ON THE FUNCTION $g(t) = \limsup_{x \to +\infty} (f(x+t) - f(x))$

by

I. CSISZÁR and P. ERDŐS

The following problem has occured in the theory of regularly increasing functions:[1]

If for a real-valued measurable function $f(x)$ the quantity

$$g(t) = \limsup_{x \to +\infty} (f(x+t) - f(x)) \tag{1}$$

is finite for every possible t, does this imply that the function $g(t)$ is bounded in every finite interval?

The aim of the present note is to show that

a) *if* (1) *is finite for every real number* t, *then* $g(t)$ *is bounded in every finite interval;*

b) *if* (1) *is finite for every positive number* t, *then* $g(t)$ *is bounded in every closed subinterval of the open half-line* $(0, +\infty)$ *but not necessarily bounded in the neighbourhood of* 0.

Case a) Suppose that $g(t)$ is not bounded in some finite interval I; then there exists a sequence of real numbers $t_n \in I$ such that $g(t_n) > n$ $(n = 1, 2, \ldots)$. Then, by (1), one could find a sequence $x_n \to +\infty$ such that

$$f(x_n + t_n) - f(x_n) > n \qquad (n = 1, 2, \ldots). \tag{2}$$

Take now an arbitrary finite interval J and consider the sets

$$\begin{aligned} Y_{1,n} &= \{y : f(x_n + y) - f(x_n) > \frac{n}{2}; \quad y \in J\}, \\ Y_{2,n} &= \{y : f(x_n + t_n) - f(x_n + y) > \frac{n}{2}; \quad y \in J\}. \end{aligned} \tag{3}$$

These sets are measurable for each n, and since by (2) $Y_{1,n} \cup Y_{2,n} = J$, we have either $\mu(Y_{1,n}) \geqq \frac{1}{2}\mu(J)$ or $\mu(Y_{2,n}) \geqq \frac{1}{2}\mu(J)$ or both (μ denotes the Lebesgue measure).

[1] The problem was told to P. ERDŐS by R. BOJANIC and J. KARAMATA. In a paper of W. MATUSZEWSKA (Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces, *Studia Math.* **21** (1962) 317—344) a proof of statement a) is given; in it, however, there is a gap: $g(t)$ is implicitely assumed to be measurable, although the measurability of $f(t)$ does not imply the measurability of $g(t)$.

Put now

$$(4) \qquad Z_n = \{z : f(x_n + t_n) - f(x_n + t_n - z) > \frac{n}{2}\,;\ t_n - z \in J\}\,.$$

Then obviously $\mu(Z_n) = \mu(Y_{2,n})$ and thus we have either

$$(5) \qquad \mu(Y_{1,n}) \geqq \frac{1}{2}\,\mu(J) \qquad \text{infinitely often}$$

or

$$(6) \qquad \mu(Z_n) \geqq \frac{1}{2}\,\mu(J) \qquad \text{infinitely often}$$

(or both), where all the $Y_{1,n}$'s and Z_n's are subsets of a fixed finite interval. This clearly implies the existence of a real number y_0 or z_0 contained in infinitely many $Y_{1,n}$ or Z_n, respectively.[2] But then — by the definitions (3), (4) and (1) — we would have $g(y_0) = +\infty$ or $g(z_0) = +\infty$, respectively, contradicting to the assumed finiteness of $g(t)$.

Case b) The first statement follows in the same way as in case a). We only have to place the interval J between the point 0 and the interval I; then, if the statement were false, one would obtain a *positive* number y_0 or z_0, with $g(y_0) = +\infty$ or $g(z_0) = +\infty$. Now we show by a counterexample, that $g(t)$ need not be bounded in the neighbourhood of 0. Let us define the function $f(x)$ in the following way:

$$f(0) = -2$$

$$f(2\,n - 1) = f(2\,n) = -\,2^{n+1} \qquad (n = 1, 2, \ldots)\,.$$

Further put for $n = 1, 2, \ldots$

$$f\left(2\,n - 1 + \frac{1}{2^k}\right) = f(2\,n - 1) + 2^k = -\,2^{n+1} + 2^k \quad (k = n, n-1, \ldots, 1)$$

and

$$f\left(2\,n - 1 - \frac{1}{2^n}\right) = f\left(2\,n - 1 + \frac{1}{2^n}\right) = -\,2^n\,.$$

For any other nonnegative value x define $f(x)$ by linear interpolation (see Figure 1). Then it is easy to see that

$$g\left(\frac{1}{2^n}\right) = \limsup_{x \to +\infty} \left(f\left(x + \frac{1}{2^n}\right) - f(x)\right) = 2^n$$

and for $t \geqq \dfrac{1}{2^n}$ we have $g(t) \leq 2^n$. Thus $g(t)$ is finite for every positive number t but obviously

$$\lim_{t \to +0} g(t) = +\infty\,.$$

[2] If A_n $(n = 1, 2, \ldots)$ are arbitrary measurable sets with $\mu(A_n) \geqq \alpha$ and $\mu(\bigcup_{n=1}^{\infty} A_n) < +\infty$, then $\mu(\limsup_{n \to \infty} A_n) = \lim_{k \to \infty} \mu(\bigcup_{n=k}^{\infty} A_n) \geqq \alpha$.

Remarks.

1. The assumption that $g(t)$ is finite for every *real* t is obviously equivalent with the assumption that

$$h(t) = \limsup_{x \to +\infty} | f(x + t) - f(x) |$$

is finite for every *positive* t. Thus also the latter condition implies the boundedness of $g(t)$ — and of $h(t)$ — in every finite interval. In the counterexample given above $h(t) = +\infty$ for every t.

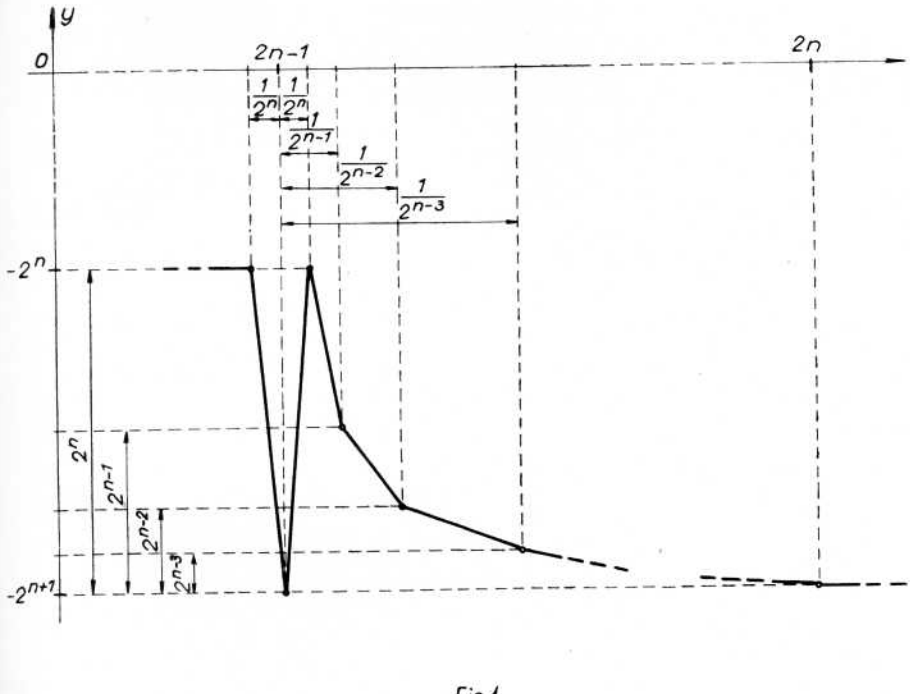

Fig. 1

2. If we do not assume the measurability of $f(x)$, then $g(t)$ can be finite for every real t without being bounded in any interval. That is the case e.g. for any non-measurable solution of the Cauchy functional equation $f(x + y) = f(x) + f(y)$.

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О ФУНКЦИИ $g(t) = \limsup_{x \to +\infty} \left(f(x+t) - f(x)\right)$

I. CSISZÁR и P. ERDŐS

Резюме

Пусть $f(x)$ — измеримая вещественная функция и положим

$$g(t) = \limsup_{x \to +\infty} \left(f(x+t) - f(x)\right).$$

Доказываются следующие установления:

а) *если функция* $g(t)$ *конечна при любом вещественном* t, *то она является ограниченной в любом конечном интервале;*

б) *если функция* $g(t)$ *конечна при любом положительном* t, *то она является ограниченной в любом закрытом подинтервале открытого интервала* $(0, +\infty)$; *но не должна быть ограниченной в окресности* 0.