ON THE REPRESENTATION OF DIRECTED GRAPHS AS UNIONS OF ORDERINGS

by

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Introduction

Consider an $m \times n$ matrix in which each row consists of a permutation of the integers $1, 2, \ldots, n$. Such matrices will be called R-matrices (they really should have been called $m \times n$ R-matrices, but where there is no danger of confusion we omit the $m \times n$). Corresponding to such a matrix R we define an oriented graph on the vertices $1, 2, \ldots, n$, in which there is an edge oriented from i to j (notation: $i \to j$) provided i precedes j in a majority of the rows of R. If i precedes j as often as j precedes i the vertices i and j are not joined by an edge. It has been known for some time [1] that every directed graph in which every pair of vertices are joined by at most one oriented edge can be realized as a graph associated with some R-matrix in this manner. The principal object of this paper is to obtain relatively sharp estimates for the smallest number $m(n)$ such that every oriented graph on n vertices corresponds to some $m \times n$ matrix of the type described.

This as well as some related problems which we will treat arise from questions concerning methods of combining individual transitive preferences on a set of alternatives by means of majority decisions. Thus we may think of the rows of the matrix R as representing orderings by individual voters, of a set of n candidates $1, 2, \ldots, n$ in order of preference. Although each voter thus expresses a set of transitive preferences, the majority opinion need not be transitive and indeed we will prove that every preference pattern (ties permitted) may be achieved by no more than $c_1\, n/\log n$ voters, (c_1 a fixed constant), i.e. $m(n) \leq c_1\, n/\log n$. On the other hand it was shown in a relatively simple way by STEARNS [2] that some preference patterns on n candidates cannot be schieved by $c_2\, n/\log n$ voters (where c_2 is another fixed positive constant) so that $m(n) > c_2\, n/\log n$.

In § 1 we consider the following problem: What is the largest number $f(n)$ such that every oriented graph on n vertices in which every pair of distinct vertices is jointed by a directed edge has at least one subgraph of $f(n)$ vertices in which the orientation is transitive, i.e. in which $i \to j$ and $j \to k$ implies $i \to k$. Our result here is that $f(n) \leq 2[\log_2 n] + 1$. STEARNS has shown that $f(n) \geq [\log_2 n] + 1$.

In § 2 we will develop some lemmas concerning oriented graphs which can be represented by $2 \times n$ R-matrices. In the voting terminology this means

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that we study the preference patterns of candidates that can be achieved by a pair of voters — we will call them a couple. The point in considering such pairs of voters is that by balancing their transitive preferences in a certain way the pair of voters can achieve a preference between certain pairs of candidates in the manner in which these pairs are to be preferred by the majority, while with respect to all other pairs the preferences of the couple cancel one another.

In § 3 we relate the graph theoretic lemmas of § 2 to the problem of estimating $m(n)$ and obtain the result

$$c_1\, n/\log n > m(n) > c_2\, n/\log n\,.$$

We conclude with a number of unsolved problems.

§ 1.

The problem discussed and partially solved here is independent of our main problem the estimation of $m(n)$. By a complete oriented graph or complete paired comparison we mean a graph in which every pair of vertices is joined by one oriented edge. As mentioned in the introduction, STEARNS has proved that every such graph on n vertices contains a subgraph on $[\log_2 n] + 1$ vertices on which the orientation is transitive. For the sake of completeness we sketch the relevant argument: Consider a complete oriented graph on n vertices. Let $w(i)$ be the number of edges oriented away from vertex i. Relabel the vertices so that $w(1) \geqq w(2) \geqq \ldots \geqq w(n)$. Since every pair of vertices contributes 1 to $\Sigma\, w(i)$ we have $\sum\limits_{i=1}^{n} w(i) = \binom{n}{2}$ so that $w(1) \geqq \frac{n-1}{2}$. To construct a transitive chain of $[\log_2 n] + 1$ vertices place vertex 1 at the beginning of the chain and use induction to find in the subgraph of $\frac{n-1}{2}$ vertices which are joined to 1 by edges oriented avay from 1, a transitive subset of $\left[\log_2\left(\frac{n-1}{2}\right)\right] + 1$ vertices. These together with the vertex 1 form the required set.

To obtain a lower bound for the largest transitive set in some complete oriented graph on n vertices, assume that every such graph has a transitive subset of k elements. Now such a transitive subset must be one of $\binom{n}{k}$ subsets of k of the vertices, and any one of these subsets in order to be transitive, can be ordered in $k!$ ways. Having fixed the transitive subset (including its order) we observe that such a transitive subset can appear in exactly $2^{\binom{n}{2}-\binom{k}{2}}$ complete directed graphs, since the complete graph is determined by the orientations on its $\binom{n}{2}$ edges, $\binom{k}{2}$ of which have already been fixed. Finally, since each of the $2^{\binom{n}{2}}$ oriented graphs has a transitive subgraph of k vertices we have

$$\binom{n}{k} k\,!\, 2^{\binom{n}{2}-\binom{k}{2}} \geqq 2^{\binom{n}{2}},$$

and using $\binom{n}{k} \leq n^k/k!$ we are lead to

$$k \leq \frac{2 \log n}{\log 2} + 1$$

which completes the proof of

Theorem 1.

$$\lfloor \log_2 n \rfloor + 1 \leq f(n) \leq 2[\log_2 n] + 1 .$$

We remark that $f(7) = 3$. That $f(7) \geq 3$ follows from the left hand side of the inequality above while $f(7) \leq 3$ is obtained by considering the directed graph on $1, 2, \ldots, 7$ in which $i \to j$ iff the number $i - j$ is a quadratic residue (mod 7). We would like to call the attention of the reader to the fact that we have been unable to disprove the conjecture that $f(n) = [\log_2 n] + 1$. In particular we cannot decide if $f(15) = 4$.

§ 2.

In what follows G will denote a directed graph in n vertices, not necessarily complete, i.e. each pair of vertices is joined by at most one directed edge. The graph H will be called bipartite and unidirected if the vertices of H can be split into two disjoint subsets A and B (one of which can be empty) such that every vertex of A is joined to every vertex of B in the direction from A to B and no other edges exist in H. Suppose the vertices of A are $a_1, \ldots, a_k$ and those of B are $b_1, b_2, \ldots, b_l$ $(k + l = p)$. A and B will be called the levels of our subgraph (A the top level, B the lower level).

Lemma 1. *A bipartite and unidirected graph H can be represented by a $2 \times p$ R-matrix.*

Proof. Consider the matrix

$$\begin{pmatrix} a_1\, a_2 & \ldots\, a_k\; b_1\, b_2 & \ldots\, b_l \\ a_k\, a_{k-1} & \ldots\, a_1\; b_l\, b_{l-1} & \ldots\, b_1 \end{pmatrix}.$$

The graph induced by this matrix has edges directed from each vertex in A to each vertex in B. However there are no edges joining vertices of A to vertices of A (or vertices of B to vertices of B) since for $i, j < k$, a_i precedes a_j in one row and follows it in the other.

Next, if a graph H can be decomposed into disjoint bipartite and undirected graphs it will be called bilevel.

Lemma 1 can be generalized to yield

Lemma 2. *A bilevel graph H with n vertices can be represented by a $2 \times n$ R-matrix.*

Proof. If the top level of H consists of the disjoint sets of vertices $A_1, A_2, \ldots, A_u$ and the lower level of the corresponding sets $B_1, B_2, \ldots, B_u$ and if $A_i = \{a_{i,1}, a_{i,2}, \ldots\}$, $B_i = \{b_{i,1}, b_{i,2}, \ldots\}$ then the required matrix has first row consisting of the elements of A_1 in some order followed by those of B_1 in some order. These are followed by the elements of A_2 in some order and the elements of B_2 in some order, etc. In the second row we have first the elements of A_n in the reverse order to that which they had in the first row, followed by the elements of B_n again in reverse order. Then come the elements of A_{n-1} followed by those of B_{n-1} again in the order opposite to that in which they appeared in the first row. We continue in this way up to the elements of A_1 in the reverse order to that in the first, row, followed finally by the elements of B_1 in reverse order. It is easily seen that this matrix induces the required graph.

We proceed to prove

Lemma 3. *If G is a directed graph with n vertices and e edges with*

$$\frac{n^2}{2^{2r+4}} < e \leq \frac{n^2}{2^{2r+1}} \text{ where } \frac{\log n}{20\,r+1} \geqq 1$$

then G contains a bipartite unidirected graph with levels A and B having $[\sqrt{n}]$ and $\left[\frac{\log n}{20r+1}\right]$ vertices respectively, and in which the valences of the vertices of A in the graph G do not exceed $16n/2^r$.

Proof. Consider first the vertices of G (if any) of valence at least $16n/2^r$. If their number is x then we must clearly have $x \cdot 16n/2^r \leq 2n^2/2^{2r+1}$ or $x \leq n/2^{r+4}$. Thus the number of edges containing two such vertices does not exceed $\binom{x}{2} \leq n^2/2^{2r+9}$. Hence if we omit all these edges there remain more than $n^2/2^{2r+5}$ edges at least one endpoint of which has valence $< 16n/2^r$. Denote the vertices of valence $< 16n/2^r$ (in G) by $v_1, v_2, \ldots, v_t$ and let their valences be $y_1, y_2, \ldots, y_t$. Clearly

$$n(1 - 2^{-(r+4)}) \leqq t \leqq n$$

and

$$\sum_{i=1}^{t} y_i > \frac{n^2}{2^{2r+5}}$$

Without loss of generality we may assume that

$$\sum_{i=1}^{t} y'_i > \frac{n^2}{2^{2r+6}}$$

where y'_i is the number of edges directed away from v_i. Let $k (k \geqq 1)$ be an indeterminate for the time being. A k-tuple of vertices will be said to belong to v_i $(1 \leqq i \leqq t)$ if every vertex of the k-tuple is adjoined to v_i by an edge directed away from v_i. There are exactly $\binom{y'_i}{k}$ k-tuples belonging to v_i. Denote by S the system of k-tuples belonging to one of the v_i $(1 \leqq i \leqq t)$ (if a k-tuple belongs to exactly r v's it occurs in S r-times). Clearly S has $\sum_{i=1}^{t} \binom{y'_i}{k}$ elements.

Now $\sum_{i=1}^{t} \binom{y'_i}{k}$ will be a minimum if all the y'_i are equal and if t is as large as possible. This is achieved by letting $t = n$ and $y'_i = \left[\frac{n}{2^{2r+6}}\right]$. Thus

$$\sum_{i=1}^{t} \binom{y'_i}{k} \geqq n \binom{\left[\frac{n}{2^{2r+6}}\right]}{k} > \frac{n^{k+1}}{k!\,(2^{2r+7})^k}.$$

Now the total number of k-tuples that can be chosen from n points equals $\binom{n}{k} \leq \frac{n^k}{k!}$ so that the same k-tuple must occur in S at least $\frac{n}{2^{(2r+7)k}}$ times. If $k = \left[\frac{\log n}{20r+1}\right]$ a simple computation shows that the same k-tuple will occur at least $[\sqrt{n}]$ times, or there will be at least $[\sqrt{n}]$ vertices form a set A each connected to each vertices of a set B which has $\left[\frac{\log n}{20\,r+1}\right]$ elements.

Note that the set A was chosen from the vertices whose valences did not exceed $16n/2^r$ so the lemma is proved.

We next prove the crucial

Lemma 4. *Let $n > n_0$. If G is a directed graph with n vertices and e edges where*

$$\frac{n^2}{2^{2r+3}} < e \leq \frac{n^2}{2^{2r+1}} \quad \text{and} \quad r < 10 \log\log n$$

then G contains a bilevel graph of at least $\frac{n \log n}{(r+1)\,2^{r+15}}$ edges.

Proof. First we omit all edges connecting vertices with valences at least $16n/2^r$. As before the number of omitted edges is at most $\frac{n^2}{2^{2r+9}}$ Hence we are left with at least

$$\frac{n^2}{2^{2r+1}}\left(\frac{1}{2^2} - \frac{1}{2^8}\right) > \frac{n^2}{2^{2r+\frac{7}{2}}}$$

edges and by Lemma 3 we have a bipartite unidirected subgraph (A_1, B_1) with levels A_1 and B_1 previously described. Since the vertices of A_1 have valence $\leq 16n/2^r$ and those of B_1 have valence $\leq n-1$ and since $r < 10 < \log\log n$ the number of edges incident to $A_1 \cup B_1$ is at most

$$n\left(\frac{16\sqrt{n}}{2^r} + \log n\right) < \frac{20\, n^{3/2}}{2^r}.$$

We remove these edges and there still remain

$$\frac{n^2}{2^{2r+\frac{7}{2}}} - \frac{20\, n^{3/2}}{2^r} > \frac{n^2}{2^{2r+4}}$$

edges, provided $n > n_0$. Lemma 3 can therefore be used again and we obtain a bipartite unidirected graph (A_2, B_2) with levels A_2 and B_2 of the required type. (In the bipartite graphs (A_i, B_i) it is not necessarily assumed that the edges go from A_i to B_i, their direction may depend on i). Now we repeat the procedure and omit the edges incident to $A_2 \cup B_2$. If we repeat this procedure $\left[\frac{\sqrt{n}}{20 \cdot 2^{r+6}}\right]$ times we are left with a graph which has at least

$$\frac{n^2}{2^{2r+\frac{7}{2}}} - \frac{20n^{3/2}}{2^r}\left[\frac{\sqrt{n}}{20 \cdot 2^{r+6}}\right] > \frac{n^2}{2^{2r+}}$$

edges. We can therefore apply Lemma 3 once more and thus obtain a bilevel graph with the components (A_i, B_i)

$$1 \leq i \leq \left[\frac{\sqrt{n}}{20 \cdot 2^{r-6}}\right] + 1$$

of at least

$$\left(\left[\frac{\sqrt{n}}{20 \cdot 2^{r+6}}\right] + 1\right)\sqrt{n}\left[\frac{\log n}{20\, r + 1}\right] > \frac{n \log n}{(r+1)\, 2^{r+15}}$$

edges and the proof of the lemma is complete.

Lemma 5. *Let G be a connected directed graph of m vertices. Then G has a bilevel subgraph of* $\left[\frac{m-1}{4}\right]$ *edges.*

Proof. We prove first that if T is a directed tree then it can be decomposed into four bilevel graphs. For this purpose consider first the corresponding undirected tree T^*. Let x_1 be an vertex of T^*. Number I all edges of T^* which can be reached from x_1 in an odd number of steps. Number II all edges which can be reached from x_1 in an even number of steps. The edges labelled I form a union of disjoint stars (a star is a tree in which all but one vertex has valency 1) which can be split into two bilevel graphs and similarly for the edges labelled II. The lemma now follows by considering for G a spanning tree T, i.e. a tree whose edges are a subset of the edges of G and which contains all the vertices of G. Such a tree clearly has $n - 1$ edges.

Lemma 6. *Let G be a directed graph of e edges. Then G contains a bilevel graph of at least* $\frac{\sqrt{e}}{8}$ *edges.*

Proof. A graph G of e edges must have at least $[\sqrt{2e}]$ vertices. Consider the connected components G_i of G having u_i vertices, $i = 1, 2, \ldots, k$.

Now by Lemma 5, each G_i contains a bilevel graph of $\frac{u_i - 1}{4}$ edges, so that G contains a bilevel graph of

$$\sum \left[\frac{u_i - 1}{4}\right] \geq \frac{\sqrt{e}}{8}$$

edges.

We are now ready to prove our main result, namely that every preference pattern on n candidates can be achieved by not more than $c_1\, n/\log n$ voters. For this purpose it will suffice, by Lemma 2, to show that the directed graph G corresponding to the preference pattern can be decomposed into edge-disjoint bilevel graphs $G_1, G_2, \ldots, G_t$, the set of whose vertices is identical with the set of vertices of G, and $t < c_1\, n/(2 \log n)$.

We are going to define the graphs

$$G_i \text{ and } G^{(i)} \quad 1 \leq i \leq \left[2^{16} \frac{n}{\log n}\right]$$

by induction. We will put $G^{(i)} = G - G_1 \cup G_2 \cup \ldots \cup G_i$ (i.e. we obtain $G^{(i)}$ by omitting from G the edges of $G_1, G_2, \ldots, G_i$). G_1 is one of the bilevel subgraphs of G having the maximun number of edges and if $G_1, \ldots, G_i$ are already defined then G_{i+1} is one of the bilevel subgraphs of $G^{(i)}$ having the maximum number of edges. Denote by e_i the number of edges of $G^{(i)}$. Let r run through the integers $r = 0, 1, \ldots, [10 \operatorname{loglog} n]$. Denote by i_r the smallest integer for which

$$e_{i_r} \leqq \frac{n^2}{2^{2r+1}}.$$

We shall prove that for $r \leqq [10 \operatorname{loglog} n]$

$$i_{r+1} - i_r \leqq 2^{15} \cdot \frac{r+1}{2^{r+1}} \cdot \frac{n}{\log n}. \tag{1}$$

If $e_{i_r} \leqq \frac{n^2}{2^{2r+3}}$ then $i_{r+1} - i_r = 0$ and (1) is satisfied thus we can assume $e_{i_r} > \frac{n^2}{2^{2r+3}}$. Let $i_r \leq j < i_{r+1}$ then $e_j > \frac{n^2}{2^{2r+3}}$ and hence by Lemma 4 $G^{(j)}$ contains a bilevel subgraph of at least

$$\frac{n \log n}{(r+1)\, 2^{r+15}}$$

edges and hence by the maximality property of G_j

$$e_j - e_{j+1} \geqq \frac{n \log n}{(r+1)\, 2^{r+15}} \tag{2}$$

(2) immediately implies (1).

From (1) we obtain that by the removal of at most

$$\sum_{0 \leqq r \leqq [10 \operatorname{loglog} n]} (i_{r+1} - i_r) < \frac{2^{15}\, n}{\log n} \sum_{r=0}^{\infty} \frac{r+1}{2^{r+1}} = \frac{2^{16}\, n}{\log n}$$

bilevel graphs G_i, $1 \leq i \leq \left[\frac{2^{16}\, n}{\log n}\right]$ we obtain a

$$G^{(t)} = G - \bigcup G_i\,, \quad 1 \leq i \leq \left[\frac{2^{16}\, n}{\log n}\right]$$

where $G^{(t)}$ has fewer than

$$\frac{n^2}{2^{20 \operatorname{loglog} n}} < \frac{n^2}{(\log n)^{13}}$$

edges.

To complete the main result we need to show that a graph with this many edges is the union of $o\left(\frac{n}{\log n}\right)$ edge — disjoint bilevel graphs and this is an almost immediate consequence of Lemma 6.

As already stated the proof of $m(n) > c_2\, n/\log n$ is relatively simple but we include it for completeness. Since each voter can vote in n ! ways the number of distinct ways in which m voters can vote is $(n!)^m$.

The number of preference patterns on n candidates is (since ties are permitted) $3^{\binom{n}{2}}$. If all these patterns can be achieved we must have $(n!)^m > 3^{\binom{n}{2}}$ from which the required result follows by a simple computation.

One might conjecture that that $m(n) \log n/n$ tends to a limit but this conjecture is clearly well beyond the methods used in this paper. We cannot even prove that $\overline{\lim_{n\to\infty}}\, m(n) \log n/n > \frac{\log 3}{2}$.

Still another problem suggested by the present considerations is to obtain good estimates for the largest number $s = s(e)$ such that every ordinary graph of e edges contains a bilevel (undirected) graph of s edges. By more complicated arguments than those used here we can prove $s > c\sqrt{e} \log e$.

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ПРЕДСТАВЛЕНИЕ УПОРЯДОЧЕННЫХ ГРАФОВ СИСТЕМАМИ ПЕРЕСТАНОВОК

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Резюме

Рассматриваем матрицу с n стольбцами и с m строками, каждая строка которой — перестановка чисел $1, 2, \ldots, n$.. С этой матрицей мы соединим упорядоченный граф следующим образом: вершины графа будут числа $1, 2, \ldots, n$. Если в большинстве строк матрицы i предшествует j, тогда граф содержит ребро упорядоченное от i до j. Если i предшествует j в стольких же строках, в скольких j предшествует i, тогда i и j не соединяются. Пусть $m(n)$ обозначает наименьшее число, такое, что из матриц с $m(n)$ строчками представимы таким образом все графы с n вершинами, в которых каждая пара вершин соединенна не более одной вершиной Stearns [2] доказал, что $m(n) > c_2\, n/\log n$.

Главный результат настоящей работы доказательство неравенства

$$m(n) \leq c_1\, n/\log n$$

(c_1 и c_2 положительные константы).