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ON DIVISIBILITY PROPERTIES OF SEQUENCES OF INTEGERS

by

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Let $a_1 < a_2 < \dots$ be an infinite sequence of integers of positive lower logarithmic density, in other words

$$\limsup_{x=+\infty} \frac{1}{\log x} \sum_{a_i < x} \frac{1}{a_i} > 0. \tag{1}$$

DAVENPORT and ERDŐS [1] proved that then there exists an infinite subsequence $a_{n_1} < a_{n_2} < \dots$ satisfying $a_{n_i} | a_{n_{i+1}}$.

In this note we will give various sharpenings of this result. The sequence $a_1 < a_2 < \dots$ will be denoted by A, an infinite subsequence $a_{n_1} < a_{n_2} < \dots$ satisfying $a_{n_i} | a_{n_{i+1}}$ will be called a chain, $c_1, c_2, \dots$ will denote positive absolute constant.

THEOREM 1. Let the sequence A satisfy (1). Then it contains a chain satisfying for infinitely many y

$$\sum_{a_{n_i} < y} 1 > c_1 (\log \log y)^{\frac{1}{2}}. \tag{2}$$

THEOREM 2. Let the sequence A satisfy

$$\limsup_{x=+\infty} \frac{1}{\log \log x} \sum_{a_n < x} \frac{1}{a_n \log a_n} = c_2 > 0. \tag{3}$$

Then it contains a chain satisfying for infinitely many x

$$\sum_{a_{n_i} < x} 1 > c_3 \log \log x. \tag{4}$$

We will not give the details of the proof of Theorem 1 since the methods of Theorem 2 can be used and Theorem 2 seems more interesting to us, but we outline the proof of the fact that Theorem 1 is best possible. Let the sequence $m_1 < m_2 < \dots$ tend to infinity sufficiently fast, our sequence A consists of the integers a for which ($\nu(a)$ denotes the number of distrinct prime factors of a)

$$\log \log m_i - (\log \log m_i)^{\frac{1}{2}} < \nu(a) < \log \log m_i + (\log \log m_i)^{\frac{1}{2}}$$

holds for some i ($i = 1, 2, \dots$). It is easy to prove by the methods of [2] that our

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sequence satisfies (1) and if the m_i tend to infinity sufficiently fast a simple computation shows that we have for every chain

$$\sum_{a_{n_i}<x} 1 < 3(\log\log x)^{\frac{1}{2}}$$

in other words Theorem 1 can not be improved.

It is easy to see that in Theorem 2 c_3 can not be greater than c_2, but perhaps the following result holds: For every sequence A there is a chain satisfying

$$(5) \qquad \limsup_{y=+\infty} \frac{1}{\log\log y} \sum_{a_{n_i}<y} 1 \geqq \limsup_{x=+\infty} \frac{1}{\log\log x} \sum_{a_n<x} \frac{1}{a_n \log a_n}.$$

We have not been able to prove or disprove (5).

Before we prove Theorem 2 we show that in general (4) will not hold for all x. In fact we shall show that to every increasing function $f(n)$ there is a sequence A of density 1 every chain of which satisfies

$$(6) \qquad a_{n_i} > f(i)$$

for infinitely many i. (6) of course implies that no lover bound can be given for the growth of $\sum_{a_{n_i}<y} 1$. We construct our sequence as follows: To each integer m we make correspond an interval $I_m = (a_m, b_m)$ where a_m and b_m are sufficiently large, also $b_m < a_{m+1}$ in other words the intervals I_m are disjoint. An integer belongs to our sequence A if and only if it is not of the form

$$mu \quad a_m < mu < b_m \quad 1 \leqq m < +\infty.$$

In other words our sequence A does not contain any multiple of m in the interval I_m, but contains all the other integers. It is easy to see that A has density 1 and that it satisfies (6), we leave the simple details of the proof to the reader.

Now we prove Theorem 2.

LEMMA 1. Let $b_1 < b_2 < \dots$ be a sequence of integers satisfying

$$\sum_i \frac{1}{b_i \log b_i} > c_4$$

Then there are two b's b_i and b_j satisfying $b_i | b_j$ and all prime factors of b_j/b_i are greater than b_i.

The lemma is almost identical with a theorem proved in [3], the condition that all prime factors of b_j/b_i are greater than b_i is not stipulated in [3].

Let k be so large that

$$(7) \qquad \sum_{i=1}^{k} \frac{1}{b_i \log b_i} > c_4$$

and let x be sufficiently large. The number of integers $y \leqq \frac{x}{b_i}$ all whose prime factors are greater than b_i is by the sieve of Eratosthenes and a well known theorem of MERTENS (c is Euler's constant)

$$(8) \qquad (1+o(1)) \frac{x}{b_i} \prod_{p \leqq b_i} \left(1-\frac{1}{p}\right) = (1+o(1)) \frac{xe^{-c}}{b_i \log b_i}.$$

Hence by (7) and (8) the number of integers not exceeding x of the form $b_i y$, where all prime factors of y are greater than b_i is greater than x. Hence there are two indices i and j $i<j$ for which

$$b_i y_1 = b_j y_2, \tag{9}$$

where all prime factors of y_1 are greater than b_i and all prime factors of y_2 are greater than b_j. But then a simple argument shows that b_i/b_j and all prime factors of b_j/b_i are greater than b_i as stated.

Consider now a sequence A satisfying (3), we split it into disjoint subsequences $\{a_i^{(r)}\} = A^{(r)}$ $1 \leqq r < +\infty$ as follows: $a_1^{(1)} = a_1$. Assume that $a_1^{(1)}, a_2^{(1)}, \ldots, a_{k-1}^{(1)}$ has already been defined. $a_k^{(1)}$ is the smallest $a_l > a_{k-1}^{(1)}$ for which $\frac{a_l}{a_i^{(1)}}$ $1 \leqq i \leqq k-1$ is never an integer all whose prime factors are greater than $a_i^{(1)}$. Suppose that the sequences $A^{(1)}, \ldots, A^{(r-1)}$ have already been defined. Let B_r be the sequence which we obtain from A by omitting all the elements of $A^{(i)}$ $(1 \leqq i \leqq r-1)$. We define $A^{(r)} = B_r^{(1)}$ as a subsequence of B_r in the same way we defined $A^{(1)}$ as a subsequence of A. Clearly $a_j^{(r)}/a_i^{(r)}$ can never be an integer all whose prime factors are greater than $a_i^{(r)}$, hence by Lemma 1 we have for every r

$$\sum_{i=1}^{+\infty} \frac{1}{a_i^{(r)} \log a_i^{(r)}} \leqq c_4. \tag{10}$$

Further to each $a_j^{(r)}$ there is an $a_i^{(r-1)}$ so that $\frac{a_j^{(r)}}{a_i^{(r-1)}}$ is an integer all whose prime factors are greater than $a_i^{(r-1)}$ (for if not then by our construction $a_j^{(r)}$ would belong to $A^{(r-1)}$). Thus if say a_n does not belong to $\bigcup_{s=1}^{r} A^{(s)}$ there is a sequence $a_{i_1}, a_{i_2}, \ldots, \ldots, a_{i_r}, a_{i_{r+1}} = a_n$, where a_{i_j} is in $A^{(j)}$ $1 \leqq j \leqq r$ and all prime factors of the integer $\frac{a_{i_{j+1}}}{a_{i_j}}$ are greater than a_{i_j}. We will call such sequences divisibility sequences of length $r+1$ belonging to a_n.

Now we can complete the proof of Theorem 2. By (3) there is a sequence x_i tending to infinity sufficiently fast for which

$$\sum_{a_n < x_i} \frac{1}{a_n \log a_n} > \frac{1}{2} c_2 \log \log x_i \tag{11}$$

Put

$$\left[\frac{1}{4c_4} c_2 \log \log x_i \right] = r_i \tag{12}$$

and define a subsequence $A^* = \{a_1^* < a_2^* < \ldots\}$ of A as follows: a_n belongs to A^* if and only if there is an i so that $a_n < x_i$ and $a_n \notin \bigcup_{j=1}^{r_i} A^{(j)}$ (clearly if such an i exists it must be unique, since if the x_i tend to infinity sufficiently fast $\bigcup_{j=1}^{r_{i+1}} A^{(j)}$ contains all the $a_n \leqq x_i$). We will denote this unique i corresponding to a_n^* by $h(a_n^*)$. By (10), (11) and (12) we have for every i

$$\sum_{a_n^* < x_i} \frac{1}{a_n^* \log a_n^*} > \frac{1}{4} c_2 \log\log x_i. \tag{13}$$

From (13) we obtain by a simple argument that the sequence a_n^* satisfies (1) hence by the theorem of DAVENPORT and ERDŐS quoted in the introduction there is an infinite subsequence of $A^* \{a_{n_1}^*, a_{n_2}^*, \ldots\}$ satisfying $a_{n_j}^* / a_{n_{j+1}}^*$. Consider now a subsequence of the $a_{n_j}^*$ say $d_1 < d_2 < \ldots$ for which $h(d_{k+1}) \geqq h(d_k)+1$. By our construction (see (12)) d_k is not contained in

$$\bigcup_{j=1}^{r_{h(d_k)}} A^{(j)} \qquad \left(r_{h(d_k)} = \left[\frac{1}{4c_4} c_2 \log\log x_{h(d_k)} \right] \right) \tag{14}$$

hence as stated previously, there is a divisibility sequence of length $r_{h(d_k)}+1$ belonging to d_k; we denote by $e_1^{(k)} < e_2^{(k)} < \ldots < e_{r_{h(d_k)}+1}^{(k)} = d_k$ the members of this divisibility sequence (they all belong to our sequence A but not necessarily to A^*). If d_k tends to infinity sufficiently fast then by (12) and (14) $r_{h(d_{k+1})} > 2d_k$ therefore at least $\frac{1}{2} r_{h(d_{k+1})}$ of the $e_i^{(k+1)}$ are greater than d_k, let $e_{s_{k+1}}^{(k+1)}$ be the least $e_i^{(k+1)}$ which is greater than d_k. By what has been said

$$s_{k+1} \leqq \tfrac{1}{2} r_{h(d_{k+1})}. \tag{15}$$

To complete our proof we now show that the infinite sequence

$$e_j^{(k)}, \quad 1 \leqq k < +\infty, \quad s_k \leqq j \leqq r_{h(d_k)}+1 \tag{16}$$

forms a chain satisfying (4). First we show that the sequence (16) satisfies (4) with $c_3 > \dfrac{1}{10c_4} c_2$ and $x = x_{h(d_k)}$. Clearly by the definition of the $e_j^{(k)}$ and $x_{h(d_k)}$

$$e_j^{(k)} \leqq d_k \leqq x_{h(d_k)}. \tag{17}$$

Hence by (12), (13), (15), (16) and (17) the number of the terms of the sequence (16) not exceeding $x_{h(d_k)}$ is greater than

$$\frac{1}{2} r_{h(d_k)} > \frac{1}{2} \left[\frac{1}{4c_4} \log\log x_{h(d_k)} \right] > \frac{1}{10c_4} \log\log x_{h(d_k)}$$

as stated.

Thus to complete our proof we only have to show that the sequence (16) really forms a chain. In other words we have to show that for each k $e_{s_{k+1}}^{(k+1)}$ is a multiple of $e_{r_{h(d_k)}+1}^{(k)} = d_k$. To show this observe that

$$e_{r_{h(d_k)}+1}^{(k+1)} = d_{k+1} = e_{s_{k+1}}^{(k+1)} \prod_{0 \leqq t \leqq r_{h(d_{k+1})} - s_{k+1}} \frac{e_{s_{k+1}+t+1}^{(k+1)}}{e_{s_{k+1}+t}^{(k+1)}}. \tag{18}$$

By our definition each prime factor of the integer

$$\frac{e_{s_{k+1}+t+1}^{(k+1}}{e_{s_{k+1}+t}^{(k+1)}}$$

is greater than $e^{(k+1)}_{s_{k+1}+t} > d_k$, hence if $d_k \nmid e^{(k+1)}_{s_{k+1}}$ we obtain from (18) that $d_k \nmid d_{k+1}$ which contradicts our assumption, hence the proof of Theorem 2 is completed. It would be easy to show that Theorem 2 holds with $c_3 > (1-\varepsilon) c_2 e^{-c}$ for every $\varepsilon > 0$

In [1] DAVENPORT and ERDŐS prove the following theorem: Let A satisfy (1), then there is a k so that

$$\limsup_{x=+\infty} \frac{1}{\log x} \sum_{a_k/a_i} \frac{1}{a_i} > 0.$$

Perhaps the following stronger result holds:

$$\limsup_{x=+\infty} \frac{1}{\log x} \sum_{\substack{a_i<x \\ a_k/a_i}} \frac{a_k}{a_i} \geqq \limsup_{x=+\infty} \frac{1}{\log x} \sum_{a_k \leqq x} \frac{1}{a_k}. \tag{19}$$

It is easy to see that if (19) is true it is best possible.

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