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On a problem of Sierpiński

(Extract from a letter to W. Sierpiński)

by

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Denote by μ_s the least integer so that every integer $> u_s$ is the sum of exactly s integers > 1 which are pairwise relatively prime. Sierpiński ([3]) proved that $u_2 = 6$, $u_3 = 17$ and $u_4 = 30$ and he asks for a determination or estimation of u_s. Denote by $f_1(s)$ the smallest integer so that every $l > f_1(s)$ is the sum of s distinct primes; $f_2(s)$ is the smallest integer so that every $l > f_2(s)$ is the sum of s distinct primes or squares of primes where a prime and its square are not both used and $f_3(s)$ is the least integer so that every $l > f_3(s)$ is the sum of s distinct integers > 1 which are pairwise relatively prime. By definition $f_3(s) = u_s$. Clearly

$$f_3(s) \leqslant f_2(s) \leqslant f_1(s).$$

Let $p_1 = 2$, $p_2 = 3, \ldots$ be the sequence of consecutive primes. Put

$$A(s) = \sum_{i=1}^{s} p_i, \quad B(s) = \sum_{i=2}^{s+1} p_i.$$

Theorem. *$f_2(s) < B(s)+C$ where C is an absolute constant independent of s.*

First we prove two lemmas.

Lemma 1. *Let C_1 be a sufficiently large absolute constant. Then*

(1) $$f_1(s) < A(s)+c_1 s \log s.$$

We shall first prove

(2) $$f_1(s) < A(s)+c_1 s \log s \log\log s$$

and then we will outline the proof of (1).

Denote by $r_k(N)$ the number of representations of N as the sum of k odd primes. It easily follows from the well-known theorem of Hardy-Littlewood-Vinogradoff ([2], p. 198), that

(3) $$r_3(N) > c_2 N^2/(\log N)^3.$$

The well-known theorem of Schnirelmann ([2], p. 52) states

$$(4) \qquad r_2(N) < \frac{c_3 N}{(\log N)^2} \prod_{p|N} \left(1+\frac{1}{p}\right) < \frac{c_4 N \log\log N}{(\log N)^2}.$$

(The last inequality of (4) follows from the prime number theorem, or from a more elementary result.)

From (4) we obtain that the number of solutions of

$$(5) \qquad N = p_{i_1} + p_{i_2} + p_{i_3}, \quad i_1 \leqslant s$$

is less than

$$(6) \qquad c_4 s N \log\log N / (\log N)^2.$$

From (6) and (3) we obtain by a simple calculation that if $N > c_1 s \log s \log\log s$ then

$$(7) \qquad N = p_u + p_v + p_w, \quad s < u < v < w$$

is solvable (since the number of solutions of $N = 2p + q$ is clearly $< cN/\log N$).

Consider now the integer

$$A(s) + t, \quad t > c_1 s \log s \log\log s.$$

Put

$$t_1 = \begin{cases} p_{s-2} + p_{s-1} + p_s + t & \text{if } t \text{ is even}, \\ 2 + p_{s-1} + p_s + t & \text{if } t \text{ is odd}. \end{cases}$$

By (7)

$$t_1 = p_u + p_v + p_w, \quad s < u < v < w$$

is solvable. Thus $A(s) + t$ is the sum of s distinct primes which proves (2).

Now we outline the proof of (1). It is easy to see that (1) will follow if we can prove that for

$$(8) \qquad c_1 s \log s < N < c_1 s \log s \log\log s$$

the number of solutions $\psi(N)$ of (5) satisfies

$$(9) \qquad \psi(N) < c_4 s N / (\log N)^2.$$

But by the above mentioned theorem of Schnirelmann

$$(10) \qquad \psi(N) \leqslant \sum_{i=1}^{s} r_2(N - p_i) < \frac{c_3 N}{(\log N)^2} \sum_{i=1}^{s} \prod_{p|(N-p_i)} \left(1+\frac{1}{p}\right).$$

Now it can be proved that if N satisfies (8) then

$$\sum_{i=1}^{s} \prod_{p|(N-p_i)} \left(1+\frac{1}{p}\right) < c_5 s. \tag{11}$$

We supress the proof of (11) since it is not quite short but uses fairly standard arguments and it is of no great importance for us to have Lemma 1 in the sharpest possible form. (9) follows immediately from (10) and (11). Hence (1) is proved and the proof of Lemma 1 is complete.

The estimation given by Lemma 1 is best possible (apart from the value of c_1), since considerations of parity shows that $B(s)-2$ can not be the sum of distinct primes and clearly

$$B(s) > A(s)+c_6 s\log s \quad \text{(since } p_s > c_7 s\log s\text{)}.$$

Perhaps $f_1(s) = B(s)+o(s\log s)$ but this I have not been able to prove. It is easy to see though that

$$\limsup_{s=\infty} \big(f_1(s)-B(s)\big) = \infty$$

and probably

$$\lim_{s=\infty} \big(f_1(s)-B(s)\big) = \infty.$$

LEMMA 2. *Put $a_k = p_k^2 - p_k$, $k \geqslant 2$. Then there exists an absolute constant A so that every even integer greater than A is the sum of distinct a_k's.*

One can easily deduce Lemma 2 from a theorem of Cassels ([1]) (it easily follows from the results on Vinogradoff ([4]) that if $0 < a < 1$ then $\binom{p}{2}a \pmod 1$ has at least one limit point different from 0, thus the theorem of Cassels can be applied). An elementary and direct proof of Lemma 2 should be possible which would have the advantage of determining the best possible value of A. Such a proof would perhaps require a considerable amount of numerical calculation and I have not carried it out.

Now we are ready to prove our Theorem. We shall in fact show that for $s > s_0(c_1)$

$$f_2(s) \leqslant B(s)+A. \tag{12}$$

Let now $n \geqslant B(s)+A$. If $n > A(s)+c_1 s\log s\log\log s$ then by Lemma 1 n is the sum of s distinct primes (we only use (2)). Thus we can assume

$$B(s)+A < n < A(s)+c_1 s\log s\log\log s.$$

Assume first $n = B(s)+2t$. Since $2t > A$, by Lemma 2

$$2t = a_{k_1}+\ldots+a_{k_r}, \quad k_1 < \ldots < k_r,$$

but $2t < c_1 s \log s \log\log s$ clearly implies that for $s > s_0 = s_0(c_1)$, $k_r \leqslant s$ (since $a_s = p_s^2 - p_s > c_1 s \log s \log\log s$). Thus

$$B(s) + 2t = \sum_{i=2}^{s+1} p_i + \sum_{i=1}^{r} a_{k_i}$$

gives a representation of $B(s)+2t$ as the sum of s distinct primes or squares of primes where p and p^2 are not both used.

Assume next $n = B(s)+2t+1$. Then $n = A(s)+2t_1$, $2t_1 < cs\log s \times \times \log\log s$. Thus the same proof again gives that n is the sum of s distinct primes of squares of primes where p and p^2 are not both used. Thus (12) and hence our Theorem is proved (the cases $s \leqslant s_0$ can be ignored because of Lemma 1).

Finally we remark that $f_3(s) \geqslant B(s)-2$ since $B(s)-2$ can not be the sum of s distinct integers >1 which are pairwise relatively prime. To see this we only have to observe that by considerations of parity no even number can occur in such a representation.

References

[1] J. W. S. Cassels, *On the representation of integers as the sums of distinct summands taken from a fixed set,* Acta Szeged 21 (1960), pp. 111 - 124.

[2] K. Prachar, *Primzahlverteilung,* Springer 1957.

[3] W. Sierpiński, *Sur les suites d'entiers deux à deux premiers entere eux,* Eenseignement Math. 10 (1964), pp. 229 - 235.

[4] I. M. Vinogradoff, *The method of trigonometrical sums in the theory of numbers,* Interscience Publishers, Chapter XI.

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