LARGE AND SMALL SUBSPACES OF HILBERT SPACE

P. Erdös, H. S. Shapiro, and A. L. Shields

In this paper we consider closed subspaces V of sequential Hilbert space ℓ_2 and of $L_2(0, 1)$. Our results are of two types: (1) if all the elements of V are "small," then V is finite-dimensional; (2) there exist infinite-dimensional subspaces V containing no small elements (except 0).

For example, Theorem 3 says that if V is a closed subspace of ℓ_2 and if $V \subset \ell_p$ for some $p < 2$, then V is finite-dimensional. On the other hand, the corollary to Theorem 4 states that there exist infinite-dimensional subspaces V of ℓ_2 none of whose nonzero elements belongs to any ℓ_p-space $(p < 2)$. [For $L_2(0, 1)$ the results are somewhat different: (1) if V is a closed subspace of $L_2(0, 1)$ and if $V \subset L_\infty$, then V is finite-dimensional. Theorem 6 gives a condition for the finite-dimensionality of V in terms of Orlicz spaces, and by Theorem 5 this condition is best possible; in particular, L_∞ cannot be replaced by L_q for any $q < \infty$. (2) There exist infinite-dimensional subspaces of L_2 none of whose nonzero elements is in any L_q-space $(q > 2)$ (Theorem 7)].

Since the elements $x \in \ell_2$ are functions $x = (x(1), x(2), \cdots)$ on the nonnegative integers, there are various ways of defining "small" elements. For example, Theorem 1 states that if all the elements $x \in V$ satisfy a condition $|x(n)| = O(\rho_n)$, where $\Sigma \rho_n^2 < \infty$, then V is finite-dimensional. On the other hand, Theorem 2 states that if $\Sigma \rho_n^2 = \infty$ then there exists an infinite-dimensional closed subspace V all of whose elements satisfy the condition $|x(n)| = O(\rho_n)$, but none of whose elements (except 0) satisfies the condition $|x(n)| = o(\rho_n)$.

Theorem 8 gives a formula for the exact dimension of any closed subspace V of ℓ_2. The paper concludes with an application of Theorem 8 to a problem involving bounded analytic functions in the unit disc: we give an elementary proof that an inner function cannot have a finite Dirichlet integral unless it is a finite Blaschke product.

We need the following compactness criterion [3, Chapter I, Section 10]:

If $\rho_n \geq 0$ and $\Sigma \rho_n^2 < \infty$, then $\{x: x \in \ell_2, \ |x(n)| \leq \rho_n\}$ is compact.

THEOREM 1. *Let* V *be a closed subspace of* ℓ_2, *and let* $\{\rho_n\}$ *be given, with* $\rho_n \geq 0$ *and* $\Sigma \rho_n^2 < \infty$. *If* $|x(n)| = O(\rho_n)$ *for all* $x \in V$, *then* V *is finite-dimensional.*

Proof. Let $V_m = \{x: x \in V, \ |x(n)| \leq m\rho_n \text{ for all } n\}$. Then V_m is compact and hence, if V were infinite-dimensional, V_m would be nowhere dense in V. But this would contradict the Baire category theorem, since $V = \bigcup V_m$.

THEOREM 2. *Let* $\rho_n > 0$, $\rho_n \to 0$ *and* $\Sigma \rho_n^2 = \infty$. *Then there exists an infinite-dimensional subspace* V *of* ℓ_2 *such that for each* $x \in V$

(i) $|x(n)| = O(\rho_n)$,

(ii) $|x(n)| = o(\rho_n) \Rightarrow x = 0$.

Received September 17, 1964.

[In other words, the elements of V are not too large, but nonetheless V contains no small elements.]

We omit the proof of the following lemma.

LEMMA 1. *If* $d_n > 0$, $d_n \to 0$, *and* $\Sigma d_n = \infty$, *then there exists an infinite subset* N *of the positive integers such that* $\Sigma_{i \in N} d_i = 1$.

Proof of Theorem 2. Let $N_1, N_2, \cdots$ be disjoint infinite subsets of the positive integers such that

$$\sum_{i \in N_j} \rho_i^2 = 1 \quad (j = 1, 2, \cdots).$$

(Apply the lemma repeatedly, each time deleting the subset N_j selected at the previous stage.)

Consider the functions $f_1, f_2, \cdots$ given by

$$f_j(i) = \begin{cases} \rho_i & (i \in N_j), \\ 0 & (i \notin N_j). \end{cases}$$

Then $\{f_j\}$ is an infinite orthonormal set in ℓ_2. Let V be the subspace spanned by it. Each $x \in V$ has the form $x = \Sigma a_j f_j$ $(\Sigma |a_j|^2 < \infty)$. Let $M = \max |a_j|$. Then

$$|x(i)| \leq M\rho_i \quad \text{for all } i,$$

which establishes (i). On the other hand, if $x \neq 0$ then at least one coefficient, say a_1, is not zero. Hence

$$\limsup_{i \to \infty} \frac{|x(i)|}{\rho_i} \geq \limsup_{i \in N_1} \frac{|a_i \rho_i|}{\rho_i} = |a_1| > 0,$$

which establishes (ii).

Remark. If all the elements of a closed subspace satisfy an O-condition, then they satisfy it uniformly. More precisely, if $|x(n)| = O(\rho_n)$ for every $x \in V$, then there exists a constant M such that

$$|x(n)| \leq M\rho_n \|x\| \quad (x \in V). \tag{1}$$

Proof. Let e_n be the n-th coordinate functional, that is, let $(x, e_n) = x(n)$ for all x. Assume $\rho_n > 0$ for all n (since (1) holds automatically for indices n for which $\rho_n = 0$).

Let $f_n = e_n/\rho_n$. By hypothesis, to each $x \in V$ there corresponds a constant c_x such that $|(x, f_n)| \leq c_x$ for all n; that is, the functionals $\{f_n\}$ are pointwise bounded on V. By the uniform-boundedness principle, there exists an M such that $\|f_n\| \leq M$ for all n.

THEOREM 3. *If* V *is a closed subspace of* ℓ_2 *and* $V \subset \ell_p$ *for some* $p < 2$, *then* V *is finite-dimensional.*

Proof. Choose $\varepsilon_n > 0$ with $\Sigma \varepsilon_n < \infty$. For $x \in V$, let

$$N_k(x) = \sum_{n \leq k} |x(n)|^p, \qquad R_k(x) = \sum_{n > k} |x(n)|^p.$$

Choose an $x_1 \in V$ such that $1 < R_0(x_1) < 1 + \varepsilon_1^p$, and an n_1 such that $N_{n_1}(x_1) \geq 1$.

If V were infinite-dimensional, then by suitably combining $n_1 + 1$ linearly independent vectors we could produce an $x_2 \in V$ such that

$$x_2(n) = 0 \quad (n \leq n_1) \qquad \text{and} \qquad 1 < R_{n_1}(x_2) < 1 + \varepsilon_2^p.$$

Now choose n_2 so that $N_{n_2}(x_2) \geq 1$.

Continuing in this manner, we construct a sequence $\{x_k\} \subset V$ and an increasing sequence of positive integers $\{n_k\}$ such that, for all k,

$$x_{k+1}(n) = 0 \quad (n \leq n_k), \qquad 1 < R_{n_k}(x_{k+1}) < 1 + \varepsilon_{k+1}^p, \qquad N_{n_k}(x_k) \geq 1.$$

Let

$$f_k(n) = \begin{cases} x_k(n) & (n \leq n_k), \\ 0 & (n > n_k), \end{cases}$$

$$g_k = x_k - f_k.$$

Then $\{f_k\}$ is a bounded orthogonal family in ℓ_2, and

$$\|f_k\|_p \geq 1, \qquad \|g_k\|_p \leq \varepsilon_k. \tag{2}$$

Let $\{a_k\}$ be a square-summable sequence of positive numbers that is not in ℓ_p, and let

$$y_1 = \sum a_k f_k, \qquad y_2 = \sum a_k g_k, \qquad y = y_1 + y_2.$$

By the Riesz-Fischer theorem, the series for y_1 converges in ℓ_2. Since $\Sigma a_k \|g_k\|_p < \infty$, the series for y_2 converges in ℓ_p and hence in ℓ_2 (the 2-norm of an element is less than or equal to the p-norm). Thus $y = \Sigma a_k x_k$, with the series converging in ℓ_2, and so $y \in V$.

However, $y_1 \notin \ell_p$. Indeed,

$$\sum |y_1(n)|^p = \sum |a_k|^p \, \|f_k\|_p^p = \infty$$

by (2). But $y_2 \in \ell_p$, and so $y \notin \ell_p$.

THEOREM 4. *If* $\rho_n \geq 0$ *and* $\Sigma \rho_n^2 = \infty$, *then there exists an infinite-dimensional subspace* V *of* ℓ_2 *such that* $\Sigma |x(n)| \rho_n = \infty$ *for all* $x \neq 0$ *in* V.

Proof. Divide the positive integers into a countable number of disjoint infinite subsets $N_1, N_2, \cdots$ such that

$$\sum_{i \in N_j} \rho_i^2 = \infty \qquad (j = 1, 2, \cdots).$$

Choose unit vectors $f_1, f_2, \cdots$ in ℓ_2 such that $f_j(i) = 0$ for $i \notin N_j$ and

$$\sum_{i \in N_j} f_j(i)\rho_i = \infty \qquad (j = 1, 2, \cdots).$$

Then $\{f_j\}$ is an infinite orthonormal set; let V be the subspace spanned by it. Let $x = \Sigma a_j f_j \in V$, $x \neq 0$. At least one coefficient, say a_1, is not zero. Thus

$$\sum |x(n)| \rho_n \geq |a_1| \sum_{i \in N} |f(i)| \rho_i = \infty.$$

COROLLARY. *There exists an infinite-dimensional subspace* V *of* ℓ_2 *none of whose nonzero elements belongs to any* ℓ_p $(p < 2)$.

Proof. Choose $\{\rho_n\} \in \ell_q$ for all $q > 2$, with $\Sigma \rho_n^2 = \infty$, and apply Theorem 4. By Hölder's inequality, no nonzero element of V can belong to any class ℓ_p $(p < 2)$.

We now consider $L_2(0, 1)$. Here the situation is quite different. Since $L_q \subset L_2$ for $q > 2$, the analogue of Theorem 3 would be that if a closed subspace V of L_2 is contained in L_q for some $q > 2$, then V is finite-dimensional. This is false, however, as the following theorem shows.

THEOREM 5. *There exists an infinite-dimensional closed subspace* V *of* $L_2(0, 1)$ *each of whose elements* f *belongs to every class* L_q $(q < \infty)$, *and in fact satisfies the condition*

$$\int \exp \{c\, |f(x)|^2\}\, dx < \infty \tag{3}$$

for every $c > 0$.

Proof. This is well known from the theory of Fourier series: let V be the subspace spanned by the Rademacher functions (see [8, Chapter V, Section 8.7]).

In Theorem 5, we cannot take $q = \infty$; in fact, condition (3) is "best possible."

THEOREM 6. *Let* V *be a closed subspace of* L_2 *over a finite measure space. Let* $\phi(x)$ *be a convex, continuous, strictly increasing function on* $[0, \infty)$ *with* $\phi(0) = 0$, *and with*

$$\lim_{x \to \infty} \phi(x)\, e^{-cx^2} = \infty \tag{4}$$

for each $c > 0$. *If* $\int \phi(|f|)\, d\mu < \infty$ *for all* $f \in V$, *then* V *is finite-dimensional.*

COROLLARY. *If* V *is a closed subspace of* L_2 *over a finite measure space, and if each function in* V *is essentially bounded, then* V *is finite-dimensional.*

Before proving the theorem, we introduce some notations about Orlicz spaces that will be used in the proof.

Let B_ϕ denote the set of measurable functions f for which

$$\int \phi(|f|)\, d\mu \leq 1.$$

E_ϕ is the set of functions f some constant multiple of which belongs to B_ϕ ($\gamma f \in B_\phi$ for some $\gamma > 0$). We do not distinguish between functions that agree almost everywhere. The proof will show that Theorem 6 is valid under the weaker hypothesis that $V \subset E_\phi$.

The Orlicz norm $\|f\|_\phi$ in E_ϕ is defined as follows:

$$\|f\|_\phi = 0 \quad \text{if and only if } f = 0 \text{ a.e.};$$

otherwise, $\|f\|_\phi$ is the reciprocal of the (unique) positive number c for which $\int \phi(c|f|)\, d\mu = 1$ (since ϕ is strictly increasing, c is well-defined).

With this norm, E_ϕ is a Banach space; B_ϕ is the unit ball. For a discussion along these lines see [4] and [7]. Using (4), we can show that $L_\infty \subset E_\phi \subset L_q$ for all $q < \infty$.

It will be convenient to modify the function ϕ somewhat. Let

$$\phi^*(x) = \max(\phi(x), x^2).$$

Then ϕ^* is a convex, continuous, strictly increasing function on $[0, \infty)$ satisfying (4) and

$$\phi^*(x) \geq x^2 \quad (x \geq 0). \tag{5}$$

Finally, $E_{\phi^*} = E_\phi$, since $\phi^*(x) = \phi(x)$ for all sufficiently large x. Since the proof of Theorem 6 will only require the hypothesis $V \subset E_\phi$, we may replace ϕ by ϕ^*. In other words, dropping the star, we may assume in what follows that the function $\phi(x)$ of Theorem 6 also satisfies (5).

LEMMA 2. $E_\phi \subset L_2$ *and* $\|f\|_2 \leq \|f\|_\phi$ *for all* $f \in E_\phi$.

Proof. It suffices to show that if $f \in B_\phi$, then $f \in L_2$ and $\|f\|_2 \leq 1$. This follows immediately from (5):

$$\int |f|^2\, d\mu \leq \int \phi(|f|)\, d\mu \leq 1.$$

We now assume that V is a closed subspace of L_2 and that $V \subset E_\phi$.

LEMMA 3. V *is a closed subspace of* E_ϕ.

Proof. Let $\{f_n\} \subset V$, $f_n \to f$ in E_ϕ. By Lemma 2, $f_n \to f$ in L_2, and hence $f \in V$.

LEMMA 4. *On* V *the* ϕ-*norm and the* 2-*norm are equivalent.*

This is a well-known result of Banach [1, Chapter III, Section 3].

Proof of Theorem 6. Assume that V is infinite-dimensional, and let $\{h_n\}$ be an orthonormal basis for V. Let

$$E_{nk} = \{x: |h_n(x)| \leq k\}.$$

We distinguish two cases, and we show that Lemma 4 is violated in both. In the first case much more is true: V cannot be a subset of L_q for any $q > 2$.

Case I. There exists a $\delta > 0$ such that

$$\inf_n \int_{E_{nk}} |h_n|^2 d\mu < 1 - \delta \quad \text{(all } k).$$

Here, for each k there exists an n such that $\int_F |h_n|^2 d\mu > \delta$, where F denotes the complement of E_{nk}. For each $q > 2$,

$$\int |h_n|^q \geq \int_F |h_n|^q > k^{q-2} \int_F |h_n|^2 > \delta k^{q-2}.$$

Since k is arbitrary, the q-norm is not equivalent to the 2-norm on V, and thus V cannot be a subset of L_q.

Case II.

$$\sup_k \inf_n \int_{E_{nk}} |h_n|^2 d\mu = 1.$$

All we really require is that

$$\int_{E_{nk}} |h_n|^2 d\mu \geq d > 0 \tag{6}$$

for some fixed k, some fixed constant d, and for infinitely many values of n.

We now show that (6) implies the existence of a positive constant δ and a sequence of measurable sets $\{F_n\}$ such that

$$\mu(F_n) \geq \delta, \qquad |h_n(x)| > \delta \quad (x \in F_n). \tag{7}$$

Indeed, fix an $\alpha > 0$ such that $\alpha^2 < \min(d, k^2)$, and choose any n for which (6) holds. Let G_n be the subset of E_{nk} where $|h_n| \leq \alpha$, and let $F_n = E_{nk} - G_n$. Then

$$\begin{aligned} d \leq \int_{E_{nk}} |h_n|^2 d\mu &= \int_{G_n} + \int_{F_n} \\ &\leq \alpha^2 \mu(G_n) + k^2 \mu(F_n) = (k^2 - \alpha^2)\mu(F_n) + \alpha^2 \mu(E_{nk}). \end{aligned}$$

Since $\mu(E_{nk})$ cannot exceed the measure of the whole space, which we take to be 1, we have the inequality

$$\mu(F_n) \geq \frac{d - \alpha^2}{k^2 - \alpha^2} > 0,$$

which establishes (7).

By considering real and imaginary parts of h_n on suitable subsets of F_n (which we continue to denote by F_n), choosing a smaller δ, and passing to a subsequence, we may assume that

$$(8) \qquad \mu(F_n) \geq \delta, \quad \Re h_n(x) \geq \delta \quad (x \in F_n,\ n = 1, 2, \cdots).$$

By a result of Visser [6], there exists a subsequence of $\{F_n\}$, which we continue to denote by $\{F_n\}$, for which

$$(9) \qquad \mu(F_1 \cap F_2 \cap \cdots \cap F_n) \geq \frac{1}{2}\delta^n \quad (n = 1, 2, \cdots).$$

Let $f_n = (h_1 + \cdots + h_n)/\sqrt{n}\delta$. Then $\|f_n\|_2 = 1/\delta$ and $\Re f_n \geq \sqrt{n}$ on a set E_n of measure at least $\frac{1}{2}\delta^n$. Choose $c > 0$ such that $\delta e^c > 1$. Then

$$(10) \qquad \int_{E_n} \exp(c\,|f_n|^2)\,d\mu \geq \frac{1}{2}\delta^n e^{cn} \to \infty \quad (n \to \infty).$$

We assert that $\|f_n\|_\phi \to \infty$. Indeed, fix $\varepsilon > 0$ and choose an N such that $\phi(\varepsilon x) \geq \exp(cx^2)$ for $x \geq N$. Then, for $n \geq N^2$, we have the relation

$$\int \phi(\varepsilon\,|f_n|)\,d\mu \geq \int_{E_n} \phi(\varepsilon\,|f_n|)\,d\mu \geq \int_{E_n} \exp(c\,|f_n|^2)\,d\mu\,;$$

the last member tends to infinity, by (10). Hence $\|f_n\|_\phi \geq 1/\varepsilon$. Thus Lemma 4 is contradicted, and this completes the proof.

We now establish a theorem analogous to Theorem 4.

THEOREM 7. *If* $h(x) \geq 0$ $(0 \leq x \leq 1)$ *and* $\int h^2\,dx = \infty$, *then there exists an infinite-dimensional subspace* V *of* $L_2(0, 1)$ *such that* $\int |fh|\,dx = \infty$ *for all* $f \in V$ $(f \neq 0)$.

Proof. Let $E_n = \{x\colon n \leq h^2(x) < n + 1\}$ $(n = 0, 1, \cdots)$ and let

$$\rho_n^2 = \int_{E_n} h^2\,dx < \infty \quad (n = 0, 1, \cdots).$$

Then $\Sigma \rho_n^2 = \infty$. Let $N_1, N_2, \cdots$ be disjoint subsets of the positive integers such that

$$\sum_{i \in N_j} \rho_i^2 = \infty \quad (j = 1, 2, \cdots),$$

and let $F_j = \bigcup_{i \in N_j} E_i$. Let $g_1, g_2, \cdots$ be nonnegative functions, with g_j supported in F_j, $\int g_j^2 = 1$, and $\int hg_j = \infty$ $(j = 1, 2, \cdots)$. Then $\{g_j\}$ is an orthonormal set. Finally, let V be the subspace spanned by $\{g_j\}$, and let $f = \Sigma a_j g_j \in V$. At least one coefficient, say a_1, is not zero. Thus

$$\int_0^1 |fh|\,dx \geq |a_1| \int_{F_1} g_1\, h\,dx = \infty .$$

COROLLARY. *There exists an infinite-dimensional subspace* V *of* $L_2(0, 1)$ *none of whose elements (except zero) is in any space* L_q $(q > 2)$.

The proof is similar to the proof of the corollary to Theorem 4.

We now return to sequential Hilbert space ℓ_2. Evaluation at the n-th coordinate is a continuous functional on any closed subspace $V \subset \ell_2$. Hence there exist elements $\lambda_1, \lambda_2, \cdots$ in V for which

$$(x, \lambda_n) = x(n) \quad (x \in V). \tag{11}$$

(If we regard V as a Hilbert space of functions on the positive integers, then $\lambda_n(j)$ is the "reproducing kernel" for V.) Thus

$$|x(n)| \leq \|x\|\,\|\lambda_n\| \quad (\text{all } n, \text{ all } x \in V).$$

THEOREM 8. *Let* V *be a closed subspace of* ℓ_2, *and let* $\{\lambda_n\} \subset V$ *be the coordinate functionals* (11). *Then*

$$\dim V = \sum \|\lambda_n\|^2$$

(finite or infinite).

Proof. Let $\{x_j\}$ be an orthonormal basis for V. Then

$$\dim V = \sum_j \|x_j\|^2 = \sum_j \sum_n |x_j(n)|^2 = \sum_n \sum_j |(x_j, \lambda_n)|^2 = \sum_n \|\lambda_n\|^2 .$$

Theorem 8 has an application to the theory of bounded analytic functions. We require a few definitions.

An *inner function* is an analytic function $\phi(z) = \sum a_n z^n$, bounded by 1 in the unit disc, whose radial boundary values have modulus 1 almost everywhere. Equivalently,

$$\sum_{n=0}^{\infty} a_n \bar{a}_{n+k} = \begin{cases} 0 & (k = 1, 2, \cdots), \\ 1 & (k = 0). \end{cases}$$

We shall show that if ϕ is an inner function, then $\sum n|a_n|^2 < \infty$ (that is, ϕ has a finite Dirichlet integral) if and only if ϕ is a finite Blaschke product. This result was proved in [5] by means of the theory of dual extremal problems. Our proof is based on Theorem 8. For a discussion of inner functions and of the Hilbert space H_2 of power series with square-summable Taylor coefficients, see [2, Chapter 5].

By ϕH_2 we denote the subspace of H_2 consisting of all multiples of ϕ. It is a closed subspace, since multiplication by ϕ is an isometry. We state the following lemma without proof.

LEMMA 5. *Let* ϕ *be an inner function, and let* $V = (\phi H_2)^{\perp}$. *Then* V *is finite-dimensional if and only if* ϕ *is a finite Blaschke product, in which case the dimension of* V *is the number of factors in the product.*

THEOREM 9. *Let* $\phi(z) = \Sigma a_n z^n$ *be an inner function. Then*

$$\sum n\,|a_n|^2 = \dim(\phi H_2)^{\perp}.$$

Thus the Dirichlet integral of ϕ *is finite (and is then an integral multiple of* π*) if and only if* ϕ *is a finite Blaschke product.*

Proof. Let W denote the subspace ϕH_2, and let V be its orthogonal complement. Let $\{e_n\}$ be the usual orthonormal basis for ℓ_2 ($e_n(j) = \delta_{nj}$) and, as in Theorem 8, let $\{\lambda_n\}$ denote the coordinate functionals in V. Finally, let $\{\mu_n\}$ be the coordinate functionals in W. Then $e_n = \lambda_n + \mu_n$.

Note that $\{z^k\phi\}$ ($k = 0, 1, \cdots$) is an orthonormal basis for W. Hence

$$\|\lambda_n\|^2 = 1 - \|\mu_n\|^2 = 1 - \sum_k |(z^k\phi, \mu_n)|^2 = 1 - \sum_{k \leq n} |a_{n-k}|^2 = \sum_{k > n} |a_k|^2,$$

since $\Sigma\,|a_k|^2 = 1$. Summing on n, we see that

$$\dim V = \sum n\,|a_n|^2.$$

Our results can be applied to other function spaces. For example, let H denote the space of entire functions $f = \Sigma a_n z^n$ with norm

$$\|f\|^2 = \sum n!\,|a_n|^2.$$

These functions all satisfy the condition

$$|f(z)|^2 = o(e^{r^2}) \qquad (r = |z|),$$

hence they all have order at most 2, and if the order is 2 then the type is at most 1/2. Suppose that V is a closed subspace of H and that

$$|f(z)|^2 = O(e^{r^2}/r^4)$$

for all $f \in V$. Then, using Cauchy's inequality for the Taylor coefficients, together with Theorem 1, we can show that V is finite-dimensional. Hence every closed infinite-dimensional subspace of H contains functions of order 2 and type 1/2.

On the other hand, our results do not answer the following question: does H_2 contain an infinite-dimensional closed subspace V with

$$|f(z)| = O\left(\frac{1}{(1 - |z|)^{1/4}}\right) \qquad (|z| < 1)$$

for all $f \in V$?

In conclusion, we mention a problem that arose in this work and was left unsettled. Let T be a bounded linear transformation from ℓ_q to ℓ_2 for some $q > 2$. Since $\ell_2 \subset \ell_q$, we may restrict T to ℓ_2, thereby obtaining a map of ℓ_2 into itself. Is this new map necessarily completely continuous?

Added in proof. Dr. Stephen Parrott has pointed out to us that this last question has an affirmative answer. In outline, the idea is to consider the adjoint map T^* from ℓ_2 to $\ell_p \subset \ell_2$. Using Theorem 3, one can show that the range of T^*, regarded as a subset of ℓ_2, contains no closed infinite-dimensional subspace. The complete continuity follows from this, *via* the polar decomposition and the spectral theorem.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.
2. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
3. L. A. Ljusternik and V. I. Sobolev, *Elements of functional analysis*, Moscow, 1951.
4. W. A. J. Luxemburg and A. C. Zaanen, *Conjugate spaces of Orlicz spaces*, Indag. Math. 18 (1956), 217-228.
5. D. J. Newman and H. S. Shapiro, *The Taylor coefficients of inner functions*, Michigan Math. J. 9 (1962), 249-255.
6. C. Visser, *On certain infinite sequences*, Nederl. Akad. Wetensch. Proc. 40 (1937), 358-367.
7. G. Weiss, *A note on Orlicz spaces*, Portugal. Math. 15 (1956), 35-47.
8. A. Zygmund, *Trigonometric series*, Cambridge University Press, 1959.

The Hungarian Academy of Sciences
and
The University of Michigan