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ON SOME EXTREMAL PROBLEMS IN GRAPH THEORY

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ABSTRACT

The author proves that if C is a sufficiently large constant then every graph of n vertices and $[Cn^{3/2}]$ edges contains a hexagon $X_1, X_2, X_3, X_4, X_5, X_6$ and a seventh vertex Y joined to X_1, X_3 and X_5. The problem is left open whether our graph contains the edges of a cube, (i.e. an eight vertex Z joined to X_2, X_4 and X_6).

Throughout this paper G, G' will denote graphs, $V(G)$ denotes the number of edges, $\pi(G)$ the number of vertices of G. $G(n;m)$ is a graph of n vertices and m edges. Vertices will be denoted by $x_1 \cdots y_1 \cdots$ edges by (x, y). $\{x_1, \cdots, x_n\}$ denotes a path whose edges are $(x_1, x_2), \cdots, (x_{n-1}, x_n)$, the vertices $x_1, \cdots, x_n$ are assumed distinct, $n - 1$ is the length of the path, similarly $(x_1, \cdots, x_n)$ is a circuit of length n whose edges are $(x_1, x_2), \cdots, (x_{n-1}, x_n), (x_n, x_1)$. $v(x)$, the valency of x is the number of edges incident to x. $G(x_1, \cdots, x_n)$ is the subgraph of G spanned by $(x_1, \cdots, x_n)$. In an even graph all circuits have even length. It is well known and easy to see that the vertices of an even graph can be divided into two classes A and B so that every edge joins a vertex of A to a vertex of B. $C, c, c_1 \cdots$ denote suitable positive absolute constants.

Recently several papers appeared which discussed various extremal problems in graph theory [1]. Denote by $f(n; k, l)$ the smallest integer for which every $G(n; f(n; k; l))$ contains *a* $G(k, l)$. Two years ago Turán asked me to determine or estimate the smallest integer m for which every $G(n;m)$ contains the various graphs determined by the vertices and edges of the regular polyhedra. For the tetrahedron the problem was solved many years ago by Turán himself [6], for the octahedron I proved several years ago that $(n^2/4) + cn^{3/2} < m < (n^2/4) + Cn^{3/2}$, details of the proof have not been published [1] and in this note we do not discuss the octahedron. The question for the dodecahedron and icosahedron seems difficult.

It is well known that $f(n; 4, 4) > cn^{3/2}$, but for a sufficiently large C every $f(n; [Cn^{3/2}])$ contains a rectangle [2]. One might conjecture that for a sufficiently large C every $G(n; [Cn^{3/2}])$ contains a cube. In fact I proved that $f(n; 8, 12) < Cn^{3/2}$, and I even showed that every $G(n; [Cn^{3/2}])$ contains a $G(8; 12)$ having the vertices,

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$x_1, x_2, x_3, x_4; y_1, y_2, y_3, y_4$ and the edges (x_i, y_j) where $\min(i,j) \leqq 2$ [3]. But at present I can not prove that it must contain a cube. I can prove the much weaker result that it contains a $G(7,9)$ consisting of a hexagon $(x_1, \cdots, x_6)$ and a vertex y joined to x_1, x_3 and x_5. To prove the existence of a cube we would need an eighth vertex z joined to x_2, x_4 and x_6, and I have not succeeded in showing this.

More precisely I am going to prove the following

THEOREM. *Let $n > n_0(k)$. Then every $G(n; 10[k^{1/2}n^{3/2}])$ contains a*

$$G(2k+1; 4k-2)$$

which has a path of length $2k\{x_1, y_1, \cdots, y_k, x_{k+1}\}$ and the further edges $(x_1, y_i), (y_1, x_j)$, $2 \leqq i \leqq k$, $3 \leqq j \leqq k+1$.

Clearly our $G(2k+1, 4k-2)$ contains for every $2 \leqq l \leqq k$ a circuit of length and another vertex joined to every second vertex of our circuit.

It seems likely that for a sufficiently large c_k every $G(n; [c_k n^{3/2}])$ contains a $G(1 + k + \binom{k}{2}; k^2)$ defined as follows: The vertices are $x_0; y_1, \cdots, y_k$; $z_{i,j}$, $1 \leqq i < j \leqq k$, x_0 is joined to all the y's and $z_{i,j}$ to y_i and y_j. I can not prove this for $k > 3$.

To prove our Theorem we need two lemmas.

LEMMA 1. *Every $G(n; m)$ has an even subgraph having at least $m/2$ edges.*

We prove the Lemma by induction for n. It is clearly true for $n \leqq 2$. Assume that it is true for $n-1$, we shall show it for n. Denote the vertices of $G(n; m)$ by $x_1, \cdots, x_n$. Since the lemma is true for $n-1$, we can split the vertices $x_1 \cdots x_{n-1}$ into two classer A and B so that the number of edges joining a vertex of A to a vertex of B is at least $\frac{1}{2}V(G(x_1, \cdots, x_{n-1}))$. Without loss of generality we can assume that the number of edges joining x_n to the vertices of B is at least $\frac{1}{2}v(x_n)$. But then the even graph spanned by the vertices $A \cup X_n$ and B has at least $\frac{1}{2}(V(G(x_{1\ldots}, x_{n-1}) + v(x_n)) \geqq (m/2)$ edges, which proves the Lemma.

By a slightly more careful induction process we can prove that if the graph $G(n; m)$ has no vertices of valency 0 then it contains an even graph having at least $\left[\frac{m}{2} + \frac{n}{4}\right]$ edges. The complete graph of n vertices $G\left(n; \binom{n}{2}\right)$ shows that this result is in general best possible. It seems probable that if we know that our $G(n; m)$ contains no triangle, the lemma can be considerably strengthened i.e. $m/2$ can perhaps be improved to cm for some $c > 1/2$, but I did not succeed in doing this.

LEMMA 2. *Every $G(n; m)$ contains a subgraph G' every vertex of which has valency (in G') greater that $[m/n]$.*

The Lemma is known [4]. The proof is very simple.

Now we can prove our Theorem. By Lemmas 1 and 2 our $G(n;10[k^{1/2}n^{3/2}])$ contains an even subgraph every vertex of which has valency greater than $5k^{1/2}n^{1/2}$. Let $x_1,\cdots,x_u;\ y_1\cdots,y_v\ u+v\leqq n$ be the vertices of G'. Let $y_1,\cdots,y_t$, $t>5k^{1/2}n^{1/2}$ be the vertices joined to x_1 and let $x_2,\cdots,x_{u'},\ u'\leqq u$ be the other x's joined to a y_i, $1\leqq i\leqq t$. G'' is the subgraph of G' spanned by $y_1,\cdots y_t, x_2\cdots x_{u'}$. Clearly each y in G'' has valency $>5k^{1/2}n^{1/2}-1>4k^{1/2}n^{1/2}$, i.e. each y_i has valency (in G') greater then $5k^{1/2}n^{1/2}$. Thus

$$V(G'')>4tk^{1/2}n^{1/2}. \tag{1}$$

Denote by $x_2,\cdots x_{u''}$ the x_i with

$$v(x_i)>2tk^{1/2}/n^{1/2}. \tag{2}$$

Let G''' be the subgraph of G'' spanned by $x_2,\cdots,x_{u''};\ y_1,\cdots,y_t$. By (1), (2) and $u''<n$ we have

$$V(G''')>V(G'')-2tk^{1/2}n^{1/2}>2tk^{1/2}n^{1/2}, \tag{3}$$

By (3) one of the y's has valency $>2k^{1/2}n^{1/2}$ (in G'''). Let this vertex be y_1 and let $x_2,\cdots x_{l+1}\ l>2k^{1/2}n^{1/2}$ be the vertices joined to y_1. Consider finally the graph $G'''(x_2,\cdots x_{l+1},\ y_2\cdots,y_t)$, each x_i has by (2) valency greater than $2t^{k^{1/2}}/n^{1/2}-1>tk^{1/2}/n^{1/2}$ $(t>4k^{1/2}n^{1/2})$. Thus by a simple computation

$$V(G'''(x_2,\cdots,x_{l+1},y_t,\cdots y_t))>\frac{tlk^{1/2}}{n}>k\pi(G'''(x_2,\cdots x_{l+1},y_2\cdots,y_t) \tag{4}$$

since by $t>4k^{1/2}n^{1/2}$,

$$l>2k^{1/2}n^{1/2}\quad\frac{tl}{t+l}>\frac{8kn}{6k^{1/2}n^{1/2}}>k^{1/2}n^{1/2}$$

and $\pi(G'''(x_2\cdots x_{l+1},y_2\cdots y_t))=l+t-1$.

From (4) we obtain by a theorem of Gallai and myself [5] that

$$G'''(x_2,\cdots x_{l+1},y_t\cdots y_t)$$

has a path of length $2k-2\{x_2,y_2,\cdots,y_k,x_{k+1}\}$. By our construction x_1 is joined to every y of our path and y_1 to every x of it. Thus finally $G'''(x_1,\cdots x_{l+1},y_1,\cdots y_k)$ satisfies the requirements of our Theorem.

The constant 10 could clearly be reduced, but I made no attempt in doing so since I am not sure if the factor $k^{1/2}$ is of the right order of magnitude.

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