A PROBLEM ON INDEPENDENT r-TUPLES

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$G(n; l)$ denotes a graph of n vertices and l edges. A set of edges is called independent if no two of them have a vertex in common. GALLAI and I [1] proved that if

$$l > \max\left(\binom{2k-1}{2},\ (k-1)(n-k+1)+\binom{k-1}{2}\right) \tag{1}$$

then $G(n; l)$ contains k independent edges. It is easy to see that the above result is best possible since the complete graph of $2k-1$ vertices and the graph of vertices $x_1, \dots, x_{k-1}; y_1, \dots, y_{n-k+1}$ and edges (x_i, x_j), $1 \le i < j \le k-1$; (x_i, y_j), $1 \le i \le k-1$, $1 \le y_j \le n-k+1$ clearly does not contain k independent edges.

By an r-graph $G^{(r)}$ we shall mean a graph whose basic elements are its vertices and r-tuples; for $r = 2$ we obtain the ordinary graphs. $G^{(r)}(n; m)$ will denote an r-graph of n vertices and m r-tuples. For $r > 2$ these generalised graphs have not yet been investigated very much. A set of r-tuples is called independent if no two of them have a vertex in common.

$f(n; r, k)$ denotes the smallest integer so that every $G^{(r)}(n; f(n; r, k))$ contains k independent r-tuples. (1) implies that

$$f(n; 2, k) = 1+\max\left(\binom{2k-1}{2},\ (k-1)(n-k+1)+\binom{k-1}{2}\right). \tag{2}$$

It does not seem easy to determine $f(n; r, k)$ for $r > 2$ and every k. For $k = 2$ KO, RADO and I [2] proved that for $n \ge 2r$

$$f(n; r, 2) = \binom{n-1}{r-1}+1. \tag{3}$$

The case $n < 2r$ is trivial since then no two r-tuples are independent.

Denote by $g(n; r, k-1)$ the number of those r-tuples formed from the elements $x_1, \dots, x_n$ each of which contain at least one of the elements $x_1, \dots, x_{k-1}$. Clearly $f(n; r, k) > g(n; r, k-1)$ and a simple computation shows that

$$g(n; r, k-1) = \sum' \binom{k-1}{i}\binom{n-k+1}{r-i} \ge (k-1)\binom{n-k+1}{r-1} \tag{4}$$

where the dash indicates that i runs from 1 to $\min(r, k-1)$.

Now we prove the following

THEOREM. *For $n > c_r k$ (c_r is a constant which depends only on r)*

$$f(n; r, k) = 1 + g(n; r, k-1).$$

The proof uses induction with respect to k. For $k = 2$ the result is known [2]. We assume that it holds for $k-1$ and prove it for k.

Let $n > c_r k$ and consider an arbitrary $G^{(r)}(n; 1+g(n; r, k-1))$. Denote by $\nu(x_i)$ the number of r-tuples in our $G^{(r)}(n; 1+g(n; r, k-1))$ which contain x_i. Without loss of generality we can assume that $\max_{1 \leq i \leq n} \nu(x_i) = \nu(x_1)$. We distinguish two cases. Assume first

$$\nu(x_1) < \frac{1+g(n; r, k-1)}{(k-1)r} \tag{5}$$

and let $R_1, \ldots, R_l$ be a maximal system of independent r-tuples of our $G^{(r)}$. We show

$$l \geq k. \tag{6}$$

If (6) would be false our r-tuples $R_1, \ldots, R_l$ would contain at most $(k-1)r$ vertices and by (5) the number of r-tuples containing any of these vertices is less than

$$1 + g(n; r, k-1).$$

Thus our $G^{(r)}(n; 1+g(n; r, k-1))$ contains an R_{l+1} which is independent of all the R_i, $1 \leq i \leq l$, which contradicts the maximality of $R_1, \ldots, R_l$, hence $l < k$ leads to a contradiction, which proves (6) and disposes of the first case.

Now we consider the second case, that is, we assume

$$\nu(x_1) \geq \frac{1+g(n; r, k-1)}{(k-1)r}. \tag{7}$$

Consider now the r-graph $G^{(r)}$ whose vertices are $x_2, \ldots, x_n$ and whose r-tuples are those r-tuples of our $G^{(r)}(n; 1+g(n; r, k-1))$ which do not contain x_1. The number of r-tuples of $G_1^{(r)}$ is clearly at least

$$1 + g(n; r, k-1) - \binom{n-1}{r-1} = 1 + g(n-1, r, k-1), \tag{8}$$

since there are at most $\binom{n-1}{r-1}$ r-tuples containing x_1. Thus by our induction hypothesis $G_1^{(r)}$ contains at least $k-1$ independent r-tuples $R_1, \ldots, R_{k-1}$. The proof of our Theorem will be complete if we succeed to show that there is an r-tuple of our $G^{(r)}(n; 1+g(n; r, k-1))$ containing x_1 which does not contain any of the $(k-1)r$ vertices of $R_1, \ldots, R_{k-1}$. To see this observe that the number of r-tuples containing x_1 and x_i is at most $\binom{n-2}{r-2}$, and therefore the number of r-tuples containing x_1 and one of the vertices of $R_1, \ldots, R_{k-1}$ is at most $(k-1)r\binom{n-2}{r-2}$. By (7) and (4) we obtain by a simple computation that for $n > c_r k$

$$(k-1)r\binom{n-2}{r-2} < v(x_1);$$

hence there is an r-tuple of our $G^{(r)}(n; 1+g(n; r, k-1))$ containing x_1 which is disjoint from $R_1, \ldots, R_{k-1}$, as stated. This completes the proof of our theorem.

It is not impossible that

$$f(n; r, k) = 1+\max\left\{\binom{rk-1}{r},\ g(n; r, k-1)\right\}. \tag{9}$$

For $r = 2$ (9) is implied by (1) and for $k=2$ (9) is proved in [2], but the general case seems elusive.

References

[1] P. Erdős and T. Gallai, On the maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hung.*, **10** (1959), 337–357.
[2] P. Erdős, Chao Ko and R. Rado, Intersection theorems for systems of finite sets, *Quarterly J. of Math.*, **12** (1961), 313–320.

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