations

(10. 22) and (10. 24) both follow from Theorem 9. 1. Finally, (10. 23) follows from Theorem 10. 7 and the fact that $\omega^{1+\mu+\nu} \Rightarrow [\omega^{1+\mu+\nu}, 1]_{\aleph_1}$ (in fact, by Theorem 10. 5, $\omega^{1+\mu+\nu} \Rightarrow [\omega^{1+\mu+\nu}, \omega]_{\aleph_1}$).

The relations (10. 17), (10. 20) and (10. 23) refer only to 2-graphs and we cannot prove the corresponding results for 3-graphs. For example, we cannot prove either

$$(?) \qquad \omega_1\omega^2 \to [\omega_1\omega^2, \omega_1]^3_{\aleph_1}$$

or

$$(?) \qquad \omega_1^2 \to [\omega_1^2, \omega_1]^3_{\aleph_1}.$$

However, we can prove

(10. 25) $$\omega_1^{m+1} \to [\omega_1, \omega_1^{m+1}]^{<\aleph_0}_{\aleph_1}$$

and

(10. 26) $$\omega_1^{m+1}\omega \to [\omega_1\omega, \omega_1^{m+1}n]^{<\aleph_0}_{\aleph_1},$$

which respectively strengthen special cases of (10. 17) and (10. 23). We omit the details of (10. 25) and (10. 26) since the method used to establish these results is rather similar to that used in the proof of the next theorem.

THEOREM 10. 8. *If* $n < \omega$, *then* $\omega_1^\omega \to [\omega_1^\omega, \omega_1^n]^{<\aleph_0}_{\aleph_1}$.

PROOF. Let $\mathscr{F} = (F, M, S)$ be a $(\aleph_1, \omega_1^\omega)$-system. We will assume that there is no $(\mathscr{F}, \aleph_1)$-free subset of S type ω_1^n and deduce that there is a $(\mathscr{F}, <\aleph_0)$-complete subset of type ω_1^ω.

Since F_μ $(\mu \in M)$ is $(\mathscr{F}, <\aleph_0)$-complete, we may assume that tp $F_\mu < \omega_1^\omega$ $(\mu \in M)$. Hence, there is $r < \omega$ and $M' \in [M]^{\aleph_1}$ such that tp $F_\mu < \omega_1^r$ $(\mu \in M')$.

Suppose there is a sub-system $\mathscr{F}'' \subset \mathscr{F}' = (F, M', S)$ such that $|\mathscr{F}''| = \aleph_1$ and tp $(S \sim Q(\mathscr{F}'')) \geqq \omega_1^{r+n}$. Then, by lemma 10. 2, S contains a $(\mathscr{F}'', \aleph_1)$-free subset of type ω_1^n. This is a contradiction since such a set is also $(\mathscr{F}, \aleph_1)$-free. Therefore,

(10. 27) $$\text{tp}\,(S \sim Q(\mathscr{F}'')) < \omega_1^{r+n} \qquad (\mathscr{F}'' \subset \mathscr{F}';\ |\mathscr{F}''| = \aleph_1).$$

Since tp $S = \omega_1^\omega$, we may write $S = \bigcup(\lambda < \omega_1^\omega) S_\lambda$ (tp), where tp $S_\lambda = \omega_1^{r+n}$ $(\lambda < \omega_1^\omega)$. Let $[0, \omega_1^\omega) = \{\lambda_0, \dots, \hat{\lambda}_{\omega_1}\}_{\neq}$. If $x \in S$ we write $M'(x) = \{\mu \in M' : x \in F_\mu\}$. Also, if $X \subset S$, we put $M'(X) = \bigcap(x \in X) M'(x) = \{\mu \in M' : X \subset F_\mu\}$ and we write $\mathscr{F}_X = (F, M'(X), S)$.

Let $\theta < \omega_1$ and suppose we have already chosen $x_\varrho \in S$ for $\varrho < \theta$ in such a way that $|M'(X)| = \aleph_1$ for every finite set $X \subset \{x_0, \dots, \hat{x}_\theta\}$. Then, by (10. 27),

$$\text{tp}\,(S \sim Q(\mathscr{F}_X)) < \omega_1^{r+n} \qquad (|X| < \aleph_0;\ X \subset \{x_0, \dots, \hat{x}_\theta\}).$$

Since there are only countably many finite sets $X \subset \{x_0, \dots, \hat{x}_\theta\}$ and since $\omega_1^{r+n} \to (\omega_1^{r+n})^1_\omega$ it follows that we can choose

$$x_\theta \in S_{\lambda_\theta} \bigcap_X Q(\mathscr{F}_X),$$

where the intersection ranges over all such finite sets X. If X is any finite subset of $\{x_0, \dots, \hat{x}_\theta\}$ then $x_\theta \in Q(\mathscr{F}_X)$, i. e. $|\{\mu \in M'(X) : x_\theta \in F_\mu\}| = \aleph_1$ and $|M'(X \cup \{x_\theta\}| = \aleph_1$. This defines, by induction, a set $Y = \{x_0, \dots, \hat{x}_{\omega_1}\} \subset S$ such that

$$|M'(X)| = \aleph_1 \qquad (X \in [Y]^{<\aleph_0}).$$

Thus Y is $(\mathscr{J}, <\aleph_0)$-complete. Also, since $x_\theta \in S_{\lambda_\theta}$ $(\theta < \omega_1)$, we have tp $Y = \omega_1^\omega$. This proves the theorem.

The relation given in Theorem 10. 8 is clearly best possible since

(10. 28) $$\omega_1^\omega \nrightarrow [\omega+1, \omega_1^\omega]^2_{\aleph_1}$$

by Theorem 9. 1. In contrast to this we have by Theorem 10. 5

(10. 29) $$\omega_1^\omega \Rightarrow [\omega, \omega_1^\omega]_{\aleph_1}.$$

In addition, if $\mu, \nu < \omega_1$, n is finite and $n \leqq \omega_1^\nu$, then by Theorems 10. 5 and 9. 1 we have respectively

(10. 30) $$\omega_1^\omega \omega^{1+\mu+\nu} \Rightarrow [\omega^{1+\nu}, \omega_1^\omega \omega^{1+\mu} n]_{\aleph_1},$$

(10. 31) $$\omega_1^\omega \omega^{1+\mu+\nu} \nrightarrow [\omega^{1+\nu}+1, \omega_1^\omega \omega^{1+\mu}]^2_{\aleph_1}.$$

These relations,[9] together with (10. 15)—(10. 24), give an analysis of (10. 1) for indecomposable ordinals $\alpha < \omega_1^{\omega+1}$. In fact, the analysis can be extended slightly to include the case $\alpha = \omega_1^{\omega+1}$. By Theorem 10. 4,

(10. 32) $$\omega_1^{\omega+1} \Rightarrow [\lambda, \omega_1^{\omega+1}]_{\aleph_1} \qquad (\lambda < \omega_1),$$

and Theorem 10. 3 gives

(10. 33) $$\omega_1^{\omega+1} \nrightarrow [\omega_1, \omega_1^\omega]_{\aleph_1}.$$

We will prove that

(10. 34) $$\omega_1^{\omega+1} \rightarrow [\omega_1^\omega \lambda, \omega_1^n]^2_{\aleph_1} \qquad (\lambda < \omega_1;\ n < \omega).$$

This is best possible since, by Theorem 10. 2,

(10. 35) $$\omega_1^{\omega+1} \nrightarrow [\omega_1^{\omega+1}, \omega_1+1]^2_{\aleph_1}.$$

Finally, to complete the discussion of (10. 1) for the case $\alpha = \omega_1^{\omega+1}$, we will also show that

(10. 36) $$\omega_1^{\omega+1} \rightarrow [\omega_1^{\omega+1}, \omega_1]^2_{\aleph_1}.$$

This seems to be as far as we can go using the present methods. We omit the details, but we can analyse relations of the form

$$\omega_1^{\omega+1}\omega \rightarrow [\beta, \gamma]^2_{\aleph_1}$$

if $\beta < \omega_1^{\omega+1}$ and Theorem 10. 2 shows that

$$\omega_1^{\omega+1}\omega \nrightarrow [\omega_1^{\omega+1}, \omega_1\omega+1]^2_{\aleph_1}.$$

However, as we remarked at the outset of this section, we cannot prove or disprove

$$(?) \qquad \omega_1^{\omega+1}\omega \rightarrow [\omega_1^{\omega+1}, \omega_1\omega]^2_{\aleph_1}.$$

[9] and (10. 33).

PROOF OF (10. 34). Let $\mathscr{J}=(F, M, S)$ be a $(\aleph_1, \omega_1^{\omega+1})$-system and suppose there is no complete sub-graph of $\mathscr{G}(\mathscr{J})$ of type $\omega_1^{\omega}\lambda$. Then $\text{tp } F_\mu < \omega_1^{\omega}\lambda$ $(\mu \in M)$. We may write $S=\bigcup(\varrho<\omega_1)S_\varrho$ (tp), where $\text{tp } S_\varrho = \omega_1^{\omega}$ $(\varrho<\omega_1)$. Since $\omega_1 \Rightarrow [\lambda, \omega_1]_{\aleph_1}$ by Theorem 10. 4, we can assume that

$$\text{(10. 37)} \qquad \text{tp } F_\mu \cap S_\varrho < \omega_1^{\omega} \qquad (\mu\in M;\ \varrho<\omega_1).$$

Suppose that there is some $\varrho<\omega_1$ such that $\text{tp}\,(S_\varrho \sim Q(\mathscr{J}))=\omega_1^{\omega}$. By (10. 37) there is $r<\omega$ and $M'\in[M]^{\aleph_1}$ such that $\text{tp } F_\mu \cap S_\varrho < \omega_1^{r}$ $(\mu\in M')$. We may write $S_\varrho \sim Q(\mathscr{J}) = \bigcup(\lambda<\omega_1^{\omega})B_\lambda$ (tp), where $\text{tp } B_\lambda=\omega_1^{r+1}$ $(\lambda<\omega_1^{\omega})$. Therefore, by Lemma 10. 2, there is a $(\mathscr{J}, \aleph_1)$-free set $X\subset S$ such that $|X\cap B_\lambda|=\aleph_1$ $(\lambda<\omega_1^{\omega})$. Therefore, $\text{tp } X \geqq \omega_1^{\omega}$. We may assume, therefore, that

$$\text{tp}\,(S_\varrho\sim Q(\mathscr{J}))<\omega_1^{\omega} \qquad (\varrho<\omega_1).$$

Since $\omega_1^{\omega}\to(\omega_1^{\omega})_2^1$, it follows that $\text{tp } S_\varrho\cap Q(\mathscr{J})=\omega_1^{\omega}$ $(\varrho<\omega_1)$. Therefore, $Q(\mathscr{J})= =\bigcup(v<\omega_1^{\omega+1})A_v$ (tp), where $\text{tp } A_v=\omega_1^{n+1}(v<\omega_1^{\omega+1})$. Since there is no complete subgraph of $\mathscr{G}(\mathscr{J})$ of type $\omega_1^{\omega+1}$, it follows from Lemma 10. 1 that S contains a $(\mathscr{J}, \aleph_1)$-free subset of type ω_1^{n+1}. This proves (10. 34).

PROOF OF (10. 36). Let $\mathscr{J}=(F, M, S)$ be a $(\aleph_1, \omega_1^{\omega+1})$-system and suppose that there is no complete subgraph of $\mathscr{G}(\mathscr{J})$ of type $\omega_1^{\omega+1}$. We must deduce that there is a $(\mathscr{J}, \aleph_1)$-free subset of type ω_1.

Suppose $\text{tp}\,(S\sim Q(\mathscr{J}))=\omega_1^{\omega+1}$. Then $S\sim Q(\mathscr{J})=\bigcup(\lambda<\omega_1)B_\lambda$ (tp), where $\text{tp } B_\lambda=\omega_1^{\omega}$ $(\lambda<\omega_1)$. Since $\text{tp } F_\mu<\omega_1^{\omega+1}$ $(\mu\in M)$, it follows that there is $\lambda(\mu)<\omega_1$ and $s(\mu)<\omega$ such that

$$\text{tp } F_\mu\cap B_\lambda<\omega_1^{s(\mu)} \qquad (\mu\in M;\ \lambda(\mu)<\lambda<\omega_1).$$

There is $M'\in[M]^{\aleph_1}$ and $s<\omega$ such that $s(\mu)=s$ $(\mu\in M')$. Therefore,

$$\text{tp } F_\mu\cap B_\lambda<\omega_1^{s} \qquad (\mu\in M';\ \lambda(\mu)<\lambda<\omega_1).$$

Let $\pi<\omega_1$ and suppose we have already defined $x_\varrho\in S$ and $\mu_\varrho\in M'$ for $\varrho<\pi$. There is some $\lambda<\omega_1$ such that $\lambda>\lambda(\mu_\varrho)$ for all $\varrho<\pi$. Therefore,

$$\text{tp } \bigcup(\varrho<\pi)F_{\mu_\varrho}\cap B_\lambda<\omega_1^{s+1}$$

and we can choose $x_\pi\in B_\lambda\sim\bigcup(\varrho<\pi)F_{\mu_\varrho}\cup\{x_\varrho\}$. Since each element of $S\sim Q(\mathscr{J})$ belongs to only countably many of the sets F_μ, it follows that there is $\mu_\pi\in M'\sim \sim\{\mu_0, \dots, \hat{\mu}_\pi\}$ such that $F_{\mu_\pi}\cap\{x_0, \dots, x_\pi\}=0$. By induction, it follows that there is a set $X=\{x_0, \dots, \hat{x}_{\omega_1}\}_{\neq}\subset S$ and $M''=\{\mu_0, \dots, \hat{\mu}_{\omega_1}\}_{\neq}\subset M$ such that $X\cap F_\mu=0$ $(\mu\in M'')$, i. e. X is a $(\mathscr{J}, \aleph_1)$-free subset of cardinal $\aleph_1$.

We may therefore assume that $\text{tp } Q(\mathscr{J})=\omega_1^{\omega+1}$. It now follows from Lemma 10. 1 that there is a set $Y\subset Q(\mathscr{J})$ which is $(\mathscr{J}, \aleph_1)$-free and $\text{tp } Y\geqq\omega_1$. This proves (10. 36).

We conclude this section by studying relations of the form

$$\text{(10. 38)} \qquad \alpha\to[\beta, \gamma]_{\aleph_0}^2$$

in the case $|\alpha|=\aleph_1$. We essentially consider only the case when α is a power of ω_1. It is convenient to use another symbol in our discussion,

$$\text{(10. 39)} \qquad \alpha\to[[\beta, \gamma]]_{\aleph_0}$$

which is related to (10. 38). The statement (10. 39) means the following is true. *Let* $\mathcal{J}=(F, M, S)$ *be any* $(\aleph_0, \alpha)$*-system of sets such that* (i) $\operatorname{tp} F_\mu < \beta$ *for all* $\mu \in M$, (ii) *each element of S belongs to only a finite number of the sets* F_μ. *Then there is a* $(\mathcal{J}, \aleph_0)$*-free subset of S of type* γ. If a system $\mathcal{J}$ satisfies (ii) we will say briefly that $\mathcal{J}$ has the *finite property*. It is easy to see how the symbol (10. 39) can be generalized but we have not investigated such problems.

If β is indecomposable then (10. 38) is stronger than (10. 39). For suppose that β is indecomposable and that (10. 38) holds. Let $\mathcal{J}=(F, M, S)$ be any $(\aleph_0, \alpha)$-system satisfying (i) and (ii). These conditions imply that each point of S is joined by edges of $\mathcal{G}(\mathcal{J})$ to a set of order type less than β. Hence, there is no complete subgraph of $\mathcal{G}(\mathcal{J})$ of type β. Now (10. 38) implies that there is a $(\mathcal{J}, \aleph_0)$-free subset of type γ.

It is an easy deduction from the partition relation

(10. 40) $$\omega_1^n \to (\omega_1^n)_\omega^1 \qquad (n < \omega)$$

that

(10. 41) $$\omega_1^n \Rightarrow [\omega_1^n, \omega_1^n]_{\aleph_0} \qquad (n < \omega).$$

Since (10. 38) is stronger than (10. 39) for indecomposable β, our Theorems 10. 9—10. 15 together with (10. 41) and (10. 49) give a complete analysis of the symbol (10. 39) in the case $\alpha = \omega_1^\lambda$ $(\lambda < \omega_1)$. We are unable to analyse (10. 38) with the same completeness and there remain several open questions.

THEOREM 10. 9. *If* α *is indecomposable,* $\omega_1^\omega \leqq \alpha < \omega_2$, *then*

$$\alpha \to [\alpha, \omega_1^\omega]_{\aleph_0}^2.$$

PROOF. Let $\mathcal{J}=(F, M, S)$ be any $(\aleph_0, \alpha)$-system of sets such that S contains no $(\mathcal{J}, \aleph_0)$-free subset of type ω_1^ω. We will deduce that there is a complete subgraph of $\mathcal{G}(\mathcal{J})$ of type α. We can assume that

$$\operatorname{tp} F_\mu < \alpha \qquad (\mu \in M).$$

Suppose that whenever $\mathcal{J}'=(F, M', S)$ is an infinite sub-system of $\mathcal{J}$ and S' is a subset of S of type α, then S' contains a $(\mathcal{J}', \aleph_0)$-free subset of type ω_1^n, for any $n < \omega$.

Let $n < \omega$ and suppose we have already defined $T_\nu \subset S$, $M_\nu \in [M]^{\aleph_0}$ and $\mu_\nu \in M$ for $\nu < n$ so that $\operatorname{tp} T_\nu = \omega_1^\nu$, $M \supset M_0 \supset \dots \supset M_{n-1}$ and so that $T_\nu \cap F_\mu = 0$ if $\nu < n$ and $\mu \in M_\nu \cup \{\mu_0, \dots, \hat{\mu}_\nu\}$. The set $S' = S \sim \bigcup(\nu < n) T_\nu \cup F_{\mu_\nu}$ has order type α. Therefore, by the assumption contained in the last paragraph, there is $T_n \subset S'$ and $M_n \in [M_{n-1}]^{\aleph_0}$ such that $\operatorname{tp} T_n = \omega_1^n$ and $T_n \cap F_\mu = 0$ $(\mu \in M_n)$. Now choose $\mu_n \in M_n \sim \sim \{\mu_0, \dots, \hat{\mu}_n\}$. The set $T = \bigcup(n < \omega) T_n$ defined inductively in this way has order type at least ω_1^ω and is $(\mathcal{J}, \aleph_0)$-free since $T \cap F_\mu = 0$ if $\mu \in \{\mu_0, \dots, \hat{\mu}_\omega\}_{\neq}$. This contradicts the initial assumption that there is no $(\mathcal{J}, \aleph_0)$-free subset of type ω_1^ω.

We may therefore assume that there are $n < \omega$, $M' \in [M]^{\aleph_0}$ and $S' \subset S$ such that S' has type α and does not contain any $(\mathcal{J}', \aleph_0)$-free subset of type ω_1^n, where $\mathcal{J}' = (F, M', S)$.

If N is any finite subset of M', then $V_N = \{x \in S' : \{\mu \in M' : x \in F_\mu\} = N\}$ is $(\mathcal{J}', \aleph_0)$-free and therefore has type less than ω_1^n. Therefore, in view of the relation (10. 40) and the fact that M' has only countably many finite subsets, the set $V = \bigcup(N \in [M']^{<\aleph_0}) V_N$ has type less than ω_1^n. Therefore, $\operatorname{tp}(S' \sim V) = \alpha$ and each element of $S' \sim V$ belongs to infinitely many of the sets F_μ ($\mu \in M'$). Therefore, since there is no $(\mathscr{J}, \aleph_0)$-free subset of S' with type ω_1^n, each element of $S' \sim V$ is joined to all the elements of $S' \sim V$ by edges of the graph $\mathscr{G}(\mathscr{J})$ except for a set of type less than ω_1^n. We may write $S' \sim V = \cup(\lambda < \alpha) B_\lambda$ (tp), where tp $B_\lambda = \omega_1^n$. Let f be any (1, 1) mapping of $[0, \omega_1)$ onto $[0, \alpha)$. Let $\theta < \omega_1$, and suppose we have already chosen elements $x_\nu \in B_{f(\nu)}$ for $\nu < \theta$. Since each point x_ν ($\nu < \theta$) is joined to all the points of $B_{f(\theta)}$ by edges of the graph $\mathscr{G}(\mathscr{J})$ apart from a set of type less than ω_1^n, it follows from (10. 40) that there is $x_\theta \in B_{f(\theta)}$ such that $\{x_\nu, x_\theta\}$ is an edge of $\mathscr{G}(\mathscr{J})$ for all $\nu < \theta$. The set $X = \{x_\theta : \theta < \omega_1\}$ defined by induction in this way is clearly a complete subgraph of $\mathscr{G}(\mathscr{J})$. Also, tp $X =$ tp $\{f(\theta): \theta < \omega_1) = \alpha$. This proves Theorem 10. 9.

The last theorem gives the best possible result in the case cf $(\alpha) = \omega$.

THEOREM 10. 10. *If* $\alpha < \omega_2$ *and* cf $(\alpha) = \omega$, *then*

$$\alpha \not\rightarrow [[\alpha, \omega_1^\omega + 1]]_{\aleph_0}.$$

PROOF. By the hypothesis, $\alpha = \alpha_0 + \ldots + \hat{\alpha}_\omega$, where $0 < \alpha_0 \leqq \alpha_1 \leqq \ldots \leqq \hat{\alpha}_\omega < \alpha < \omega_2$. Let $S = \cup(\lambda < \omega) S_\lambda$ (tp), where tp $S_\lambda = \alpha_\lambda$ ($\lambda < \omega$). Then tp $S = \alpha$. In view of the partition relation

$$\alpha_\lambda \not\rightarrow (\omega_1, \omega_1^2, \ldots, \omega_1^{\hat{\omega}})^1 \qquad (\lambda < \omega)$$

already referred to in the proof of Theorem 10. 3, it follows that there are disjoint sets $A_{\lambda\nu} \subset S_\lambda$ ($\nu < \omega$) such that tp $A_{\lambda\nu} < \omega_1^\nu$ ($\nu < \omega$) and $S_\lambda = \cup(\nu < \omega) A_{\lambda\nu}$ ($\lambda < \omega$). Let $M = [0, \omega)$ and consider the set-system $\mathscr{J} = (F, M, S)$, where

$$F_\mu = \cup(\lambda < \mu < \nu) A_{\lambda\nu} \qquad (\mu < \omega).$$

Clearly, tp $F_\mu < \alpha_0 + \ldots + \alpha_\mu < \alpha$ ($\mu < \omega$) and each element of S belongs to only finitely many of the sets F_μ. Suppose that C is any subset of S of type $\omega_1^\omega + 1$. Then C is not cofinal with S and there is some $\lambda < \omega$ such that tp $C \cap S_\lambda \geqq \omega_1^\omega$. Let $\{\mu_0, \mu_1, \ldots\}_<$ be any infinite subset of M. Then there is $k < \omega$ such that $\mu_k > \lambda$. If $C \cap F_{\mu_k} = 0$ then $C \cap S_\lambda \subset \cup(\nu \leqq \mu_k) A_{\lambda\nu}$ and we obtain the contradiction that tp $C \cap S_\lambda < \omega_1^\omega$. This proves that C is not $(\mathscr{J}, \aleph_0)$-free and completes the proof of Theorem 10. 10.

The condition cf $(\alpha) = \omega$ in Theorem 10. 10 is a necessary one.

THEOREM 10: 11. *If* $\omega \leqq \lambda < \omega_2$, *then* $\omega_1^{\lambda+1} \rightarrow [[\omega_1^{\lambda+1}, \omega_1^{\omega+1}]]_{\aleph_0}$.

PROOF. Let $\mathscr{J} = (F, M, S)$ be any $(\aleph_0, \omega_1^{\lambda+1})$-system which has the finite property and is such that tp $F_\mu < \omega_1^{\lambda+1}$ ($\mu \in M$). If $x \in S$ we write $M(x) = \{\mu \in M: x \in F_\mu\}$.

Let $n < \omega$ and suppose that we have already defined $N_\varrho, \mu_\varrho, T_\varrho$ for $\varrho < n$ so that

(10. 42) $$N_\varrho \in [M]^{<\aleph_0},$$

(10. 43) $$\mu_\varrho \in M \sim \cup(\sigma \leqq \varrho) N_\sigma,$$

(10. 44) $$\mu_\varrho \neq \mu_\sigma \qquad (\sigma < \varrho),$$

(10. 45) $$T_\varrho \subset S \sim \cup(\sigma < \varrho) F_{\mu_\sigma},$$

(10. 46) T_ϱ *is a cofinal subset of* S *of type* $\omega_1^{\varrho+1}$,

(10. 47) $$M(x) = N_\varrho \qquad (x \in T_\varrho).$$

We may write
$$S\sim\bigcup(\varrho<n)F_{\mu_\varrho}=\bigcup(\nu<\omega_1)P_\nu\,(\mathrm{tp}),$$
where $\mathrm{tp}\,P_\nu=\omega_1^\lambda$ $(\nu<\omega_1)$. For each $\nu<\omega_1$ there is, by (10. 40), a finite set $N_{n_\nu}\subset M$ and a set $Q_\nu\subset P_\nu$ such that $\mathrm{tp}\,Q_\nu=\omega_1^n$ and
$$M(x)=N_{n_\nu}\qquad(x\in Q_\nu).$$
Hence there is some finite set $N_n\subset M$ such that $N_{n_\nu}=N_n$ for $\aleph_1$ values of ν. Put $T_n=\bigcup(N_{n_\nu}=N_n)Q_\nu$. Then $\mathrm{tp}\,T_n=\omega_1^{n+1}$ and T_n is cofinal with S. Now choose $\mu_n\in M\sim\bigcup(\varrho\leqq n)N_n\cup\{\mu_0,\dots,\hat{\mu}_n\}$. The formulae (10. 42)—(10. 47) also hold if $\varrho=n$ and we can assume, by induction, that these hold for all $\varrho<\omega$.

If $\sigma<\varrho$, then $F_\varrho\cap F_{\mu_\sigma}=0$ by (10. 45). Also, if $\sigma\geqq\varrho$, then $\mu_\sigma\notin N_\varrho$ by (10. 43) and (10. 47) again implies that $T_\varrho\cap F_{\mu_\sigma}=0$. Hence, by (10. 44) the set $T=\bigcup(\varrho<\omega)T_\varrho$ is $(\mathscr{J},\aleph_0)$-free. Moreover, (10. 46) implies that $\mathrm{tp}\,T\geqq\omega_1^{\omega+1}$. This proves Theorem 10. 11.

The next theorem shows that the result of Theorem 10. 11 is best possible in the case $\mathrm{cf}(\lambda)=\omega$.

THEOREM 10. 12. *If* $\lambda<\omega_2$ *and* $\mathrm{cf}(\lambda)=\omega$, *then*
$$\omega_1^{\lambda+1}\nrightarrow[[\omega_1^\lambda,\omega_1^{\omega+1}+1]]_{\aleph_0}.$$

PROOF. Let $S=\bigcup(\nu<\omega_1)S_\nu\,(\mathrm{tp})$, where $\mathrm{tp}\,S_\nu=\omega_1^\lambda$. By theorem 10. 10 there is, for each $\nu<\omega_1$, a $(\aleph_0,\omega_1^\lambda)$-system $\mathscr{J}_\nu=(F^{(\nu)},M,S_\nu)$ which has the finite property and is such that $\mathrm{tp}\,F_\mu^\nu=\delta_\mu<\omega_1^\lambda$ $(\mu\in M)$ and there is no $(\mathscr{J}_\nu,\aleph_0)$-free subset of S_ν of type $\omega_1^\omega+1$. Consider the set system $\mathscr{J}=(F,M,S)$ where
$$F_\mu=\bigcup(\nu<\omega_1)F_\mu^{(\nu)}\qquad(\mu\in M).$$
Clearly $\mathscr{J}$ has the finite property and $\mathrm{tp}\,F_\mu=\delta_\mu\omega_1<\omega_1^\lambda$. Also, if C is any subset of S of type $\omega_1^{\omega+1}+1$, then there is some $\nu<\omega_1$, so that $\mathrm{tp}\,C\cap S_\nu\geqq\omega_1^\omega+1$. Therefore, C is not $(\mathscr{J}_\nu,\aleph_0)$-free and this implies that C is not $(\mathscr{J},\aleph_0)$-free.

In contrast with the last result, it is possible to strengthen Theorem 10. 11 in the case $\mathrm{cf}(\lambda)\neq\omega$.

THEOREM 10. 13. *If* $\gamma<\omega_1$, $\omega<\lambda<\omega_2$, $\mathrm{cf}(\lambda)\neq\omega$, $\mathrm{cf}(\lambda^-)\neq\omega$, *then*
$$\omega_1^\lambda\rightarrow[[\omega_1^\lambda,\omega_1^{\omega+1}\gamma]]_{\aleph_0}.$$

PROOF. The hypothesis implies that
$$\omega_1^\lambda=\sum(\nu<\omega_1)\omega_1^{\lambda_\nu+1}$$
where $\omega\leqq\lambda_0\leqq\lambda_1\leqq\dots\leqq\hat{\lambda}_{\omega_1}<\lambda$. Let $\mathscr{J}=(F,M,S)$ be any $(\aleph_0,\omega_1^\lambda)$-system such that $\mathrm{tp}\,F_\mu<\omega_1^\lambda$ $(\mu\in M)$ and $\mathscr{J}$ has the finite property. We want to prove that S contains a $(\mathscr{J},\aleph_0)$-free subset of type $\omega_1^{\omega+1}\gamma$. We can assume $\gamma>0$.

We may write $S=\bigcup(\nu<\omega_1)S_\nu\,(\mathrm{tp})$ where $\mathrm{tp}\,S_\nu=\omega_1^{\lambda_\nu+1}$. There is some $\delta<\omega_1$ such that

$$\mathrm{tp}\,F_\mu\cap S_\nu<\omega_1^{\lambda_\nu+1}\qquad(\delta\leqq\nu<\omega_1;\ \mu\in M).\tag{10. 48}$$

We can assume that if $\gamma' < \gamma$, S' is a subset of S of type ω_1^λ and $\mathscr{J}'$ is an infinite subsystem of $\mathscr{J}$, then S' contains a $(\mathscr{J}, \aleph_0)$-free subset of type $\omega_1^{\omega+1}\gamma$, which is not cofinal with S.

Case 1. $\gamma = \gamma' + 1$. There are, by assumption, $\varrho \in [\delta, \omega_1)$, $M_0 \in [M]^{\aleph_0}$ and $T \subset S$ such that $\text{tp } T = \omega_1^{\omega+1}\gamma$, $T \subset \bigcup(\nu < \varrho)S_\nu$ and $T \cap F_\mu = 0$ $(\mu \in M_0)$. By (10. 48) and Theorem 10. 11 it follows that there are $T' \subset S_\varrho$ and $M' \in [M_0]^{\aleph_0}$ such that $\text{tp } T' = \omega_1^{\omega+1}$, and $T' \cap F_\mu = 0$ $(\mu \in M')$. Then the set $T \cup T'$ is $(\mathscr{J}, \aleph_0)$-free, is not cofinal with S and has type $\omega_1^{\omega+1}\gamma$.

Case 2. $\gamma = \gamma_0 + \ldots + \hat{\gamma}_\omega$, where $\gamma_0 \leqq \gamma_1 \leqq \ldots \leqq \hat{\gamma}_\omega < \gamma$. Our assumption implies that there are $T_\sigma, M_\sigma, \mu_\sigma$ for $\sigma < \omega$ such that $\text{tp } T_\varrho = \omega_1^{\omega+1}\gamma_\sigma$, T_σ precedes T_τ in the ordering of S if $\sigma < \tau$, $M_\sigma \in [M]^{\aleph_0}$, $M_0 \supset M_1 \supset \ldots$, $\mu_\sigma \in M_\sigma \sim \{\mu_0, \ldots, \hat{\mu}_\sigma\}$ and

$$T_\sigma \cap F_\mu = 0 \quad \text{if} \quad \mu \in M_\sigma \cup \{\mu_0, \ldots, \hat{\mu}_\sigma\}.$$

Then the set $T = \bigcup(\sigma < \omega)T_\sigma$ has order type $\omega_1^{\omega+1}\gamma$ and is disjoint from the sets F_μ if $\mu \in \{\mu_0, \ldots, \hat{\mu}_\omega\}$ and is not cofinal with S. This completes the proof of Theorem 10. 13.

It follows immediately from Theorem 10. 13 that

$$(10.\ 49) \qquad \omega_1^\lambda \to [[\omega_1^\mu, \omega_1^{\omega+1}\gamma]]_{\aleph_0} \quad \textit{if} \quad \omega < \mu + 1 < \lambda < \omega_2;\ \gamma < \omega_1.$$

We will prove (Theorem 10. 14) that $\omega_1^{\omega+1}\gamma$ cannot be replaced by $\omega_1^{\omega+2}$ in (10. 49).

We need first a lemma.

LEMMA 10. 3. *Let* $\text{tp } S = \omega_1^{\lambda+1}$, $M = [0, \omega)$ and let $M_0, \ldots, \hat{M}_\omega$ *be countably many infinite subsets of* M. *Then there is a set-system* $\mathscr{J} = (F, M, S)$ *such that* $\text{tp } F_\mu < \omega_1^\mu$ $(\mu < \omega)$, $\mathscr{J}$ *has the finite property and*

$$\text{tp}\left(S \sim \bigcup(\mu \in M_i) F_\mu\right) < \omega_1^\omega \qquad (i < \omega).$$

PROOF. Let $i, j < \omega$. Suppose that $m_{i'j'} < \omega$ has been defined for all pairs (i', j') which alphabetically precede (i, j) and such that $i' + j' \leqq i + j$. Then we choose $m_{ij} \in M_i$ so that m_{ij} is larger than all the $m_{i'j'}$, so far defined. Then $M_i' = \{m_{ij}: j < \omega\}_< \subset M_i$ and the sets M_i' are mutually disjoint.

Since $\omega_1^\lambda \nrightarrow (1, \omega_1, \omega_1^2, \ldots, \omega_1^{\hat{\omega}})^1$, it follows that there are disjoint sets $C_n \subset S$ $(n < \omega)$ such that $S = \bigcup(n < \omega)C_n$ and $\text{tp } C_n < \omega_1^n$. If $\mu = m_{ij}$ for some $i, j < \omega$ with $j \geqq 1$, then we define

$$(10.\ 50) \qquad F_\mu = \bigcup(m_{i,\,j-1} \leqq n \leqq m_{ij})C_n.$$

If the integer $\mu \neq m_{ij}$ for any integers i, j (with $j \geqq 1$), then we put $F_\mu = 0$.

The system $\mathscr{J} = (F, M, S)$ so defined has the properties described in the lemma. Since $\text{tp } C_n < \omega_1^n$, it follows from (10. 50) that $\text{tp } F_\mu < \omega_1^\mu$ for all $\mu \in M$. Also, since the sets C_n are disjoint and the inequality

$$m_{i,j-1} \leqq n \leqq m_{i,j}$$

for fixed n is satisfied for only a finite number of pairs it follows that $\mathscr{J}$ has the finite property. Finally, for $i < \omega$ we have

$$\bigcup(\mu \in M_i)F_\mu \supset \bigcup(j < \omega)F_{m_{ij}} = \bigcup(m_{i0} \leqq n < \omega)C_n$$

so that $\text{tp}\left(S \sim \bigcup(\mu \in M_i)F_\mu\right) < \omega_1^{m_{i0}} < \omega_1^\omega$.

(∗) THEOREM 10. 14. *If* $\alpha<\omega_2$, *then* $\alpha\nrightarrow[[\omega_1^\omega, \omega_1^{\omega+2}]]_{\aleph_0}$.

PROOF. There is no loss of generality if we assume that $\alpha=\omega_1^\beta$ and that $\omega+1<\beta<\omega_2$. We are going to assume that if $\gamma<\beta$ then the following is true:

$\mathscr{P}_\gamma$: If $M=[0,\omega)$ *and* tp $S=\omega_1^\gamma$, *then there is a set-system* $\mathscr{J}=(F, M, S)$ *such that* $\mathscr{J}$ *has the finite property*, tp $F_\mu<\omega_1^\mu$ $(\mu<\omega)$ *and such that there is no* $(\mathscr{J}, \aleph_0)$*-free set of type* $\omega_1^{\omega+2}$.

The statement $\mathscr{P}_\gamma$ clearly implies that

$$\omega_1^\gamma\nrightarrow[[\omega_1^\omega, \omega_1^{\omega+2}]]_{\aleph_0}. \tag{10. 51}$$

We want to deduce that $\mathscr{P}_\beta$ is true.

Case 1. cf $(\beta)=\omega$. Let $S=\bigcup(\nu<\omega)S_\nu$ (tp), where

$$\text{tp } S_\nu=\omega_1^{\gamma_\nu+1}<\omega_1^\beta=\text{tp } S.$$

By our assumption, there are set systems $\mathscr{J}_\nu=(F^{(\nu)}, M, S_\nu)$ for $\nu<\omega$ such that $\mathscr{J}_\nu$ has the finite property, tp $F_\mu^\nu<\omega_1^\mu$ $(\mu<\omega)$ and there is no $(\mathscr{J}_\nu, \aleph_0)$-free subset of type $\omega_1^{\omega+2}$. Now put $F_\mu=\bigcup(\nu<\omega)F_\mu^{(\nu)}$ $(\mu<\omega)$. The system $\mathscr{J}=(F, M, S)$ clearly has the finite property and tp $F_\mu<\omega_1^\mu$ $(\mu<\omega)$. Also, if $C\subset S$ has order type $\omega_1^{\omega+2}$, then there is $\nu<\omega$ such that tp $C\cap S_\nu=\omega_1^{\omega+2}$. If follows that C is not $(\mathscr{J}_\nu, \aleph_0)$-free and hence not $(\mathscr{J}, \aleph_0)$-free. This proves that $\mathscr{P}_\beta$ holds in this case.

Case 2. cf $(\omega_1^\beta)=\omega_1$. Then we may write $S=\bigcup(\nu<\omega_1)S_\nu$ (tp) where tp $S_\nu=\omega_1^{\gamma_\nu}<\omega_1^\beta=$ tp S. Let $M_0, M_1, \dots, \hat{M}_{\omega_1}$ be all infinite subsets of M.

By the assumption $\mathscr{P}_\gamma$ $(\gamma<\beta)$ it follows that there are sets $F'_{\nu\mu}\subset S_\nu$ $(\nu<\omega_1;\ \mu<\omega)$ such that each point of S_ν belongs to only a finite number of the sets $F'_{\nu\mu}$, tp $F'_{\nu\mu}<\omega_1^\mu$, and such that

$$\text{tp}\left(S_\nu\sim\bigcup(\mu\in M')F'_{\nu\mu}\right)<\omega_1^{\omega+2} \qquad (\nu<\omega_1;\ M'\in[M]^{\aleph_0}). \tag{10. 52}$$

Also, by Lemma 10. 3, there are sets $F''_{\nu\mu}\subset S_\nu$ $(\nu<\omega_1;\ \mu<\omega)$ such that points of S_ν belong to only a finite number of the sets $F''_{\nu\mu}$ $(\mu<\omega)$, tp $F''_{\nu\mu}<\omega_1^\mu$ $(\mu<\omega)$ and so that

$$\text{tp}\left(S_\nu\sim\bigcup(\mu\in M_\varrho)F''_{\nu\mu}\right)<\omega_1^\omega \qquad (\varrho<\nu<\omega_1). \tag{10. 53}$$

Now put $F_0=0$ and $F_{\mu+1}=\bigcup(\nu<\omega)F'_{\nu\mu}\cup F''_{\nu\mu}$ $(\mu<\omega)$.

The set system $\mathscr{J}=(F, M, S)$ so defined has the finite property and tp $F_\mu<\omega_1^\mu$ $(\mu<\omega)$. Also, if $M'\in[M]^{\aleph_0}$, then $M'=M_\varrho$ for some $\varrho<\omega_1$ and therefore, by (10. 52) and (10. 53),

$$\text{tp}\left(S\sim\bigcup(\mu\in M')F_\mu\right)=\delta_0+\delta_1+\dots+\hat{\delta}_{\omega_1}=\delta,$$

where $\delta_\nu<\omega_1^{\omega+2}$ if $\nu\leqq\varrho$ and $\delta_\nu<\omega_1^\omega$ if $\varrho<\nu<\omega_1$. Hence $\delta<\omega_1^{\omega+2}$ and so there is no $(\mathscr{J}, \aleph_0)$-free subset of S of type $\omega_1^{\omega+2}$. This proves that $\mathscr{P}_\beta$ holds in case 2.

By induction $\mathscr{P}_\beta$ holds for all $\beta<\omega_2$ and hence (10. 51) holds with $\gamma=\beta$. This completes the proof of Theorem 10. 14.

The next theorem gives a strengthening of Theorem 10. 9 in the case $\alpha=\omega_1^\lambda$ and cf $\lambda\neq\omega$.

THEOREM 10. 15. *If* $\gamma < \omega_1^{\omega+1}$, $\omega < \lambda < \omega_2$, $\mathrm{cf}(\lambda) \neq \omega$ *then*

$$\omega_1^\lambda \to [\omega_1^\lambda, \gamma]^2_{\aleph_0}.$$

PROOF. Let $\mathscr{J} = (F, M, S)$ be any $(\aleph_0, \omega_1^\lambda)$-system. Suppose that there is no complete subgraph of $\mathscr{G}(\mathscr{J})$ of type ω_1^λ. Then $\mathrm{tp}\, F_\mu < \omega_1^\lambda$ $(\mu \in M)$. We will deduce that S contains a $(\mathscr{J}, \aleph_0)$-free subset of type γ.

If $\mathrm{tp}\,(S \sim Q(\mathscr{J})) = \omega_1^\lambda$, then Theorems 10. 11 and 10. 13 imply that S contains a $(\mathscr{J}, \aleph_0)$-free subset of type at least $\omega_1^{\omega+1}$. Therefore, we may assume that $\mathrm{tp}\, Q(\mathscr{J}) = \omega_1^\lambda$.

Let $n < \omega$. Suppose that there is a set $S' \subset Q(\mathscr{J})$ such that $\mathrm{tp}\, S' = \omega_1^\lambda$ and S' contains no $(\mathscr{J}, \aleph_0)$-free subset of type ω_1^n. Then each point of S' is joined by edges of the graph $\mathscr{G}(\mathscr{J})$ to all the points of S' except for a set of type less than ω_1^n. We may write $S' = \bigcup(\nu < \omega_1^\lambda) S_\nu$ (tp), where $\mathrm{tp}\, S_\nu = \omega_1^n$. Let f be a (1, 1)-mapping of $[0, \omega_1^\lambda)$ onto $[0, \omega_1)$. Let $\theta < \omega_1$, and suppose that we have already defined $x_\varphi \in S_{f(\varphi)}$ for $\varphi < \theta$. By (10. 40) it follows that there is some $x_\theta \in S_{f(\theta)}$ so that $\{x_\varphi, x_\theta\}$ is an edge of $\mathscr{G}(\mathscr{J})$ for all $\varphi < \theta$. The set $X = \{x_\theta : \theta < \omega_1\}$ defined by induction in this way is a complete subgraph of $\mathscr{G}(\mathscr{J})$ of type ω_1^λ. This contradiction proves that any subset of $Q(\mathscr{J})$ of type ω_1^λ contains a $(\mathscr{J}, \aleph_0)$-free subset of type ω_1^n.

Using a simple inductive argument it follows that if $S' \subset Q(\mathscr{J})$ and $\mathrm{tp}\, S' = \omega_1^\lambda$, then there are μ_n, N_n, T_n for $n < \omega$ such that $N_n \in [M]^{\aleph_0}$, $N_0 \supset N_1 \supset \ldots$, $\mu_n \in N_n \sim \sim \{\mu_0, \ldots, \hat{\mu}_n\}$, $\mathrm{tp}\, T_n = \omega_1^n$, $T_n \subset S' \sim \bigcup(\varrho < n) F_\mu$ and

$$T_n \cap F_\mu = 0 \qquad (\mu \in N_n).$$

The set $T = \bigcup(n < \omega) T_n$ is $(\mathscr{J}, \aleph_0)$-free and has order type at least ω_1^ω. Therefore any subset of $Q(\mathscr{J})$ of type ω_1^λ contains a $(\mathscr{J}, \aleph_0)$-free set of type ω_1^ω.

There is $\delta < \omega_1$ such that $\gamma < \omega_1^\omega \delta$. Suppose that any subset of $Q(\mathscr{J})$ of type ω_1^λ contains a $(\mathscr{J}, \aleph_0)$-free subset of type $\omega_1^\omega \varepsilon$ for all $\varepsilon < \delta$. Our proof will be complete if we deduce that any subset $S^* \subset Q(\mathscr{J})$ of type ω_1^λ also contains a $(\mathscr{J}, \aleph_0)$-free subset of type $\omega_1^\omega \delta$.

Case 1. $\delta = \varepsilon + 1$. By the induction hypothesis there are $A \subset S^*$ and $M' \in [M]^{\aleph_0}$ such that $\mathrm{tp}\, A = \omega_1^\omega \varepsilon$ and $A \cap F_n = 0$ $(\mu \in M')$. The set A is not cofinal with S^* and so there is a set $S^{**} \subset S^*$ such that A precedes S^{**} in the ordering of S and $\mathrm{tp}\, S^{**} = \omega_1^\lambda$. There are $B \subset S^{**}$ and $M'' \in [M']^{\aleph_0}$ so that $\mathrm{tp}\, B = \omega_1^\omega$ and $B \cap F_\mu = 0$ $(\mu \in M'')$. Therefore, the set $A \cup B$ is $(\mathscr{J}, \aleph_0)$-free and has type $\omega_1^\omega \delta$.

Case 2. $\delta = \delta_0 + \ldots + \hat{\delta}_\omega$, where $\delta_\nu < \delta$ $(\nu < \omega)$. Using a similar argument we find successively K_n, M_n, μ_n for $n < \omega$ such that $M_n \in [M]^{\aleph_0}$, $M_0 \supset M_1 \supset \ldots$, $\mathrm{tp}\, K_n = \omega_1^\omega \delta_n$, $K_n \cap F_\mu = 0$ if $\mu \in M_n \cup \{\mu_0, \ldots, \hat{\mu}_n\}$, $\mu_n \in M_n \sim \{\mu_0, \ldots, \hat{\mu}_n\}$ and K_n precedes K_{n+1} in the ordering of S. The set $K = \bigcup(n < \omega) K_n$ has order type $\omega_1^\omega \delta$ and is $(\mathscr{J}, \aleph_0)$-free. This completes the proof of Theorem 10. 15.

There still remain a number of unsolved questions in connection with the symbol (10. 38). For example, we do not know if the last theorem can be improved to strengthen Theorems 10. 11 and 10.13 to similar relations of the form (10. 38). Thus we do not know if the relation

$$(?) \quad \omega_1^\lambda \to [\omega_1^\lambda, \omega_1^{\omega+1}]^2_{\aleph_0} \qquad (\omega < \lambda < \omega_2;\ \mathrm{cf}(\lambda) \neq \omega)$$

is true. In fact we cannot even establish the following weaker result

PROBLEM 6. (?) $\omega_1^\lambda \to [\omega_1^\omega, \omega_1^{\omega+1}]^2_{\aleph_0}$ $(\omega < \lambda < \omega_2)$.

We have proved (Theorem 10. 14) that

$$(*) \qquad \omega_1^\lambda \nrightarrow [\omega_1^\omega, \omega_1^{\omega+2}]^2_{\aleph_0} \qquad (\lambda < \omega_2)$$

but there is still a gap between this negative result and Problem 6. For example, we formulate

PROBLEM 7. (?) $\omega_1^{\omega+2} \to [\omega_1^{\omega+\nu}, \omega_1^{\omega+1}\alpha]^2_{\aleph_0}$ $(\nu = 1$ or $2)$.

§ 11. The relations (2.9) **and** (2.10). In this section and the next three we study the relations

$$(11.\ 1) \qquad (a, m, n, c)^{<k} \to s$$

and

$$(11.\ 2) \qquad (a, m, n, c)^{<k} \twoheadrightarrow s$$

defined in § 2. Our discussion is not complete and we shall mention a number of unsolved problems of this kind. In this section we establish a few general results. Finite and denumerable problems are discussed in § 12 and § 13, and in § 14 the symbols (11. 1) and (11. 2) are discussed for arbitrary cardinals.

Theorems 11. 1 and 11. 2 remain valid if $\to$ is replaced throughout by $\twoheadrightarrow$.

THEOREM 11. 1. *Let* $m \leqq m_1$; $n \geqq n_1$; $c \geqq c_1$; $s \leqq s_1$; $k \geqq k_1$. *Then* $(a, m, n, c)^{<k} \to s$ *implies* $(a, m_1, n_1, c_1)^{<k_1} \to s_1$.

This follows immediately from the definition of (11. 1).

THEOREM 11. 2. (i) *If* $m \geqq \aleph_0$ *and* $(a, m, m, c)^{<k} \to s$, *then* $(a, m', m', c)^{<k} \to s$. (ii) *If* $a \geqq \aleph_0$ *and* $(a, m, n, a)^{<k} \to s$, *then* $(a', m, n, a')^{<k} \to s$.

PROOF. (i) Let $\mathscr{I} = (F, M, S)$ be any (m', a)-system of subsets of S such that there is no $(\mathscr{I}, m')$-free subset of S of cardinal c. Let $N = \bigcup(\mu \in M)N_\mu$, where the sets N_μ are mutually disjoint, $0 < |N_\mu| < m$ $(\mu \in M)$ and $|N| = m$. Consider the (m, a)-system $\mathscr{I}^* = (F^*, N, S)$, where $F_\nu^* = F_\mu$ if $\nu \in N_\mu$ $(\mu \in M)$.

If $X \subset S$ is $(\mathscr{I}^*, m)$-free, then there is $N' \in [N]^m$ such that $F_\nu^* \cap X = 0$ $(\nu \in N')$. Put $M' = \{\mu \in M: N' \cap N_\mu \neq 0\}$. Then $|M'| = m'$ and $F_\mu \cap X = 0$ $(\mu \in M')$, i. e. X is also $(\mathscr{I}, m')$-free. This shows that there is no $(\mathscr{I}^*, m)$-free subset of S of cardinal c. Therefore, by the hypothesis $(a, m, m, c)^{<k} \to s$, it follows that $S = \bigcup(\lambda \leqq \theta)A_\lambda$, where $|\theta| < s$; $|A_0| < a$ and the sets A_λ $(0 < \lambda \leqq \theta)$ are $(\mathscr{I}^*, <k)$-complete. This proves the result since any $(\mathscr{I}^*, <k)$-complete set is also $(\mathscr{I}, <k)$-complete.

(ii) Let $\mathscr{I} = (F, M, S)$ be any (m, a')-system such that S contains no $(\mathscr{I}, n)$-free subset of cardinal a'. There are mutually disjoint sets T_λ $(\lambda \in S)$ such that $T = \bigcup(\lambda \in S)T_\lambda$ has cardinal a, $|T_\lambda| < a$ $(\lambda \in S)$ and such that

$$(11.\ 3) \qquad |\bigcup(\lambda \in S')T_\lambda| = a \qquad (S' \in [S]^{a'}).$$

Consider the (m, a)-system $\mathscr{I}^* = (F^*, M, T)$, where

$$F_\mu^* = \bigcup(\lambda \in F_\mu)T_\lambda \qquad (\mu \in M).$$

If $X \subset T$ is $(\mathcal{J}^*, n)$-free, then there is $N \in [M]^n$ such that $X \cap F^*_\mu = 0$ $(\mu \in N)$. This implies that the set $Y = \{\lambda \in S: T_\lambda \cap X \neq 0\}$ is $(\mathcal{J}, n)$-free, since $Y \cap F_\mu = 0$ $(\mu \in N)$. Therefore, $|Y| < a'$ and $|X| < a$. Hence T contains no $(\mathcal{J}^*, n)$-free subset of cardinal a. By the hypothesis $(a, m, n, a)^{<k} \to s$, it follows that $T = \bigcup (\varrho \leqq \theta) A_\varrho$, where $|\theta| < s$; $|A_0| < a$ and the sets A_ϱ $(0 < \varrho \leqq \theta)$ are $(\mathcal{J}^*, <k)$-complete. Put $B_\varrho = \{\lambda \in S: T_\lambda \cap \cap A_\varrho \neq 0\}$ $(0 < \varrho \leqq \theta)$, $B_0 = S \sim \bigcup (0 < \varrho \leqq \theta) B_\varrho$. If $\lambda \in B_0$, then $T_\lambda \cap A_\varrho = 0$ $(0 < \varrho \leqq \theta)$. Therefore, $\bigcup (\lambda \in B_0) T_\lambda \subset A_0$. Since $|A_0| < a$, it follows from (11. 3) that $|B_0| < a'$. Let $0 < \varrho \leqq \theta$ and let $U \in [B_\varrho]^{<k}$. Then there is a set $V \in [A_\varrho]^{<k}$ such that $V \cap T_\lambda \neq 0$ for all $\lambda \in U$. Since A_ϱ is $(\mathcal{J}^*, <k)$-complete, it follows that there is some $\mu \in M$ such that $V \subset F^*_\mu = \bigcup (\lambda \in F_\mu) T_\lambda$. Hence $U \subset F_\mu$. Therefore, B_ϱ is $(\mathcal{J}, <k)$-complete if $0 < \varrho \leqq \theta$. The result now follows since $S = \bigcup (\varrho \leqq \theta) B_\varrho$.

The next theorem establishes certain connections between (11. 1), (11. 2) and the polarized partition relation (2. 14). We write

$$\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} c \\ d \end{bmatrix}^{1,1}_\lambda \tag{11. 4}$$

if the following is true. *Let A, B be disjoint sets, $|A| = a$, $|B| = b$; $[A, B]^{1,1} = \bigcup (\varrho < \lambda) K_\varrho$; $K_\mu \cap K_\varrho = 0$ $(\mu < \varrho < \lambda)$. Then there are $\pi < \lambda$, $C \in [A]^c$, $D \in [B]^d$ such that $[C, D]^{1,1} \cap K_\pi = 0$.* In the particular case when $\lambda = 2$ it will be noticed that (11. 4) is equivalent to the polarized partition relation

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} c & c \\ d & d \end{pmatrix}^{1,1}.$$

The first part of Theorem 11. 3 is expressed in terms of (11. 4), but we only apply the result when $\lambda = 2$.

THEOREM 11. 3. (i) *Let $a \geqq c \geqq \aleph_0$; $c' > s$. If $(a, m, n, c)^2 \to s$, then* $\begin{bmatrix} a \\ m \end{bmatrix} \Rightarrow \begin{bmatrix} c \\ n \end{bmatrix}^{1,1}_{\omega(s)}$.

(ii) *If $a \geqq c \geqq \aleph_0$, then the relations $(a, m, n, c)^2 \to 2$ and* $\begin{pmatrix} a \\ m \end{pmatrix} \to \begin{pmatrix} c & c \\ n & n \end{pmatrix}^{1,1}$ *are equivalent.*

PROOF. (i) Let $\pi = \omega(s)$. Suppose that $\begin{bmatrix} a \\ m \end{bmatrix} \not\Rightarrow \begin{bmatrix} c \\ n \end{bmatrix}^{1,1}_\pi$. Let A, M be disjoint sets such that $|A| = a$, $|M| = m$. Then there is a function $\chi: A \times M \to [0, \pi)$ such that

$$\{\chi(\alpha, \mu): \alpha \in C, \mu \in N\} = [0, \pi) \qquad (C \in [A]^c; N \in [M]^n). \tag{11. 5}$$

Put $S = \bigcup (\alpha \in A) S_\alpha$, where $S_\alpha = \{(\alpha, \nu): \nu < \pi\}$. Then $|S| = a \cdot s = a$. Consider the system of sets $\mathcal{J} = (F, M, S)$, where

$$F_\mu = \bigcup (\alpha \in A) \{(\alpha, \chi(\alpha, \mu))\} \qquad (\mu \in M).$$

Let $X \in [S]^c$, $N \in [M]^n$. It follows from the hypothesis $c' > s$ that there are $\nu_1 < \pi$ and $C \in [A]^c$ such that

$$(\alpha, \nu_1) \in X \qquad (\alpha \in C).$$

By (11. 5) there are $\alpha_1 \in C$ and $\mu_1 \in N$ such that $\chi(\alpha_1, \mu_1) = v_1$. Therefore, $X \cap F_{\mu_1} \neq 0$. This proves that S does not contain any $(\mathcal{J}, n)$-free subset of power c. Let $T \in [S]^{<a}$. Then $S_\alpha \subset S \sim T$ for some $\alpha \in A$. Since S_α is a complete subgraph of the complementary graph of $\mathcal{G}_2(\mathcal{J})$, it follows that $S \sim T$ is not the union of fewer than s complete subgraphs of $\mathcal{G}_2(\mathcal{J})$. Therefore, $(a, m, n, c)^2 \nrightarrow s$. This proves (i).

(ii) If $a \geqq c \geqq \aleph_0$ and $(a, m, n, c)^2 \to 2$, then by (i) $\begin{bmatrix} a \\ m \end{bmatrix} \Rightarrow \begin{bmatrix} c \\ n \end{bmatrix}_2^{1,1}$. As we have already remarked, this is the same as $\begin{pmatrix} a \\ m \end{pmatrix} \to \begin{pmatrix} c & c \\ n & n \end{pmatrix}^{1,1}$.

Suppose now that $a \geqq c \geqq \aleph_0$ and that $(a, m, n, c)^2 \nrightarrow 2$. Then there is a (m, a)-system $\mathcal{J} = (F, M, S)$ such that S has no $(\mathcal{J}, n)$-free subset of power c and such that $|S \sim A_1| = a$ whenever A_1 is a complete subgraph of $\mathcal{G}_2(\mathcal{J})$.

It follows by transfinite induction that there are disjoint sets $\{x_\alpha, y_\alpha\}_{\neq} \subset S$ $(\alpha < \omega(a))$ which are edges of $\mathcal{G}^*$ the complementary graph of $\mathcal{G}_2(\mathcal{J})$. Let $W = [0, \omega(a))$ and put $W \times M = K_0 \cup K_1$, where $(\alpha, \mu) \in K_0$ if and only if $\alpha \in W$, $\mu \in M$ and $x_\alpha \notin F_\mu$.

Let $U \in [W]^c$, $N \in [M]^n$. We want to show that $U \times N \not\subset K_\varrho$ ($\varrho = 0$ or 1). Since the sets $\{x_\alpha : \alpha \in U\}$ and $\{y_\alpha : \alpha \in U\}$ have cardinal c, they are not $(\mathcal{J}, n)$-free. Hence, there are $\alpha_1, \alpha_2 \in U$ and $\mu_1, \mu_2 \in N$ such that $x_{\alpha_1} \in F_{\mu_1}$ and $y_{\alpha_2} \in F_{\mu_2}$. Therefore, $(\alpha_1, \mu_1) \in K_1$. Also, since $\{x_{\alpha_2}, y_{\alpha_2}\}_{\neq}$ is an edge of $\mathcal{G}^*$, it follows that $x_{\alpha_2} \notin F_{\mu_2}$, i. e. $(\alpha_2, \mu_2) \in K_0$. This proves that $\begin{pmatrix} a \\ m \end{pmatrix} \nrightarrow \begin{pmatrix} c & c \\ n & n \end{pmatrix}^{1,1}$ and completes the proof of Theorem 11. 3.

Since Theorem 11. 3 is the first occasion that we have mentioned the polarized partition relations it is convenient to collect here the known results which we employ in § 14. It is proved in [2] that

$$(11.6)\qquad \begin{pmatrix} a^+ \\ a \end{pmatrix} \to \begin{pmatrix} a & a \\ a & a \end{pmatrix}^{1,1} \quad \textit{if} \quad a' = \aleph_0,$$

and

$$(11.7)\qquad \begin{pmatrix} a^+ \\ a^+ \end{pmatrix} \to \begin{pmatrix} a & a \\ a & a \end{pmatrix}^{1,1} \quad \textit{for any} \quad a \geqq \aleph_0.$$

Also, if $Z(a) = \{a, a', a^+, (a')^+\}$, then

$$(11.8)\qquad \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a & a \\ b & b \end{pmatrix}^{1,1} \quad \textit{if and only if} \quad Z(a) \cap Z(b) = 0.$$

§ 12. A finite problem. If $a \geqq \aleph_0$ and m, n, k, s are finite, then the relation $(a, m, n, c)^k \to s$ does not essentially depend upon a and c. In fact, as we show in Theorem 12. 1, the last relation is equivalent to (2. 11).

THEOREM 12. 1. *Let* $a \geqq \aleph_0, a \geqq c \geqq 1; k, m, n, s < \aleph_0$. *Then the relations* $(a, m, n, c)^k \to s$ *and* $(m, n)^k \to s$ *are equivalent.*

PROOF. Suppose $(a, m, n, c)^k \nrightarrow s$. Then there is a (m, a)-system $\mathcal{J}=(F, M, S)$ such that S contains no $(\mathcal{J}, n)$-free subset of cardinal c and such that S is not the union of a set of cardinal $<a$ and $s-1$ $(\mathcal{J}, k)$-complete sets. Put

$$S'=\bigcup_N (S \sim \bigcup(\mu \in N) F_\mu)$$

where the outer union extends over all sets $N \in [M]^n$. Since the sets $S \sim \bigcup(\mu \in N) F_\mu$ are $(\mathcal{J}, n)$-free, it follows that $|S'|<a$. Therefore, $S \sim S'$ is not the union of $s-1$ $(\mathcal{J}, k)$-complete sets. Since $S \sim S'$ is covered by the union of any n of the m sets F_μ $(\mu \in M)$, it follows from the definition of (2. 11) that $(m, n)^k \nrightarrow s$.

Now suppose that $(m, n)^k \nrightarrow s$. Then there is a set S_1 and a system of m sets $\mathcal{J}_1=(F, M, S_1)$ such that S_1 is covered by the union of any n of the m sets F_μ $(\mu \in M)$ and such that S_1 is not the union of $s-1$ complete subgraphs of $\mathcal{G}_k(\mathcal{J}_1)$. Therefore, the complementary k-graph $\mathcal{G}_k^*$ of $\mathcal{G}_k(\mathcal{J}_1)$ is not $(s-1)$-chromatic. It follows from a result in [1] that there is a *finite* set $S_2 \subset S_1$ such that $\mathcal{G}_k^*$ is not even $(s-1)$-chromatic when restricted to S_2. Therefore, there is no loss of generality if we assume that S_1 is finite. Let $|A|=a$ and let $T=A \times S_1$. Then $|T|=a \cdot |S_1|=a$. Consider the (m, a)-system $\mathcal{J}'=(F', M, T)$, where

$$F'_\mu = A \times F_\mu \qquad (\mu \in M).$$

Then T is covered by any n of the m sets F'_μ and, since $c \geqq 1$, T contains no $(\mathcal{J}', n)$-free subset of power c. Let $T=\bigcup(\lambda<s) T_\lambda$, where $|T_0|<a$. Then there is some $\alpha \in A$ such that

$$\{\alpha\} \times S_1 \subset \bigcup(0<\lambda<s) T_\lambda .$$

If T_λ is a complete subgraph of $\mathcal{G}_k(\mathcal{J}')$ then D_λ, the projection of T_λ on S_1, is a complete subgraph of $\mathcal{G}_k(\mathcal{J}_1)$. Since $S_1=D_1 \cup \ldots \cup \hat{D}_s$, it follows that not all of the sets $T_1, \ldots, \hat{T}_s$ are complete subgraphs of $\mathcal{G}_k(\mathcal{J}')$. Therefore, $(a, m, n, c)^k \nrightarrow s$. This completes the proof of Theorem 12. 1.

It is clear from the definition that $(m, n)^k \rightarrow s$ if $n<s$. In the more interesting case $n \geqq s$ we have the following result.

THEOREM 12. 2. *Let* $m, n, k, s<\aleph_0$; $n \geqq s$. *Then* $(m, n)^k \rightarrow s$ *holds if* $m \geqq k(n-s+1)+s-1$.

PROOF. We establish first the special case

$$(kn-k+1, n)^k \rightarrow 2. \tag{12. 1}$$

Let $F_\mu \subset S$ $(\mu<kn-k+1)$ and suppose that S is covered by the union of any n of these sets F_μ. Then, if $x \in S$, $\{x\}$ is disjoint from at most $n-1$ of the F_μ. Hence, if $X \in [S]^{\leqq k}$ then there is some $\mu<k(n-1)+1$ such that $X \subset F_\mu$. This shows that the k-graph (S, E) is complete, where $E=\bigcup [F_\mu]^k$. Hence (12. 1).

Let $M=[0, m)$, where $m \geqq k(n-s+1)+s-1$, and let $\mathcal{J}'=(F', M, S')$ be a system of sets such that S' is covered by the union of any n of the sets F'_μ $(\mu<m)$. Let $T=S' \sim F'_0 \cup \ldots \cup F'_{s-3}$. Then T is covered by the union of any $n-s+2$ of the sets F'_μ $(s-2 \leqq \mu<m)$. Therefore T is a complete subgraph of $\mathcal{G}_k(\mathcal{J}')$ since

$$(m-s+2, n-s+2)^k \rightarrow 2$$

by (12. 1). Hence, $S = T \cup F_0 \cup \ldots \cup F_{s-3}$ is the union of $s-1$ complete subgraphs of $\mathcal{G}_k(\mathcal{J}')$. This proves that $(m, n)^k \to s$.

We conjecture that Theorem 12. 2 gives the best possible result, i. e.

PROBLEM 8. (?) $(m, n)^k \nrightarrow s$ *if* $m = k(n-s+1)+s-2$ *and* $n \geqq s$.

For $k=1$ this conjecture is clearly true. In Theorems 12. 3 and 12. 4 we confirm the conjecture in some other special cases.

There is a certain connection between Problem 8, in the case $k=2$, and a conjecture of KNESER. Let $S = [\{1, 2, \ldots, 2n+p\}]^n$ and let $\mathcal{G} = (S, E)$, where

$$E = \{\{\Pi_1, \Pi_2\} : \Pi_1, \Pi_2 \in S; \Pi_1 \cap \Pi_2 = 0\}.$$

KNESER [6] conjectured that the graph $\mathcal{G}$ is not $(p+1)$-chromatic. This means that S cannot be expressed as the union of $p+1$ complete subgraphs of $\mathcal{G}^*$, the complementary graph of $\mathcal{G}$. Put $M = \{1, 2, \ldots, 2n+p\}$ and

$$F_i = \{\Pi \in S : i \in \Pi\} \qquad (i \in M).$$

Then the graph $\mathcal{G}(\mathcal{J})$ associated with the set-system $\mathcal{J} = (F, M, S)$ is the same as $\mathcal{G}^*$. Also the union of any $n+p+1$ of the sets F_i covers S. We know by Theorem 12. 2 that

$$(2n+p, n+p+1)^2 \to p+3$$

and this implies that KNESER's graph $\mathcal{G}$ is $(p+2)$-chromatic. KNESER's conjecture is equivalent to showing that $p+3$ cannot be replaced by $p+2$ in the last relation.

THEOREM 12. 3. *Let* $2 \leqq n, k, s < \aleph_0$. *Then*

$$(kn-k, n)^k \nrightarrow 2, \tag{12. 2}$$

$$(2n-1, n+1)^2 \nrightarrow 3, \tag{12. 3}$$

$$(s+2, s+1)^2 \nrightarrow s. \tag{12. 4}$$

PROOF. Let $S = [0, k)$ and let $\mathcal{J}$ be the system of $kn-k$ subsets of S given by F_i $(i < kn-k)$, where

$$F_i = S \sim \{j\} \quad \textit{if} \quad j < k \quad \textit{and} \quad j(n-1) \leqq i < (j+1)(n-1).$$

Each element of S is disjoint from exactly $n-1$ of the sets F_i and so S is covered by the union of any n of these sets. Since $|S| = k$ and $S \not\subset F_i$ for any $i < kn-k$, S is not a complete subgraph of $\mathcal{G}_k(\mathcal{J})$. This proves (12. 2).

Let $m = 2n-1$; $S' = M = [0, m)$. Consider the system $\mathcal{J}' = (F', M, S')$, where

$$F'_\mu = \{\lambda : \lambda < m; \mu \not\equiv \lambda n + i \pmod m \text{ for all } i < n\} \qquad (\mu < m).$$

Each element $\lambda \in S'$ is a member of exactly $n-1$ of the $2n-1$ sets F'_μ and so the union of any $n+1$ of these sets covers S'. If $\lambda < m-1$ and $\mu < m$, then $\{\lambda, \lambda+1\} \not\subset F_\mu$. Otherwise μ would be incongruent (modulo m) to the numbers $\lambda n + i$ and $(\lambda+1)n+i$ for all $i < n$, which is impossible. Similarly, $\{0, m-1\}$ is not an edge of the graph $\mathcal{G}(\mathcal{J}')$. Hence, $0, 1, \ldots, m-1, 0$ is a cycle in $\mathcal{G}^*$ the complementary graph of $\mathcal{G}(\mathcal{J}')$. Since m is odd this implies that $\mathcal{G}^*$ is not 2-chromatic. Therefore, S' is not the union of two complete subgraphs of $\mathcal{G}(\mathcal{J}')$. This proves (12. 3).

Let $M_s=\{1, 2, ..., s+2\}$, $T_s=[M_s]^2$ and let $\mathcal{F}^{(s)}=(F^{(s)}, M_s, T_s)$ be the system of sets given by

$$F_\mu^{(s)}=\{\{\lambda, \mu\}: \lambda \in M_s \sim \{\mu\}\} \qquad (\mu \in M_s).$$

Each element of T_s belongs to exactly 2 of the $s+2$ sets $F_\mu^{(s)}$ and so T_s is covered by the union of any $s+1$ of these sets. Therefore, in order to prove (12. 4) it is enough to prove

(12. 5) *T_s is not the union of $s-1$ complete subgraphs of $\mathcal{G}(\mathcal{F}^{(s)})$.*

We will prove (12. 5) by induction on s. Clearly (12. 5) holds if $s=2$ since the elements $\{1, 2\}$ and $\{3, 4\}$ of T_2 are not joined by an edge of $\mathcal{G}(\mathcal{F}^{(2)})$. Now assume that $s>2$ and that T_{s-1} is not the union of $s-2$ complete subgraps of $\mathcal{G}(\mathcal{F}^{(s-1)})$.

Let $T_s=A_1 \cup ... \cup A_{s-1}$ and suppose that A_i is a complete subgraph of $\mathcal{G}(\mathcal{F}^{(s)})$ for $1 \leq i < s$. Since $|T_s|=\frac{1}{2}(s+1)(s+2)>3(s-1)$, at least one of the sets A_i contains 4 or more elements. We can assume that $\{X_1, X_2, X_3, X_4\}_{\neq} \subset A_{s-1}$. Since $|X_i|=2$ and $X_i \cap X_j \neq 0$ $(1 \leq i \leq j \leq 4)$, a simple argument shows that $\bigcap(1 \leq i \leq 4) X_i \neq 0$. In view of the symmetry of the graph on T_s, we can assume that $s+2 \in X_i$ $(1 \leq i \leq 4)$, i. e. $X_i=\{\lambda_i, s+2\}$ $(1 \leq i \leq 4)$ where $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}_{\neq} \subset [1, s+2)$. Since A_{s-1} is a complete subgraph of $\mathcal{G}(\mathcal{F}^{(s)})$, it follows that $T_{s-1} \cap A_{s-1}=0$. Hence $T_{s-1}= =\bigcup(1 \leq i < s-1) T_{s-1} \cap A_i$. The graph $\mathcal{G}(\mathcal{F}^{(s)})$ restricted to T_{s-1} is identical with $\mathcal{G}(\mathcal{F}^{(s-1)})$ and so $T_{s-1} \cap A_i$ is a complete subgraph of $\mathcal{G}(\mathcal{F}^{(s-1)})$. This contradicts the induction hypothesis. Therefore, (12. 5) holds for $s \geq 2$ and this implies (12.4).

By Theorem 12. 2 we have that $(3n-4, n)^3 \to 3$ for any integer n. We can show that this result is best possible in the cases $n=3, 4, 5$ but we give details only for the case $n=5$.

THEOREM 12. 4. $(10, 5)^3 \nrightarrow 3$.

PROOF. Let $S=[\{0, 1, ..., 9\}]^4$. Consider the system $\mathcal{F}$ of 10 subsets of S given by

$$F_i=\{X \in S: i \notin X\} \qquad (i<10).$$

Since each element $X \in S$ is a member of exactly 6 of the sets F_i, it follows that S is covered by any 5 of these sets. We will assume that $S=U \cup V$ and that U, V are complete subgraphs of $\mathcal{G}_3(\mathcal{F})$ and derive a contradiction.

We first show that if $X_1, X_2 \in U$ then $X_1 \cap X_2 \neq 0$. If this is not the case then U contains 2 elements of S which are disjoint. In view of the symmetry, we may suppose that $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\} \in U$. If $\{i, j\}_{\neq} \subset [0, 8)$, then the three elements $\{0, 1, 2, 3\}$, $\{4, 5, 6, 7\}$, $\{i, j, 8, 9\}$ of S do not form an edge of $\mathcal{G}_3(\mathcal{F})$. Therefore, $\{i, j, 8, 9\} \in V$ $(0 \leq i < j \leq 7)$. Similarly, the elements $\{0, 1, 4, 9\}$, $\{2, 3, 5, 9\}$ and $\{6, 7, 8, 9\}$ do not form an edge of $\mathcal{G}_3(\mathcal{F})$. Since V is a complete subgraph and $\{6, 7, 8, 9\} \in V$ we can assume, by symmetry, that $\{0, 1, 4, 9\} \in U$. Since $\{0, 1, 2, 3\}$, $\{0, 1, 4, 9\} \in U$, it follows that $\{5, 6, 7, 8\} \in V$. Also, $\{2, 3, 4, 8\} \in V$ since $\{0, 1, 4, 9\}$ and $\{4, 5, 6, 7\} \in U$. Hence, $\{2, 3, 4, 8\}$, $\{5, 6, 7, 8\}$ and $\{0, 1, 8, 9\} \in V$. This contradicts the assumption that V is complete.

Thus, we may suppose that if X, Y are disjoint elements of S, then one of these is in U and the other is in V. Suppose $\{0, 1, 2, 3\} \in U$. Then $\{4, 5, 6, 7\} \in V$; $\{8, 9, 0, 1\} \in U$; $\{2, 3, 4, 5\} \in V$; $\{6, 7, 8, 9\} \in U$. This is a contradiction since $\{0, 1, 2, 3\} \cap$ $\cap \{6, 7, 8, 9\}=0$. Therefore, $(10, 5)^3 \nrightarrow 3$.

The simplest problems which are not answered by the above theorems are[10]

$$(?)\quad (8,6)^2 \nrightarrow 4,$$

$$(?)\quad (13,6)^3 \nrightarrow 3.$$

§ 13. The denumerable case. From Theorem 6.1 we have, in particular, that

$$\aleph_0 \to [\aleph_0, \aleph_0]_{\aleph_0}^{<\aleph_0}.$$

The next theorem is a strengthening of this relation.

THEOREM 13. 1. $(\aleph_0, \aleph_0, \aleph_0, \aleph_0)^{<\aleph_0} \to \aleph_0$.

PROOF. Let $\mathscr{J} = (F, M, S)$ be any $(\aleph_0, \aleph_0)$-system which is such that S contains no infinite $(\mathscr{J}, \aleph_0)$-free subset. We will assume that S cannot be expressed as the union of a finite set and a finite number of $(\mathscr{J}, <\aleph_0)$-complete sets and deduce a contradiction.

Put $M_0 = M$. Let $\lambda < \omega$ and suppose we have already defined $M_\lambda \in [M]^{\aleph_0}$ and elements $x_\nu \in S$ and $\mu_\nu \in M$ for $\nu < \lambda$. Let

$$T = S \sim \bigcup(\nu < \lambda) F_{\mu_\nu} \cup \{x_\nu\}.$$

If $T \subset P(\mathscr{J}_\lambda)$, where $\mathscr{J}_\lambda = (F, M_\lambda, S)$, then T is $(\mathscr{J}, <\aleph_0)$-complete by (2. 4). Therefore, S is the union of the finite set $\{x_0, \ldots, x_\lambda\}$ and the $(\mathscr{J}, <\aleph_0)$-complete sets $T, F_{\mu_0}, \ldots, \hat{F}_{\mu_\lambda}$. This contradicts our assumption. Therefore, we can suppose that $T \not\subset P(\mathscr{J}_\lambda)$. Hence, there are $x_\lambda \in T$ and $M_{\lambda+1} \in [M_\lambda]^{\aleph_0}$ such that $x_\lambda \notin F_\mu$ for all $\mu \in M_{\lambda+1}$. Now choose $\mu_\lambda \in M_{\lambda+1} \sim \{\mu_0, \ldots, \hat{\mu}_\lambda\}$. In this way we define by induction $X = \{x_\lambda : \lambda < \omega\}_{\neq} \subset S$ and $M' = \{\mu_\lambda : \lambda < \omega\}_{\neq} \subset M$ so that $X \cap F_\mu = 0$ for all $\mu \in M'$. This contradicts the fact that S does not contain an infinite $(\mathscr{J}, \aleph_0)$-free subset and the theorem follows.

It will follow from Theorem 13. 5 that

$$(13.\ 1) \qquad (\aleph_0, \aleph_0, \aleph_0, \aleph_0)^{<\aleph_0} \nrightarrow n \qquad (n < \aleph_0)$$

so that Theorem 13. 1 gives the best possible result. However, if instead of $(<\aleph_0)$-graphs we consider only k-graphs where k is finite then we obtain the following much stronger result.

THEOREM 13. 2. *If* $1 \leq k < \aleph_0$, *then* $(\aleph_0, \aleph_0, \aleph_0, \aleph_0)^k \to 3$.

PROOF. Let $\mathscr{J} = (F, M, S)$ be any $(\aleph_0, \aleph_0)$-system such that S does not contain an infinite $(\mathscr{J}, \aleph_0)$-free subset. Suppose that S is not the union of a finite set and two complete subgraphs of $\mathscr{G}_k(\mathscr{J})$. Then, if T is any finite subset of S, the k-graph $\mathscr{G}^*$ complementary to $\mathscr{G}_k(\mathscr{J})$ is not 2-chromatic on $S \sim T$.

Let $\lambda < \omega$ and suppose the finite sets $K_0, \ldots, \hat{K}_\lambda \subset S$ have already been defined. Since $\mathscr{G}^*$ is not 2-chromatic on $A = S \sim \bigcup(\nu < \lambda) K_\nu$, it follows by the theorem of ERDŐS and de BRUIJN [1] already referred to, that $\mathscr{G}^*$ is not 2-chromatic on some finite subset $K_\lambda \subset A$. Hence, there are disjoint finite sets $K_\lambda \subset S$ $(\lambda < \omega)$ such that $\mathscr{G}^*$ is not 2-chromatic on each K_λ.

[10] R. K. GUY has now settled these relations and, in collaboration with E. C. MILNER, has confirmed the conjecture contained in Problem 8 in a number of other cases.

There are sets $K'_\lambda \subset K_\lambda$ and $M_\lambda \in [M]^{\aleph_0}$ for $\lambda < \omega$ such that $M \supset M_0 \supset M_1 \supset \ldots$ and

$$K_\lambda \cap F_\mu = K'_\lambda \qquad (\lambda < \omega;\ \mu \in M_\lambda).$$

Since K'_λ is a complete subgraph of $\mathcal{G}_k(\mathcal{J})$ for $\lambda < \omega$ and since $\mathcal{G}^*$ is not 2-chromatic on K_λ, it follows that there is $P_\lambda = \{y_{\lambda 1}, \ldots, y_{\lambda k}\}_{\neq} \subset K_\lambda \sim K'_\lambda$ which is a k-edge of $\mathcal{G}^*$. Therefore,

$$P_\lambda \sim F_\mu \neq 0 \qquad (\lambda < \omega; \mu \in M).$$

For $\lambda < \omega$ choose $\mu_\lambda \in M_\lambda \sim \{\mu_0, \ldots, \hat{\mu}_\lambda\}$ and write $F'_\lambda = F_{\mu_\lambda}$. Consider the partition

$$[W]^2 = L_1 \cup \ldots \cup L_k,$$

where $W = [0, \omega)$ and $\{\lambda, \nu\}_< \in L_\pi$ if and only if $y_{\nu\pi} \in P_\nu \sim F'_\lambda$. Since $\aleph_0 \to (\aleph_0)^2_k$ by RAMSEY's theorem [8], it follows that there are $\varrho \in \{1, 2, \ldots, k\}$ and $N \in [W]^{\aleph_0}$ such that $[N]^2 \subset L_\varrho$.

Let $\lambda, \nu \in N$. If $\lambda < \nu$, then $y_{\nu\varrho} \in P_\nu \sim F'_\lambda$ and so $y_{\nu\varrho} \notin F'_\lambda$. If $\lambda \geqq \nu$, then $\mu_\lambda \in M_\lambda \subset M_\nu$ and therefore $K_\nu \cap F'_\lambda = K'_\nu$ and $y_{\nu\varrho} \notin F'_\lambda$. Thus, if $Y = \{y_{\nu\varrho}\colon \nu \in N\}$ and $M' = \{\mu_\lambda : \lambda \in N\}$, then $Y \cap F_\mu = 0$ for $\mu \in M'$. Therefore, Y is an infinite $(\mathcal{J}, \aleph_0)$-free subset of S. This contradiction proves Theorem 13. 2.

Trivially, of course, we have that $(\aleph_0, \aleph_0, \aleph_0, \aleph_0)^1 \to 2$, but for $k > 1$ the last theorem gives the best possible result. In fact, we have the following negative theorem.

THEOREM 13. 3. *If* $m \geqq \aleph_0$, *then* $(m, m, m, m)^2 \nrightarrow 2$.

PROOF. Put $\alpha = \omega(m)$. Let $M = [0, \alpha)$ and $S = \{(\varrho, \lambda)\colon \varrho < 2; \lambda < \alpha\}$ and consider the system of sets $\mathcal{J} = (F, M, S)$, where

$$F_\mu = \{(0, \lambda)\colon \lambda < \mu\} \cup \{(1, \lambda)\colon \lambda > \mu\} \qquad (\mu < \alpha).$$

Let $M' \in [M]^m$ and let $T = \cup(\mu \in M')F_\mu$. If $\lambda < \alpha$, then there is $\mu \in M'$ so that $\lambda < \mu$. Therefore, $\{(0, \lambda)\colon \lambda < \alpha\} \subset T$. Also, if $\gamma \in M'$, then $S \sim T \subset \{(1, \lambda)\colon \lambda \leqq \gamma\}$. Thus $|S \sim T| < m$, i. e. S does not contain any $(\mathcal{J}, m)$-free subset of power m.

If $A \subset S$ and $|A| < m$, then there are $\lambda, \mu < \alpha$ such that $\lambda < \mu$ and $(0, \mu), (1, \lambda) \in S \sim A$. Since $\{(0, \mu), (1, \lambda)\} \not\subset F_\nu$ for any $\nu < \alpha$, it follows that $S \sim A$ is not a complete subgraph of $\mathcal{G}(\mathcal{J})$. This proves the theorem.

The next two theorems are included here by way of contrast with Theorems 13. 1—3.

THEOREM 13. 4. *If* $n < \aleph_0$, *then* $(\aleph_0, \aleph_0, \aleph_0, n)^{<\aleph_0} \to 2$.

PROOF. Let $\mathcal{J} = (F, M, S)$ be any $(\aleph_0, \aleph_0)$-system of sets such that S does not contain a $(\mathcal{J}, \aleph_0)$-free subset of n elements. Suppose that S is not the union of a finite set and a $(\mathcal{J}, <\aleph_0)$-complete set.

Put $M_0 = M$. Let $\lambda < n$ and suppose that we have already defined $x_\nu \in S$ for $\nu < \lambda$ and an infinite set $M_\lambda \subset M$. As in the proof of Theorem 13. 1 we can assume that $T = S \sim \{x_0, \ldots, \hat{x}_\lambda\} \not\subset P(\mathcal{J}_\lambda)$, where $\mathcal{J}_\lambda = (F, M_\lambda, S)$, otherwise S is the union of the finite set $\{x_0, \ldots, \hat{x}_\lambda\}$ and the $(\mathcal{J}, <\aleph_0)$-complete set T. Hence, there is $x_\lambda \in T$ and $M_{\lambda+1} \in [M_\lambda]^{\aleph_0}$ such that $x_\lambda \notin F_\mu$ $(\mu \in M_{\lambda+1})$. After n steps we obtain a set $X = \{x_0, \ldots, \hat{x}_n\} \subset S$ and an infinite set $M_n \subset M$ such that $X \cap F_\mu = 0$ for all $\mu \in M_n$. This contradiction proves the result.

THEOREM 13. 5. *If* $1 \leqq n < \aleph_0$, *then*

$$(13.\,2) \qquad (\aleph_0, \aleph_0, n, \aleph_0)^{<\aleph_0} \to n+1,$$

$$(13.\,3) \qquad (\aleph_0, \aleph_0, n, \aleph_0)^{<\aleph_0} \nrightarrow n.$$

REMARK. Theorem 11. 1 and (13. 3) imply (13. 1).
We prove first a simple lemma.

LEMMA 13. 1. *Let* $n \leqq t < \omega$ *and let* K *be a set with* $\binom{t}{n}$ *elements. Then there are* t *sets* $A_1, \ldots, A_t \subset K$ *such that* K *is covered by the union of any* $n+1$ *of these sets but not by the union of any* n *of them.*

PROOF. Let $K = \{k_1, \ldots, k_p\}_{\neq}$, where $p = \binom{t}{n}$. Let $[\{1, 2, \ldots, t\}]^n = \{B_1, \ldots, B_p\}_{\neq}$ and let

$$A_\tau = K \sim \bigcup(\tau \in B_\nu;\ 1 \leqq \nu \leqq p)\{k_\nu\} \qquad (1 \leqq \tau \leqq t).$$

Since each element of K is disjoint from exactly n of the sets A_τ, it follows that K is contained in the union of any $n+1$ of these sets. On the other hand, for any $\varrho(1 \leqq \varrho \leqq p)$, $\bigcup(\tau \in B_\varrho)A_\tau = K \sim \{k_\varrho\}$, so that the union of any n of the t sets A_τ has a non-empty complement in K.

PROOF OF THEOREM 13. 5. Let $\mathcal{F} = (F, M, S)$ be any $(\aleph_0, \aleph_0)$-system such that S does not contain a $(\mathcal{F}, n)$-free subset of power $\aleph_0$. Let $N \in [M]^n$. Then $S \sim \sim \bigcup(\mu \in N)F_\mu$ is finite. Hence S is the union of a finite set and n sets which are $(\mathcal{F}, <\aleph_0)$-complete. This proves (13. 2).

Let $T = \bigcup(n \leqq m < \omega)K_m$, where $|K_m| = \binom{m}{n-1}$ and the sets K_m $(n \leqq m < \omega)$ are disjoint. By the lemma, there are m sets $A_{m0}, \ldots, \hat{A}_{mm} \subset K_m$ such that K_m is covered by the union of any n of these but not by the union of any $n-1$. Put $A_{m\mu} = 0$ if $m \leqq \mu < \omega$. Let $M^* = [0, \omega)$ and consider the set-system $\mathcal{F}^* = (F^*, M^*, T)$, where

$$F_\nu^* = \bigcup(n \leqq m < \omega)A_{m\nu} \qquad (\nu < \omega).$$

Let $N \in [M]^n$. Then, by definition of the sets F_ν^*, it follows that $K_m \subset \bigcup(\nu \in N)F_\nu^*$ provided $m > \nu$ *for all* $\nu \in N$. Therefore, $T \sim \bigcup(\nu \in N)F_\nu^*$ is a finite set and so T does not contain any $(\mathcal{F}^*, n)$-free infinite subset. Let Y be any finite subset of T and suppose that $T \sim Y = X_1 \cup \ldots \cup X_{n-1}$, where each X_i is a $(\mathcal{F}^*, <\aleph_0)$-complete set. There is an integer r such that $Y \subset \bigcup(n \leqq m < r)K_m$. Therefore, $K_r = \bigcup(1 \leqq i < n)K_r \cap X_i$. Since X_i $(1 \leqq i < n)$ is $(\mathcal{F}^*, <\aleph_0)$-complete, it follows that there is some $\nu_i \in M^*$ such that $K_r \cap X_i \subset F_{\nu_i} \cap K_r = A_{r\nu_i}$. Let $N' = \{\nu_i\colon 1 \leqq i < n\}$. Then $K_r \subset \bigcup(\nu \in N')A_{r\nu}$. This contradicts the definition of the sets $A_{r\nu}$ since $|N'| \leqq n-1$. This proves (13. 4).

§ 14. The general case. In this section we study relations of the forms (2. 9) and (2. 10) for arbitrary cardinals. Most of our results refer only to 2-graphs and the methods we employ do not seem to extend to k-graphs for $k > 2$. For example, it follows from Theorem 14. 1 that

$$(\aleph_1, \aleph_1, \aleph_1, \aleph_1)^2 \to \aleph_1,$$

but we cannot answer

PROBLEM 9. (?) $(\aleph_1, \aleph_1, \aleph_1, \aleph_1)^3 \to \aleph_1$.

THEOREM 14. 1. *Let a be any cardinal,* $m \geqq \aleph_0$. *If* $m = n^+$ *or if* $m' = \aleph_0$, *then*

$$(a, m, m, m)^2 \twoheadrightarrow m. \tag{14. 1}$$

PROOF. Let $\mathscr{J} = (F, M, S)$ be any (m, a)-system such that S does not contain any $(\mathscr{J}, m)$-free subset of power m.

If $a < m$, then (14. 1) is obvious since each element of S is a complete subgraph of $\mathscr{G}(\mathscr{J})$. Therefore, we can assume that $a \geqq m$. Also, an examination of the proof of Theorem 13. 2 reveals that (14. 1) holds in the case $m = \aleph_0$. For the argument used there was essentially that if $m = \aleph_0$ and S is not the union of a finite set and two complete subgraphs of $\mathscr{G}(\mathscr{J})$, then S contains an infinite $(\mathscr{J}, m)$-free set. Since this is not the case it follows that S is in fact the union of a finite set and two complete subgraphs of $\mathscr{G}(\mathscr{J})$. Since each element of S is a complete subgraph of $\mathscr{G}(\mathscr{J})$, it follows that (14. 1) holds for $m = \aleph_0$ and any a. Therefore, we can assume that $a \geqq m > \aleph_0$.

For the rest of the proof we will assume that S is not the union of fewer than m complete subgraphs of $\mathscr{G}(\mathscr{J})$ and deduce a contradiction.

If $x \in S$ and $M' \subset M$, we write

$$M'(x) = \{\mu \in M' : x \in F_\mu\}.$$

Case 1. $m = n^+$. Let $S_0 = \{x \in S : |M(x)| < m\}$ and $S_1 = S \sim S_0$. Since there is no $(\mathscr{J}, m)$-free set of power m, it follows that each point of S_1 is joined by edges of $\mathscr{G}(\mathscr{J})$ to all the points of S apart from a set of power less than m. Therefore, each connected component of $\mathscr{G}^*$, the complementary graph of $\mathscr{G}(\mathscr{J})$, restricted to S_1 has power less than m. Let T_ϱ $(\varrho < \theta)$ be the connected components of the graph $\mathscr{G}^*$ restricted to S_1. Then S_1 is the union of these disjoint sets T_ϱ and $0 < |T_\varrho| \leqq n$ $(\varrho < \theta)$. Also, $[T_\varrho, T_\sigma]^{1,1} \subset \bigcup(\mu \in M)[F_\mu]^2$ $(\varrho < \sigma < \theta)$. For $\varrho < \theta$, let $T_\varrho = \{x_{\varrho\lambda} : \lambda < \omega(n)\}$ (we do not assert that the $x_{\varrho\lambda}$ are distinct for different values of λ). Put $X_\lambda = \{x_{\varrho\lambda} : \varrho < \theta\}$ for $\lambda < \omega(n)$. Then X_λ is a complete subgraph of $\mathscr{G}(\mathscr{J})$. Therefore, $S_1 = \bigcup(\lambda < \omega(n))X_\lambda$ is the union of fewer than m complete subgraphs of $\mathscr{G}(\mathscr{J})$.

Let $\varphi < \omega(m)$ and suppose that we have already defined $x_\lambda \in S_0$ and $\mu_\lambda \in M$ for $\lambda < \varphi$. If

$$T = S_0 \sim \bigcup(\lambda < \varphi)F_{\mu\lambda} \cup \{x_\lambda\}$$

is empty, then S_0 (and hence S) is the union of fewer than m complete subgraphs of $\mathscr{G}(\mathscr{J})$. This contradiction proves that there is some $x_\varphi \in T$. Also, since each x_λ $(\lambda \leqq \varphi)$ is contained in at most n of the sets F_μ $(\mu \in M)$, it follows that there is $\mu_\varphi \in M \sim \{\mu_0, \ldots, \hat{\mu}_\varphi\}$ such that $\{x_0, \ldots, x_\varphi\} \cap F_{\mu\varphi} = 0$. It follows by transfinite induction that there are sets $X = \{x_\varphi : \varphi < \omega(m)\}_{\neq} \subset S_0$ and $M_1 = \{\mu_\varphi : \varphi < \omega(m)\}_{\neq} \subset \subset M$ such that $X \cap F_\mu = 0$ for $\mu \in M_1$. This contradicts the fact that there is no $(\mathscr{J}, m)$-free subset of S of power m. This proves that (14. 1) holds in the case $m = n^+$.

Case 2. $m' = \aleph_0$. By the earlier remarks, we can assume that $m > \aleph_0$. Therefore, we may suppose that

$$m = m_0 + m_1 + \ldots + \hat{m}_\omega,$$

where $\aleph_0 < m_0 < m_1 < \ldots < \hat{m}_\omega < m$ and $m_k' = m_k$ for $k < \omega$.

Let $k<\omega$, $S'\subset S$, $M'\in[M]^m$ and let $\mathcal{J}'=(F, M', S)$. We will show that if S' does not contain a $(\mathcal{J}', m)$-free subset of power m_k, then S' is the union of fewer than m_k complete subgraphs of $\mathcal{G}(\mathcal{J})$.

Put $S_0'=\{x\in S': |M'(x)|<m\}$ and $S_1'=S'\sim S_0'$. If S' does not contain a $(\mathcal{J}', m)$-free subset of power m_k, then each point of S_1' is joined by edges of the graph $\mathcal{G}(\mathcal{J}')$ to all the points of S' except for a set of power less than m_k. Since every edge of $\mathcal{G}(\mathcal{J}')$ is also an edge of $\mathcal{G}(\mathcal{J})$ it follows that the connected components of $\mathcal{G}^*$, the complementary graph of $\mathcal{G}(\mathcal{J})$, when restricted to S_1' each have power less than m_k (we use here the fact that $m_k'=m_k$). Using an argument similar to that applied to the set S_1 in *case* 1, it follows that S_1' is the union of fewer than m_k complete subgraphs of $\mathcal{G}(\mathcal{J})$.

Suppose that $|S_0'|\geqq m_k$. For each $x\in S_0'$, there is an integer $n(x)$ such that $|M'(x)|<$ $<m_{n(x)}$. Therefore, since $m_k'=m_k>\aleph_0$, there is a set $S_0''\subset S_0'$ and an integer n such that $|S_0''|=m_k$ and $|M'(x)|<m_n$ for all $x\in S_0''$. The set S_0'' intersects at most $m_k\cdot m_n$ of the sets F_μ $(\mu\in M')$ and is, therefore, $(\mathcal{J}', m)$-free. Hence, if S' does not contain a $(\mathcal{J}', m)$-free subset of power m_k, then $|S_0'|<m_k$. It now follows that S' is the union of fewer than m_k complete subgraphs of $\mathcal{G}(\mathcal{J})$, and this proves our earlier assertion.

Put $M_0=M$. Let $k<\omega$ and suppose we have defined already $M_k\in[M]^m$ and sets $S_\varkappa^*\subset S$ and $M_\varkappa^*\subset M$ for $\varkappa<k$ such that $|S_\varkappa^*|=|M_\varkappa^*|=m_\varkappa$ $(\varkappa<k)$ and

$$(14.2)\qquad S_\varkappa^*\cap F_\mu=0 \quad \textit{if} \quad \mu\in M_\lambda^* \quad \textit{and} \quad \lambda,\varkappa<k.$$

Put $S_k'=S\sim\bigcup(\mu\in M_\varkappa^*;\ \varkappa<k)F_\mu$ and let $\mathcal{J}_k=(F, M_k, S)$. Clearly, S_k' is not the union of fewer than m_k complete subgraphs of $\mathcal{G}(\mathcal{J})$ otherwise, by the definition of S_k', S is also the union of fewer than m_k complete subgraphs which is contrary to our initial assumption. Therefore, in the light of our remarks above, S_k' must contain a $(\mathcal{J}_k, m)$-free subset of power m_k. Hence there are $S_k^*\subset S_k'$ and $M_{k+1}\subset M_k$ such that $|S_k^*|=m_k$, $|M_{k+1}|=m$ and

$$S_k^*\cap F_\mu=0 \qquad (\mu\in M_{k+1}).$$

Now let M_k^* be any subset of M_{k+1} of cardinal power m_k. It is easy to see that (14. 2) remains true if we replace k by $k+1$. Therefore, by induction, there are sets $S_k^*\subset S$ and $M_k^*\subset M$ for $k<\omega$ such that $|S_k^*|=|M_k^*|=m_k$ and (14. 2) holds for all $k<\omega$. If $S^*=\bigcup(k<\omega)S_k^*$ and $M^*=\bigcup(k<\omega)M_k$, it follows that $S^*\cap F_\mu=0$ for all $\mu\in M^*$. This contradicts the fact that S does not contain a $(\mathcal{J}, m)$-free subset of power m. This proves that (14. 1) holds also in the case $m'=\aleph_0$ and concludes the proof of the theorem.

We do not know if (14. 1) holds also in the case $m>m'>\aleph_0$. Thus we formulate

PROBLEM 10. (?) $(\aleph_{\omega_1}, \aleph_{\omega_1}, \aleph_{\omega_1}, \aleph_{\omega_1})^2\to\aleph_{\omega_1}$.

In some cases the next theorem gives stronger results than Theorem 14. 1.

(∗) THEOREM 14. 2. *If* $m, n\geqq\aleph_0$, *then the relation*

$$(14.3)\qquad (n, m, m, n)^2\to 2$$

holds if and only if $\{m, m^+, m', (m')^+\}\cap\{n, n^+, n', (n')^+\}=0$.

PROOF. This is an immediate consequence of Theorem 11. 3 (ii) and (11. 8).

The next theorem shows that (14. 3) can be extended to more general graphs if m, n satisfy more stringent conditions.

(∗) THEOREM 14. 3. (i) *If* $n' > m^+$, *then* $(n, m, m, n)^{<m'} \to 2$. (ii) *If* $m' > n^+$, *then* $(n, m, m, n)^n \to 2$.

PROOF. Let $\mathcal{J} = (F, M, S)$ be a (m, n)-system such that S does not contain a $(\mathcal{J}, m)$-free subset of power n.

(i) By this assumption, we have that

$$|S \sim \bigcup(\mu \in M')F_\mu| < n \quad \textit{whenever} \quad M' \in [M]^m.$$

Since M contains only m^+ distinct subsets of power m and $n' > m^+$, it follows that the cardinal of the set

$$T = \bigcup_{M'}(S \sim \bigcup(\mu \in M')F_\mu)$$

is less than n (the outer union extending over all subsets $M' \subset M$ of power m). Therefore, $|S \sim T| = n$. Let X be any subset of $S \sim T$ such that $|X| < m'$,. Suppose $X \not\subset F_\mu$ for all $\mu \in M$. Then there is $M_1 \in [M]^m$ and an element $x \in X$ such that $x \notin F_\mu$ for all $\mu \in M_1$. This contradicts the hypothesis that $X \subset S \sim T$. Hence $S \sim T$ is $(\mathcal{J}, < m')$-complete and the result follows.

(ii) If $m' > n^+$, then there is $A \subset S$ and $M_2 \in [M]^m$ such that $F_\mu = A$ for all $\mu \in M_2$. Since the set $S \sim A$ is $(\mathcal{J}, m)$-free, it has power less than n. Now (ii) follows since A is $(\mathcal{J}, n)$-complete.

From Theorem 14. 2 it follows that

$$(\aleph_\omega, \aleph_n, \aleph_n, \aleph_\omega)^2 \to 2 \quad \textit{if} \quad 2 \leq n < \omega$$

and

$$(\aleph_\omega, \aleph_i, \aleph_i, \aleph_\omega)^2 \nrightarrow 2 \quad \textit{if} \quad i = 0 \textit{ or } 1. \tag{14. 4}$$

Theorem 14. 5, which is a partial strengthening of Theorem 14. 1, also shows that (14. 4) is best possible in the case $i = 0$, i. e.

$$(\aleph_\omega, \aleph_0, \aleph_0, \aleph_\omega)^2 \to 3. \tag{14. 5}$$

In Theorem 14. 5 we obtain an even stronger result that (14. 4) in the case $i = 1$.

(∗) THEOREM 14. 4. *If* $m' = \aleph_0 > k$, *then* $(m, m, m, m)^k \to 3$.

REMARK. In particular we have $(\aleph_\omega, \aleph_\omega, \aleph_\omega, \aleph_\omega)^2 \to 3$ and this implies (14. 5) by Theorem 11. 2 (i).

PROOF OF THEOREM 14. 4. The case $m = \aleph_0$ has already been dealt with in Theorem 13. 2. We therefore assume that $m > \aleph_0$. Then there are cardinals m_λ $(\lambda < \omega)$ such that $m = m_0 + m_1 + \dots$ and $m_\lambda = m_\lambda' > (\aleph_0 + m_0 + \dots + \hat{m}_\lambda)^+$ $(\lambda < \omega)$.

Let $\mathcal{J} = (F, M, S)$ be a (m, m)-system which contains no $(\mathcal{J}, m)$-free subset of power m. We will assume that S is not the union of a set of power less than m and two complete subgraphs of $\mathcal{G}_k(\mathcal{J})$ and we will obtain a contradiction.

The first part of our argument is similar to that used in the proof of Theorem 13. 2. Our last assumption implies that $\mathcal{G}^*$, the k-graph complementary to $\mathcal{G}_k(\mathcal{J})$, is not 2-chromatic on any set $S' \subset S$ if $|S \sim S'| < m$. It follows that there are m disjoint finite sets K_ϱ $(\varrho \in R)$, where $|R| = m$, such that $\mathscr{G}^*$ is not 2-chromatic on K_ϱ. We can assume that the index set R is disjoint from M.

For $\lambda < \omega$ there are $R_\lambda \subset R$ and $n_\lambda < \omega$ such that $|R_\lambda| = m_{2\lambda}$ and $|K_\varrho| = n_\lambda$ for $\varrho \in R_\lambda$. It follows from (11. 8) and the fact that $\aleph_1 < m_{2\lambda} = m'_{2\lambda} < m$, that

$$(14.6) \qquad \binom{m_{2\lambda}}{m} \to \binom{m_{2\lambda}, \dots, m_{2\lambda}}{m, \dots, m}^{1,1}_{t_\lambda} \qquad (\lambda < \omega),$$

where $t_\lambda = 2^{n_\lambda}$ is the number of distinct subsets of K_ϱ if $\varrho \in R_\lambda$. It follows from (14. 6) that there are sets $M^{(\lambda)} \subset M$ and $R'_\lambda \subset R_\lambda$ for $\lambda < \omega$ such that $M \supset M^{(0)} \supset M^{(1)} \supset \dots$, $|M^{(\lambda)}| = m$, $|R'_\lambda| = m_{2\lambda}$ and such that

$$K_\varrho \cap F_\mu = K'_\varrho \qquad (\varrho \in R'_\lambda;\ \mu \in M^{(\lambda)};\ \lambda < \omega).$$

Choose sets $M_\lambda \subset M^{(\lambda)}$ such that $|M_\lambda| = m_{2\lambda+1}$ for $\lambda < \omega$. Then

$$(14.7) \qquad K_\varrho \cap F_\mu = K'_\varrho \qquad (\varrho \in R'_\sigma;\ \mu \in M_\lambda;\ \sigma \leqq \lambda < \omega).$$

Put $R' = \bigcup(\lambda < \omega) R'_\lambda$ and $M' = \bigcup(\lambda < \omega) M_\lambda$.

Since K'_ϱ is a complete subgraph of $\mathscr{G}_k(\mathscr{J})$ for $\varrho \in R'$, and since $\mathscr{G}^*$ is not 2-chromatic on K_ϱ, it follows that there is a set

$$P_\varrho = \{y_{\varrho 1}, \dots, y_{\varrho k}\}_{\neq} \subset K_\varrho \sim K'_\varrho \qquad (\varrho \in R')$$

which is a k-edge of $\mathscr{G}^*$. This means that

$$(14.8) \qquad P_\varrho \sim F_\mu \neq 0 \qquad (\varrho \in R';\ \mu \in M).$$

Put $T_{2\lambda} = R'_\lambda$ and $T_{2\lambda+1} = M_\lambda$ for $\lambda < \omega$ and put $T = \bigcup(\lambda < \omega) T_\lambda = R' \cup M'$. Now consider the partition

$$[T]^2 = J_0 \cup J_1 \cup \dots \cup J_k,$$

where $J_0 = [R']^2 \cup [M']^2$ and $\{\varrho, \mu\}_{\neq} \in J_\pi$ if $1 \leqq \pi \leqq k$, $\varrho \in R'$ $\mu \in M'$ and $y_{\varrho\pi} \notin F_\mu$. In view of (14. 8), every element of $[T]^2$ is in at least one of the sets J_i $(i \leqq k)$.

By Lemma 6. 1 there are disjoint sets $T'_\lambda \subset T_\lambda$ for $\lambda < \omega$ such that $|T'_\lambda| = m_\lambda$ and

$$[T'_\alpha, T'_\beta]^{1,1} \subset J_{\pi(\alpha,\beta)} \quad \textit{for} \quad \alpha < \beta < \omega,$$

where $\pi(\alpha, \beta) \leqq k$ (for $\alpha < \beta < \omega$). Now put $R''_\lambda = T'_{2\lambda}$ and $M''_\lambda = T'_{2\lambda+1}$ for $\lambda < \omega$. Then $R''_\lambda \subset R'_\lambda$, $M''_\lambda \subset M_\lambda$ for $\lambda < \omega$ and $|R''_\lambda| = m_{2\lambda}$, $|M''_\lambda| = m_{2\lambda+1}$. If $\varrho \in R''_\lambda$ and $\mu \in M''_\sigma$, where $\lambda, \sigma < \omega$, then $\{\varrho, \mu\}_{\neq} \notin J_0$, by definition of J_0. In fact, if $\{\alpha, \beta\}_< = \{2\lambda, 2\sigma+1\}_{\neq}$, then $\{\varrho, \mu\} \in J_\pi$ and

$$y_{\varrho\pi} \notin F_\mu,$$

where $\pi = \pi(\alpha, \beta) \in [1, k+1)$.

By RAMSEY's Theorem [8], there is an integer q $(1 \leqq q \leqq k)$ and an infinite set $N \subset [0, \omega)$ such that

$$\pi(2\sigma+1, 2\lambda) = q \quad \textit{for} \quad \{\sigma, \lambda\}_< \subset N.$$

By the last paragraph, this implies that

$$(14.9) \qquad y_{\varrho q} \notin F_\mu \qquad (\varrho \in R''_\lambda;\ \mu \in M''_\sigma;\ \{\sigma, \lambda\}_< \subset N).$$

Let $R^* = \bigcup(\lambda\in N)R''_\lambda$ and $M^* = \bigcup(\lambda\in N)M''_\sigma$. Then $|R^*|=|M^*|=m$. Suppose $\varrho\in R^*$ and $\mu\in M^*$. Then $\varrho\in R''_\lambda$ and $\mu\in M''_\sigma$ for some $\lambda,\sigma\in N$. If $\lambda<\sigma$, then

$$y_{\varrho q}\notin F_\mu \tag{14. 10}$$

by (14. 9). On the other hand, if $\sigma\leqq\lambda$, then (14. 10) again holds since, in this case, $y_{\varrho q}\in P_\varrho\subset K_\varrho\sim K'_\varrho$ and (14. 10) follows from (14. 7). Therefore, (14. 10) holds for arbitrary $\varrho\in R^*$ and $\mu\in M^*$. This implies that the set $Y=\{y_{\varrho q}\colon \varrho\in R^*\}$ is $(\mathscr{J}, m)$-free. Since $|Y|=m$ this is a contradiction and the theorem follows.

By using a very similar method to that used in the proofs of Theorems 13. 2 and 14. 4, we can prove that

$$(m, m, m, m)^k\to 3 \qquad (k<\omega) \tag{14. 11}$$

if m is an inaccessible cardinal satisfying the Ramsey-type of partition relation

$$m\to(m, m)^2. \tag{14. 12}$$

It is well known[11] [5] that if m satisfies (14. 12), then m is inaccessible and there is a non-trivial 2-valued m-additive measure function defined on the subsets of a set of power m. We *cannot* prove, however, that if there is no such non-trivial measure for the cardinal m then (14. 11) is false.

For inaccessible cardinals we have also the following result.

(∗) THEOREM 14. 5. *If m is a regular limit number, then*

$$(m, m^+, m^+, m)^2\nrightarrow m.$$

REMARK. This gives, in particular, that

$$(\aleph_0, \aleph_1, \aleph_1, \aleph_0)^2\nrightarrow\aleph_0.$$

Therefore, by Theorem 11. 2 (ii),

$$(\aleph_\omega, \aleph_1, \aleph_1, \aleph_\omega)^2\nrightarrow\aleph_0 \tag{14. 13}$$

and this is stronger than (14. 4) when $i=1$. We do not know if (14. 13) is best possible.

PROBLEM 11. (?) $(\aleph_\omega, \aleph_1, \aleph_1, \aleph_\omega)^2\to\aleph_1$.

PROOF OF THEOREM 14. 5. Let $\alpha=\omega(m)$ and $\beta=\omega(m^+)$. Since m is a regular limit number,

$$m'=m=m_0+m_1+\ldots+\hat{m}_\alpha,$$

where $m_0<m_1<\ldots<\hat{m}_\alpha<m$.

Let $S=\bigcup(\lambda<\alpha)S_\lambda$, where $|S_\lambda|=m_\lambda\ (\lambda<\alpha)$. Then $|S|=m$. Let $\{B_0, B_1, \ldots, \hat{B}_\beta\}_{\neq}$ be the set of all those sets $B\subset S$ such that $|B|=m$ and $|B\cap S_\lambda|\leqq 1$ for all $\lambda<\alpha$. There are $m^+=m_0\cdot m_1\ldots\hat{m}_\alpha$ such sets by (∗). Let $M=[\alpha,\beta)$. By a method similar to that used in the proof of Theorem 9. 1 we will show that there are sets $F_\mu\subset S$ $(\mu\in M)$ such that $|F_\mu\cap S_\lambda|\leqq 1$ $(\lambda<\alpha)$ and $F_\mu\cap B_\varrho\neq 0$ (if $\varrho<\mu$).

[11] In the notation of [5], the property P_2 implies $P_3^{(2)}$.

Let $\mu \in M$ and let $\{C_0, \ldots, \hat{C}_\alpha\}_{\neq} = \{B_0, \ldots, \hat{B}_\mu\}$. Since $|C_\varrho| = m$ for $\varrho < \alpha$ (and each C is a B), there are ordinals $\lambda(\varrho) < \alpha$ such that $\lambda(0) < \lambda(1) < \ldots < \hat{\lambda}(\alpha) < \alpha$ and

$$C_\varrho \cap S_{\lambda(\varrho)} \neq 0 \qquad (\varrho < \alpha).$$

Put $F_\mu = \bigcup(\varrho < \alpha) C_\varrho \cap S_{\lambda(\varrho)}$. These sets F_μ $(\mu \in M)$ have the properties described in the last paragraph.

Consider the set system $\mathscr{J} = (F, M, S)$. If $S' \in [S]^m$, then there is $\sigma < \beta$ such that $B_\sigma \subset S'$. This implies that $S' \cap F_\mu \neq 0$ if $\mu > \sigma$. Hence there is no $(\mathscr{J}, m)$-free subset of S of power m. Now suppose that S is the union of sets $T, T_0, T_1, \ldots, \hat{T}_\theta$ where $\theta < \alpha$ and $|T| < m$. Then there are $\lambda < \alpha$ and $\gamma < \theta$ such that $|S_\lambda \cap T_\gamma| > 1$. Otherwise, we would have the contradiction that $|S_\lambda| \leqq |T| + |\theta|$ for all $\lambda < \alpha$. Since $|F_\mu \cap S_\lambda| \leqq 1$ for all $\mu \in M$, it follows that T_γ is not a complete subgraph of $\mathscr{G}(\mathscr{J})$. This proves that S is not the union of a set of power less than m and fewer than m complete subgraphs of $\mathscr{G}(\mathscr{J})$.

We use the next lemma to establish further negative relations of the form (2. 9), but the result seems to have an interest independently of the problems discussed in this paper.[12]

$(*)$ LEMMA 14. 1. *Let $|S| = m^+ > \aleph_0$ and let $\mathscr{E} = [S]^2$ be the set of edges of the complete graph on S. Then it is possible to colour the edges $e \in \mathscr{E}$ with m^+ colours in such a way that whenever $X, Y \subset S$, $|X| = m$, $|Y| = m^+$, then there is some $x \in X$ such that all the m^+ colours occur among the edges $[\{x\}, Y]^{1,1}$ which join x to the elements of Y.*

PROOF. Let $\alpha = \omega(m)$, $\beta = \omega(m^+)$ and let $S = \{x_0, \ldots, \hat{x}_\beta\}_{\neq}$. We will colour the edges by assigning to each $e \in \mathscr{E}$ an ordinal number $\theta(e) < \beta$.

Put $M = [0, \beta)$, $F = M^S$, $D = [S]^m \times F$. If $K = (B, f) \in D$, then $B \subset S$, $|B| = m$ and f is a function

$$f: S \to M.$$

We write $S(K) = B$ and $f_K = f$. Let $K_0, K_1, \ldots, \hat{K}_\beta$ be a well-ordering of the elements of D. We now describe the colouring of the elements of $\mathscr{E}$ in the following way.

Let $\mu < \beta$. Then the power of the set

$$J = \{\lambda: \lambda < \mu,\ S(K_\lambda) \subset \{x_0, \ldots, \hat{x}_\mu\}\}$$

it at most m. Therefore, there is some $s \leqq \alpha$ such that $\{K_\lambda: \lambda \in J\} = \{L_0, L_1, \ldots, \hat{L}_s\}_{\neq}$. Since $|S(K)| = m$ for each $K \in D$, it follows that we can choose elements z_σ $(\sigma < s)$ such that

$$z_\sigma \in S(L_\sigma) \sim \{z_0, \ldots, \hat{z}_\sigma\} \qquad (\sigma < s).$$

We colour the edge $\{z_\sigma, x_\mu\}_< \in \mathscr{E}$ with the ordinal $f_{L_\sigma}(z_\sigma)$ for $\sigma < s$. The remaining edges which have x_μ as a last element, i. e. $\{z, x_\mu\}_<$ for $z \in \{x_0, \ldots, \hat{x}_\mu\} \sim \{z_0, \ldots, \hat{z}_s\}$, may be coloured arbitrarily, e. g with the ordinal 0. Since μ is an arbitrary ordinal $< \beta$, this procedure defines a colouring of the elements of $\mathscr{E}$. It remains to prove that the colouring described has the properties stated in the Lemma.

[12] Lemma 14. 1 is a strengthening of Theorem 17A [2].

Let $X, Y \subset S, |X| = m, |Y| = m^+$. Suppose that for each element $x \in X$ there is one colour $\varrho(x) < \beta$ which does not occur among the edges $[\{x\}, Y]^{1,1}$. There is some $f \in F$ such that $f(x) = \varrho(x)$ for $x \in X$. Also, there is $\gamma < \beta$ such that $K_\gamma = (X, f)$. Since $|Y| = m^+$, there is $\delta < \beta$ such that $\delta > \gamma$, $x_\delta \in Y$ and $X \subset \{x_0, \ldots, \hat{x}_\delta\}$. With the colouring described above, there is some $x' \in X$ such that the edge $\{x', x_\delta\}_<$ is coloured $f(x') = \varrho(x')$. This contradicts the definition of ϱ and the lemma follows.

We use Lemma 14. 1 to prove the next two theorems.

(∗) THEOREM 14. 6. *Let* $m \geqq \aleph_0, m > n$. *Then*

$$(m^+, m^+, m^+, m^+)^2 \nrightarrow m \tag{14. 14}$$

and

$$(m^+, m^+, m^+, m)^2 \nrightarrow n. \tag{14. 15}$$

PROOF. We will prove (14. 14) and (14. 15) together. Where the details of the two proofs differ we will refer to them respectively as cases (i) and (ii).

Let $|A| = |B| = m^+$, $A \cap B = 0$. Put $\beta = \omega(m^+)$. Suppose the edges of the complete graph on $A \cup B$ are coloured in the manner described in Lemma 14. 1. The m^+ different colours being denoted by the ordinals less than β. If $\lambda \in A$ and $\mu \in B$, then $\chi(\lambda, \mu)$ denotes the colour of the edge joining λ and μ. For case (i) we define $\pi = \omega(m)$ and for case (ii) we define $\pi = \omega(n)$.

Let $S = \bigcup(\lambda \in A) S_\lambda$, where $S_\lambda = \{(\lambda, \nu) : \nu < \pi\}$ $(\lambda \in A)$. Consider the (m^+, m^+)-system $\mathscr{J} = (F, B, S)$, where

$$F_\mu = \{(\lambda, \nu) \in S : \chi(\lambda, \mu) = \nu\} \qquad (\mu \in B).$$

Since there is no edge of the graph $\mathscr{G}(\mathscr{J})$ in each of the sets S_λ $(\lambda \in A)$, it follows that S is not the union of a set of power less than m^+ and fewer than π complete subgraphs of $\mathscr{G}(\mathscr{J})$.

Let B' be any subset of B of power m^+. Also, let S' be an arbitrary subset of S such that $|S'| = m^+$ in case (i) and $|S'| = m$ in case (ii). In both cases $|S'| > |\pi|$. This implies that the set $A' = \{\lambda \in A : S' \cap S_\lambda \neq 0\}$ has cardinal at least m. By the property of the colouring on $[A \cup B]^2$, it follows that there is $\lambda' \in A'$ such that $\{\chi(\lambda', \mu) : \mu \in B'\} = [0, \beta)$. There are $\varrho < \pi$ and $\mu' \in B'$ such that $(\lambda', \varrho) \in S'$ and $\chi(\lambda', \mu') = \varrho$. Therefore, $S' \cap F_{\mu'} \neq 0$. This proves that there is no $(\mathscr{J}, m^+)$-free subset of S with power m^+ in case (i) or with power m in case (ii). This proves (14. 14) and (14. 15).

The condition that $m > n$ for the relation (14. 15) to hold is not necessary in the case of an inaccessible cardinal m. Thus we prove the following result.

(∗) THEOREM 14. 7. *If m is a regular limit number, then*

$$(m^+, m^+, m^+, m)^2 \nrightarrow m.$$

PROOF. Let $\alpha = \omega(m)$, $\beta = \omega(m^+)$. There are cardinals $m_\varrho < m$ $(\varrho < \alpha)$ such that $m_0 < m_1 < m_2 < \ldots < \hat{m}_\alpha < m$.

Let $A = \{(\varrho, \sigma) : \varrho < \alpha; \sigma < \beta\}$ and let B be any set of cardinal m^+ which is disjoint from A. As in the proof of the last theorem, consider a colouring of the elements of $[A \cup B]^2$ with m^+ colours (the ordinals less than β) so that the condition described in Lemma 14. 1 holds. The colour of the edge joining $\lambda \in A$ and $\mu \in B$ is denoted by $\chi(\lambda, \mu)$.

Let $S=\cup(\varrho<\alpha;\sigma<\beta)S_{\varrho\sigma}$, where $S_{\varrho\sigma}=\{(\varrho,\sigma,\tau)\colon\tau<\omega(m_\varrho)\}$. Consider the (m^+,m^+)-system $\mathscr{J}=(F,B,S)$, where

$$F_\mu=\{(\varrho,\sigma,\tau)\in S\colon\chi((\varrho,\sigma),\mu)=\tau\}$$

for $\mu\in B$.

The sets $S_{\varrho\sigma}((\varrho,\sigma)\in A)$ are complete subgraphs of the complementary graph of $\mathscr{G}(\mathscr{J})$. Therefore S is not the union of a set of power less than m^+ and fewer than m complete subgraphs of $\mathscr{G}(\mathscr{J})$. Let $S'\in[S]^m$ and $B'\in[B]^{m^+}$. Since $m'=m$, it follows that there is $A'\in[A]^m$ such that $S'\cap S_{\varrho\sigma}\neq 0$ for $(\varrho,\sigma)\in A'$. By the property of χ, there is some element $(\varrho',\sigma')\in A'$ such that $\{\chi((\varrho',\sigma'),\mu)\colon\mu\in B'\}=[0,\beta)$. It follows that there are $\tau<\omega(m_{\varrho'})$ and $\mu'\in B'$ such that $(\varrho',\sigma',\tau)\in S'$ and $\chi((\varrho',\sigma'),\mu')=\tau$, i. e. $S'\cap F_{\mu'}\neq 0$. This proves that there is no $(\mathscr{J},m^+)$-free subset of S of power m.

We do not know if the last theorem can be extended to the case of singular limit numbers.

PROBLEM 12. (?) $(\aleph_{\nu+1},\aleph_{\nu+1},\aleph_{\nu+1},\aleph_\nu)^2\nrightarrow\aleph_\nu$ $(\nu=\omega$ *or* $\omega_1)$.

We can, however, prove that the condition $m>n$ for (14. 15) to hold is a necessary one in the case of a non-limit number m. Thus we have that

$$(\aleph_{\nu+2},\aleph_{\nu+2},\aleph_{\nu+2},\aleph_{\nu+1})^2\rightarrow\aleph_{\nu+1}\qquad(\nu\geqq 0).$$

This is a special case of the following theorem.

THEOREM 14. 8. *If* $\aleph_\lambda'>\aleph_{\nu+1}$, *then*

$$(a,\aleph_\lambda,\aleph_\lambda,\aleph_{\nu+1})^2\twoheadrightarrow\aleph_{\nu+1}.\tag{14. 16}$$

PROOF. Let $\mathscr{J}=(F,M,S)$ be a $(\aleph_\lambda,a)$-system such that there is no $(\mathscr{J},\aleph_\lambda)$-free subset of power $\aleph_{\nu+1}$.

Let $S_0=S\sim Q(\mathscr{J})$, i. e. each element of S_0 is a member of fewer than $\aleph_\lambda$ of the sets F_μ $(\mu\in M)$. If S_0 contains a subset S' of power $\aleph_{\nu+1}$, then S' has a non-empty intersection with fewer than $\aleph_\lambda$ of the sets F_μ $(\mu\in M)$ since $\aleph_\lambda'>\aleph_{\nu+1}$. Therefore, S' is $(\mathscr{J},\aleph_\lambda)$-free. This is a contradiction and proves that $|S_0|<\aleph_{\nu+1}$.

If $x\in Q(\mathscr{J})$, then x is joined by edges of the graph $\mathscr{G}(\mathscr{J})$ to all the points of S apart from a set of power less than $\aleph_{\nu+1}$. This follows from the fact that there is no $(\mathscr{J},\aleph_\lambda)$-free subset of power $\aleph_{\nu+1}$. It follows that the connected components of the complementary graph of $\mathscr{G}(\mathscr{J})$ each have power at most $\aleph_\nu$ when restricted to $Q(\mathscr{J})$. Hence $Q(\mathscr{J})$ is the union of $\aleph_\nu$ complete subgraphs of $\mathscr{G}(\mathscr{J})$. The theorem now follows since each element of S_0 is a complete subgraph of $\mathscr{G}(\mathscr{J})$.

In contrast with (14. 16) we have that

$$(a,\aleph_\lambda,\aleph_\lambda,\aleph_\nu)^2\rightarrow 2\quad\text{if}\quad a\geqq\aleph_\nu,\ \aleph_\lambda'>\aleph_{\nu+1}.$$

This follows directly from Theorem 11. 3 (ii) and (11. 8). Similarly, we have that

$$(a,\aleph_\nu,\aleph_\nu,\aleph_\lambda)^2\rightarrow 2\quad\text{if}\quad a\geqq\aleph_\lambda,\ \aleph_\lambda'>\aleph_{\nu+1}.$$

By Theorem 11. 1 this relation implies

$$(a,\aleph_\lambda,\aleph_\nu,\aleph_\lambda)^2\rightarrow 2\quad\text{if}\quad a\geqq\aleph_\lambda,\ \aleph_\lambda'>\aleph_{\nu+1}.\tag{14. 17}$$

The condition $\aleph'_\lambda > \aleph_{\nu+1}$ is essential in (14. 17) since

(14. 18) $$(m^+, m^+, m, m^+)^2 \nrightarrow m \quad \textit{if} \quad m \geqq \aleph_0.$$

This last result is a direct consequence of Theorem 11. 3 (i) and Lemma 14. 1 which implies that

$$\begin{bmatrix} m^+ \\ m^+ \end{bmatrix} \Rightarrow \begin{bmatrix} m^+ \\ m \end{bmatrix}^{1,1}_{\omega(m)}$$

(in fact the suffix $\omega(m)$ can even be replaced by $\omega(m^+)$).

If the final m^+ on the left side of (14. 18) is reduced, then we obtain a positive result. By (11. 7) and Theorem 11. 3 (ii)

$$(m^+, m^+, m, m)^2 \to 2.$$

Similarly, using (11. 6) instead of (11. 7), we deduce the relations

$$(m^+, m, m, m)^2 \to 2 \quad \textit{if} \quad m' = \aleph_0$$

and

$$(m, m^+, m, m)^2 \to 2 \quad \textit{if} \quad m' = \aleph_0.$$

However we are unable to establish either of the relations stated in the next problem

PROBLEM 13. (?) $(\aleph_2, \aleph_1, \aleph_1, a)^2 \to 2 \qquad (a = \aleph_0 \text{ or } \aleph_1)$.

We conclude this section by giving another proof of (14. 18) which employs a different idea to that used to establish Lemma 14. 1.

PROOF OF (14. 18). Let $\alpha = \omega(m)$, $\beta = \omega(m^+)$. Let $S = \cup(\lambda < \beta) S_\lambda$, where $S_\lambda = \{x_{\lambda\nu}: \nu < \alpha\}$. Then $|S| = m^+$. Put $M = [0, \beta)$ and let $[M]^m = \{N_0, \ldots, \hat{N}_\beta\}_{\neq}$. We define a (m^+, m^+)-system $\mathscr{J} = (F, M, S)$ by defining the intersection of the sets F_μ $(\mu \in M)$ with each of the sets S_λ $(\lambda < \beta)$.

Let $\lambda < \beta$. Then there is some $\pi = \pi(\lambda) \leqq \alpha$ such that $\{N_0, \ldots, \hat{N}_\lambda\} = \{M_{\lambda\nu}: \nu < \pi\}_{\neq}$. Since each of the sets $M_{\lambda\nu}$ $(\nu < \pi)$ has cardinal power m, it follows that there are ordinals $\mu_{\lambda\nu\tau}$ $(\nu < \pi(\lambda); \tau < \alpha)$ such that

$$\mu_{\lambda\nu\tau} \in M_{\lambda\nu} \qquad (\tau < \alpha)$$

and such that $\mu_{\lambda\nu\tau} \neq \mu_{\lambda\nu'\tau'}$ unless $\nu = \nu'$ and $\tau = \tau'$. We define the sets F_μ for $\mu \in M$ so that

$$x_{\lambda\tau} \in F_\mu \quad \textit{if and only if} \quad \mu = \mu_{\lambda\nu\tau} \quad \textit{for some} \quad \nu < \pi(\lambda).$$

From this definition of the sets F_μ $(\mu \in M)$, it is clear that $S_\lambda \subset \cup(\mu \in M_{\lambda\nu}) F_\mu$ for $\nu < \pi(\lambda)$. Therefore,

$$S_\lambda \subset \cup(\mu \in N_\theta) F_\mu \quad \textit{if} \quad \lambda > \theta.$$

This implies that there is no $(\mathscr{J}, m)$-free subset of S with power m^+.

Since the ordinals $\mu_{\lambda\nu\tau}$ are distinct for $\nu < \pi(\lambda)$, $\tau < \alpha$ and a fixed value of $\lambda (< \beta)$, it follows that

$$|F_\mu \cap S_\lambda| \leqq 1 \qquad (\mu \in M, \lambda < \beta).$$

Therefore, the set S_λ $(\lambda < \beta)$ does not contain an edge of the graph $\mathscr{G}(\mathscr{J})$. Hence S cannot be expressed as the union of a set of power less than m^+ and fewer than m complete subgraphs of $\mathscr{G}(\mathscr{J})$. This proves (14. 18).

§ 15. In this final section we establish a few results and state a few problems concerning the symbol

$$(m, \alpha, \beta)^2 \to n \tag{15.1}$$

defined in § 2. We consider only the case when α is infinite and indecomposable. Clearly (15. 1) holds if $|\beta| < n \leqq |\alpha|$ since a set of type α (indecomposable) can be decomposed into $|\alpha|$ disjoint sets each of type α.

THEOREM 15. 1. *If α is infinite and indecomposable and $\alpha \to (\alpha, \beta)^2$, then for any $m \geqq 1$*

$$(m, \alpha, \beta)^2 \to 3. \tag{15. 2}$$

PROOF. Let $\mathscr{F} = (F, M, S)$ be any (m, α)-system of sets such that $\mathscr{G}(\mathscr{F})$ contains no complete subgraph of type β. The hypothesis $\alpha \to (\alpha, \beta)^2$ implies that the complementary graph of $\mathscr{G}(\mathscr{F})$ contains a complete subgraph of type α, i. e. there is a set $T \subset S$ such that $\operatorname{tp} T = \alpha$ and $|T \cap F_\mu| \leqq 1$ $(\mu \in M)$. We may write $T = T_0 \cup T_1$, where T_0, T_1 are disjoint sets of type α. Put $T_2 = S$. Then S is the union of the three sets T_ϱ $(\varrho < 3)$ which each have type α and for each $\mu \in M$ there is $\varrho(\mu) < 2$ such that $F_\mu \cap T_{\varrho(\mu)} = 0$. This proves (15. 2).

An immediate deduction from Theorem 15. 1 and the partition relations (3. 4) and (3. 5) that, for arbitrary m,

$$(m, \omega, \omega)^2 \to 3 \tag{15. 3}$$

$$(m, \omega^2, \beta)^2 \to 3 \quad \text{if} \quad \beta < \omega. \tag{15. 4}$$

SPECKER showed [9] that $\omega^3 \nrightarrow (\omega^3, 3)^2$, and a simple modification of SPECKER's construction gives that

$$(m, \omega^3, 4)^2 \nrightarrow 3 \quad \text{if} \quad m \geqq \aleph_0.$$

However, we cannot answer

PROBLEM 14. (?) $(\aleph_0, \omega^3, 5)^2 \nrightarrow 4$.

The relations (15. 3) and (15. 4) are best possible in the sense that the number 3 cannot be replaced by 2. This follows from the following simple result.

THEOREM 15. 2. *If $m \geqq |\alpha| > 0$, then $(m, \alpha, 3)^2 \nrightarrow 2$.*

PROOF. We may clearly assume $|\alpha| \geqq \aleph_0$. Let S be an ordered set of type α and let $x_0 \in S$. Consider the graph $\mathscr{G} = (S, E)$ formed by joining x_0 to all other points of S by edges. $\mathscr{G}$ does not contain any triangle. Let $\mathscr{F} = (F, M, S)$ be any (m, α)-system which is such that $\cup(\mu \in M)\{F_\mu\} = E$. Such a system exists if $m \geqq |E| = |\alpha|$.

Suppose that $S = S_0 \cup S_1$ and $S_\varrho \neq 0$ $(\varrho < 2)$. Since x_0 is in one of the sets S_ϱ, there is some $\mu \in M$ such that $F_\mu \cap S_\varrho \neq 0$ $(\varrho < 2)$. This proves the result.

The next result should be contrasted with (15. 3) and (15. 4) (and Problem 14).

THEOREM 15. 3. *Let $1 < \lambda < \omega_1$; $n < \aleph_0 \leqq m$. Then*

$$(m, \omega^\lambda, \omega)^2 \nrightarrow n.$$

PROOF. Let $\operatorname{tp} S=\omega^\lambda$. Since $1<\lambda<\omega_1$ we may write $S=\bigcup(\nu<\omega)S_\nu\,(\mathrm{tp})$, where $|S_\nu|=\aleph_0$ for $\nu<\omega$. Let $S_\nu=\{x_{\nu\varrho}:\varrho<\omega\}_{\neq}$. Let $\mathscr{J}=(F,M,S)$ be a (m,ω^λ)-system which is such that $\{F_\mu:\mu\in M\}$ coincides with the set of all finite subsets of S of the form

$$\{x_{\nu_0\varrho_0},x_{\nu_1\varrho_1},\dots,x_{\nu_k\varrho_k}\}_<$$

where $k<\omega$; $\nu_0<\nu_1<\dots<\nu_k$ and $\varrho_0>\varrho_1>\dots>\varrho_k$. It is easy to see that the graph $\mathscr{G}(\mathscr{J})$ does not contain an infinite complete subgraph.

Suppose that $S=A_0\cup\dots\cup A_n$, where $\operatorname{tp} A_i=\omega^\lambda$ $(i<n)$. Then $n>0$ and we can choose integers ν_i $(i<n)$ so that $\nu_0<\nu_1<\dots<\nu_n$ and $|S_{\nu_i}\cap A_i|=\aleph_0$ $(i<n)$. Now we can choose integers $\varrho_{n-1},\dots,\varrho_1,\varrho_0$ successively so that $\varrho_{n-1}<\dots<\varrho_0$ and $x_{\nu_i\varrho_i}\in A_i$ $(i<n)$. The set $\{x_{\nu_i\varrho_i}:i<n\}$ belongs to the set-system $\mathscr{J}$ and intersects each of the sets A_i $(i<n)$. This proves the theorem.

The integer n in Theorem 15. 3 clearly cannot be replaced by $\aleph_0$. In fact for any cardinal $a\geqq\aleph_0$ we have the trivial relation

$$(m,a,a)^2\to a.$$

We have the following stronger result in the case of singular cardinals.

(*) THEOREM 15. 4. *If $a>a'$, then for any m,*

$$(m,a,a)^2\to(a')^+.$$

PROOF. Let $\alpha=\omega(a')$. Then there are cardinals $a_\lambda<a$ for $\lambda<\alpha$ such that $a=a_0+a_1+\dots+\hat{a}_\alpha$ and $a_\lambda=a'_\lambda>(a'+a_0+\dots+\hat{a}_\lambda)^+$ for $\lambda<\alpha$. Let $\mathscr{J}=(F,M,S)$ be any (m,a)-system such that $\mathscr{G}(\mathscr{J})$ does not contain a complete subgraph of power a. Let $\beta=\omega((a')^+)$. We want to prove that there are sets $T_\theta\in[S]^a$ for $\theta<\beta$ such that $S=\bigcup(\theta<\beta)T_\theta$ and each set F_μ $(\mu\in M)$ is disjoint from at least one of the sets T_θ $(\theta<\beta)$.

Suppose there is some $\lambda<\alpha$ such that $\mathscr{G}(\mathscr{J})$ does not contain a complete subgraph of power a_λ. Since[13]

$$a\to(a_\lambda,a_\varrho)^2\qquad(\varrho<\alpha)$$

it follows that the complementary graph to $\mathscr{G}(\mathscr{J})$ contains a complete subgraph of power a_ϱ for each $\varrho<\alpha$. Hence there are sets $S_\varrho\in[S]^{a_\varrho}$ for $\varrho<\alpha$ such that

$$|F_\mu\cap S_\varrho|\leqq 1\qquad(\mu\in M;\ \varrho<\alpha).\tag{15. 5}$$

Each set S_ϱ $(\varrho<\alpha)$ can be expressed as the union of $(a')^+$ disjoint sets $S_{\varrho\theta}$ $(\theta<\beta)$ such that $|S_{\varrho\theta}|=a_\varrho$. Put $T_0=S$ and $T_{1+\theta}=\bigcup(\varrho<\alpha)S_{\varrho\theta}$ for $\theta<\beta$. Then $S=\bigcup(\theta<\beta)T_\theta$ and each set T_θ has power a. Also, by (15. 5), if $\mu\in M$ and $\varrho<\alpha$, then there is $\theta(\mu,\varrho)<\beta$ such that

$$F_\mu\cap S_{\varrho\theta}=0\quad\textit{if}\quad\theta>\theta(\mu,\varrho).$$

There is $\varphi=\varphi(\mu)<\beta$ such that $\varphi>\theta(\mu,\varrho)$ for all $\varrho<\alpha$. It follows that $F_\mu\cap T_\varphi=0$.

Therefore, we may assume that there are sets $S'_\lambda\subset S$ for $\lambda<\alpha$ such that S'_λ is a complete subgraph of $\mathscr{G}(\mathscr{J})=(S,E)$ of power a_λ. By Lemma 6. 1, there are

[13] E. g. see [3], Theorem 34, Corollary 2.

disjoint sets $S''_\lambda \subset S'_\lambda$ $(\lambda < \alpha)$ and a function $f:[[0, \alpha)]^2 \to \{0, 1\}$ such that $|S''_\lambda| = a_\lambda$ $(\lambda < \alpha)$ and

$$(15.6)\qquad \begin{cases} [S''_\lambda, S''_\nu]^{1,1} \cap E = 0 & \text{if } f(\lambda, \nu) = 0, \\ [S''_\lambda, S''_\nu]^{1,1} \subset E & \text{if } f(\lambda, \nu) = 1. \end{cases}$$

Put $T_\theta = S$ for $\theta \in [\alpha, \beta)$ and $T_\theta = \bigcup (\theta \leqq \lambda < \alpha) S''_\lambda$ for $\theta < \alpha$. Then $|T_\theta| = a$ for $\theta < \beta$. Let $\mu \in M$. Suppose that $F_\mu \cap T_\theta \neq 0$ for all $\theta < \alpha$. Then there are ordinals $\lambda_\nu < \alpha$ such that $\lambda_0 < \lambda_1 < \ldots < \hat{\lambda}_\alpha < \alpha$ and

$$(15.7)\qquad F_\mu \cap S''_{\lambda_\nu} \neq 0 \qquad (\nu < \alpha).$$

Put $S^* = \bigcup (\nu < \alpha) S''_{\lambda_\nu}$. Then $|S^*| = a$ and S^* is a complete subgraph of $\mathcal{G}(\mathcal{J})$ by (15. 6) and (15. 7). This contradiction proves that F_μ is disjoint from at least one of the sets T_θ $(\theta < \alpha)$ and completes the proof of Theorem 15. 4.

The next theorem shows that the relation given in Theorem 15. 4 is essentially best possible in the sense that $(a')^+$ cannot be replaced by a'.

(∗) THEOREM 15. 5. *If* $m > \aleph_\alpha > \aleph'_\alpha = \aleph_\beta$, *then*

$$(m, \omega_\alpha, \omega_\beta + 1)^2 \nrightarrow \aleph_\beta.$$

PROOF. Let $S = \bigcup (\lambda < \omega_\beta) S_\lambda$ (tp); tp $S_\lambda <$ tp $S = \omega_\alpha$ $(\lambda < \omega_\beta)$. Let $\mathcal{J} = (F, M, S)$ be a system of sets such that $\{F_\mu : \mu \in M\}$ is identical with the set of all sets $F \subset S$ which are such that $|F| = \aleph_\beta$ and $|F \cap S_\lambda| \leqq 1$ for all $\lambda < \omega_\beta$. Clearly, there is no subset of S of order type $\omega_\beta + 1$ which is a complete subgraph of $\mathcal{G}(\mathcal{J})$. Suppose that $S = \bigcup (\lambda < \omega_\beta) A_\lambda$, and $|A_\lambda| = \aleph_\alpha$ for $\lambda < \omega_\beta$. Then there are ordinals $\nu_\lambda < \omega_\beta$ such that $\nu_0 < \nu_1 < \ldots < \hat{\nu}_{\omega_\beta} < \omega_\beta$ and such that $A_\lambda \cap S_{\nu_\lambda} \neq 0$ $(\lambda < \omega_\beta)$. Choose $x_\lambda \in A_\lambda \cap S_{\nu_\lambda}$ for $\lambda < \omega_\beta$ and let $F = \{x_\lambda : \lambda < \omega_\beta\}$. There is some $\mu \in M$ such that $F_\mu = F$ and so $F_\mu \cap A_\lambda \neq 0$ for all $\lambda < \omega_\beta$. This proves the theorem.

We do not have comparable results to Theorems 15. 4 and 15. 5 for regular cardinals. Thus, if $a = b^+$, then we have nothing between the trivial relation

$$(m, a, a)^2 \to a$$

already mentioned and the following result.

(∗) THEOREM 15. 6. *If* $m \geqq a = b^+ \geqq \aleph_0$, *then* $(m, a, a)^2 \nrightarrow \aleph_0$.

PROOF. It follows from Lemma 14. 1 that if $|S| = a$, then there is a graph $\mathcal{J} = (S, E)$ such that whenever $A, B \subset S$, $|A| = a$ and $|B| = b$, then there are elements $x, y \in A$ and $z \in B$ such that $\{x, z\} \in E$ and $\{y, z\}_{\neq} \notin E$. Since the graph contains no complete subgraph of power a, it follows by (∗) that there are just a complete subgraphs of $\mathcal{G}$. Hence, if $m \geqq a$, there is a (m, a)-system $\mathcal{J} = (F, M, S)$ such that $\{F_\mu : \mu \in M\}$ is the set of all the complete subgraphs of $\mathcal{G}$.

Let $S = \bigcup (\nu < \omega) A_\nu^{(0)}$, where $|A_\nu^{(0)}| = a$ $(\nu < \omega)$. We want to show that there is some $\mu \in M$ so that $F_\mu \cap A_\nu^{(0)} \neq 0$ for all $\nu < \omega$. Let $n < \omega$ and suppose that we have already defined $x_0, \ldots, \hat{x}_n \in S$ and $A_\nu^{(n)} \subset A_\nu^{(0)}$ for $n \leqq \nu < \omega$ so that $x_\varrho \in A_\varrho^{(n)}$ $(\varrho < n)$, $|A_\nu^{(n)}| = a$ $(n \leqq \nu < \omega)$ and

$$\{x_\varrho, y\}_{\neq} \in E \quad \text{if} \quad \varrho < n \quad \text{and} \quad y \in \bigcup (\nu \geqq n) A_\nu^{(n)}.$$

There is some element $x_n \in A_n^{(n)}$ such that

$$A_\nu^{(n+1)} = \{y \in A_\nu^{(n)} \colon \{x_n, y\}_{\neq} \in E\}$$

has power a for all $\nu > n$. If this were not the case then for each $x \in A_n^{(n)}$ there is some $\nu(x) \in [n+1, \omega)$ so that x is joined to at most b points of $A_{\nu(x)}^{(n)}$ by edges of $\mathcal{G}$. There is $X \in [A_n^{(n)}]^b$ such that $\nu(x) = \nu_0$ for all $x \in X$. Hence

$$Y = A_{\nu_0}^{(n)} \sim \bigcup (x \in X)\{y \colon \{x, y\}_{\neq} \in E\}$$

has power a and

$$[X, Y]^{1,1} \cap E = \varnothing.$$

This contradicts the definition of the graph $\mathcal{G}$. It follows by induction that there are elements $x_n \in A_n^{(0)}$ $(n < \omega)$ such that $F = \{x_0, \ldots, \hat{x}_\omega\}$ is a complete subgraph of $\mathcal{G}$. This completes the proof.

The first problem of this kind which we cannot settle is

PROBLEM 15. (?) $(m, \aleph_2, \aleph_2)^2 \to \aleph_1$.

It follows from a result in [2][14] that if $(*)$ holds and $a \geqq \aleph_0$, then there is a graph $\mathcal{G} = (S, E)$ such that $|S| = a^+$, $\mathcal{G}$ contains no complete subgraph of power $\aleph_2$ and is such that whenever $A \in [S]^a$, $B \in [S]^{a^+}$, then there is an edge of $\mathcal{G}$ joining some point of A to a point of B. Using this result and the same method employed in the proof of Theorem 15. 6 we can prove

$$(*) \qquad (m, a^+, \aleph_2) \nrightarrow \aleph_0 \qquad (m \geqq a^+).$$

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[14] It is proved in [2] (Theorem 10) that $a^+ \nrightarrow (a^+3, 3, \ldots)_{\aleph_0}$. This, together with the fact that $\aleph_2 \to (3, 3, \ldots)^2_{\aleph_0}$, implies that $a^+ \nrightarrow (\aleph_2, a^+,)^2$. In fact, the proof used in [2] of this relation actually gives the stronger result stated in the text.