ON THE NUMBER OF TRIANGLES CONTAINED IN CERTAIN GRAPHS

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Let $G(n;m)$ denote a graph of n vertices and m edges. Vertices of G will be denoted by $x_1, \ldots, y_1 \ldots$; edges will be denoted by (x, y) and triangles by (x, y, z). $(G - x_1 - x_2 - \ldots - x_k)$ will denote the graph G from which the vertices $x_1, \ldots, x_k$ and all edges incident to them have been omitted. $G - (x_i, x_j)$ denotes the graph G from which the edge (x_i, x_j) has been omitted.

A special case of a well known theorem of Turán states that every $G(n; \left[\frac{n^2}{4}\right] + 1)$ contains a triangle and that there is a unique $G(n; \left[\frac{n^2}{4}\right])$ which contains no triangle [3]. Rademacher proved that every $G(n; \left[\frac{n^2}{4}\right] + 1)$ contains at least $\left[\frac{n}{2}\right]$ triangles (Rademacher's proof was not published). I gave a very simple proof of Rademacher's theorem [1] and recently proved that if $k < c_1 n$ then every $G(n; \left[\frac{n^2}{4}\right] + k)$ contains at least $k\left[\frac{n}{2}\right]$ triangles and that $k\left[\frac{n}{2}\right]$ is best possible [2]. In [2] I conjectured that this holds for $k < \left[\frac{n}{2}\right]$.

Recently I observed that if a $G(n; \left[\frac{n^2}{4}\right])$ contains a triangle it contains at least $\left[\frac{n}{2}\right] - 1$ triangles. More generally we shall prove the following:

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THEOREM 1. Let $\ell \geq 0$. If a $G(n; \left[\frac{n^2}{4}\right] - \ell)$ contains a triangle then it contains at least $\left[\frac{n}{2}\right] - \ell - 1$ triangles.

The theorem is trivial if $\ell \geq \left[\frac{n}{2}\right] - 2$; thus we can henceforth assume

$$0 \leq \ell \leq \left[\frac{n}{2}\right] - 3 . \tag{1}$$

First we show that the theorem is best possible. Let the vertices of our $G(n; \left[\frac{n^2}{4}\right] - \ell)$ be $x_1, \ldots, x_{[(n+1)/2]}$; $y_1, \ldots, y_{[n/2]}$. Its edges are: (x_1, x_2); (x_i, y_j) for $2 \leq i \leq \left[\frac{n+1}{2}\right]$, $1 \leq j \leq \left[\frac{n}{2}\right]$; (x_1, y_j) for $1 \leq j \leq \left[\frac{n}{2}\right] - \ell - 1$. Clearly the graph has n vertices, $\left[\frac{n^2}{4}\right] - \ell$ edges, and contains only the $\left[\frac{n}{2}\right] - \ell - 1$ triangles (x_1, x_2, y_j), $1 \leq j \leq \left[\frac{n}{2}\right] - \ell - 1$.

Now we prove theorem 1. First we need the following simple lemma.

LEMMA. Assume that G has n vertices $x_1, \ldots, x_n$ and that it contains the triangle (x_1, x_2, x_3). Further assume that there are $n+r$ edges each incident to at least one of x_1, x_2 and x_3. Then G contains at least r triangles (x_i, x_j, x_k) with

$$1 \leq i < j \leq 3 < k . \tag{2}$$

We prove the Lemma by induction with respect to r. For $r = 1$ the lemma is evident, for in this case there are $n - 2$ edges connecting $x_4, \ldots, x_n$ with x_1, x_2, x_3 and thus at least one x_k, $k > 3$, is adjacent to two of the x_j, $j \leq 3$; thus there is at least one triangle of the form (2).

Let now $r > 1$ and assume that the lemma holds for $r-1$. Just as in the case $r=1$, there is at least one triangle (x_i, x_j, x_k) satisfying (2). In the graph $G - (x_i, x_k)$ there are $n+r-1$ edges incident to x_1, x_2 and x_3, so by our induction hypothesis it contains at least $r-1$ triangles. G contains further the r-th triangle (x_i, x_j, x_k). Hence the proof of our lemma is complete.

Now we prove Theorem 1. The theorem is trivial for $n \leq 5$. By the assumption of Theorem 1 our $G(n; \left[\frac{n^2}{4}\right] - \ell)$ contains a triangle, say (x_1, x_2, x_3). Assume first that

$$(G(n; \left[\frac{n^2}{4}\right] - \ell) - x_1 - x_2 - x_3)$$

has not more than $\left[\frac{(n-3)^2}{4}\right]$ edges. In this case there are at least

$$\left[\frac{n^2}{4}\right] - \left[\frac{(n-3)^2}{4}\right] - \ell = n + \left[\frac{n}{2}\right] - 2 - \ell$$

edges incident to x_1, x_2 and x_3. Thus by our lemma there are at least $\left[\frac{n}{2}\right] - 2 - \ell$ triangles in our graph which satisfy (2). Together with (x_1, x_2, x_3) this gives the required $\left[\frac{n}{2}\right] - \ell - 1$ triangles and hence our theorem is proved in this case.

Assume next that

$$(G(n; \left[\frac{n^2}{4}\right] - \ell) - x_1 - x_2 - x_3)$$

has more than $\left[\frac{(n-3)^2}{4}\right]$ edges. By Rademacher's theorem it then has at least $\left[\frac{n-3}{2}\right]$ triangles and together with (x_1, x_2, x_3)

we obtain that $G(n; \left[\frac{n^2}{4}\right] - \ell)$ has at least $\left[\frac{n-1}{2}\right] \geq \left[\frac{n}{2}\right] - \ell - 1$ triangles ($\ell \geq 0$). This completes the proof of Theorem 1.

Our proof used Rademacher's theorem, but the latter would be quite easy to prove by our method. In fact an induction argument easily gives the following theorem.

THEOREM 2. Consider a graph $G(n; \left[\frac{n^2}{4}\right] + \ell)$ and assume that if $\ell \leq 0$ it contains at least one triangle. Then it contains at least $\left[\frac{n}{2}\right] + \ell - 1$ triangles.

For $\ell \leq 0$ this is our Theorem 1; for $\ell = 1$ it is Rademacher's theorem; for $\ell > 1$ [2] contains a sharper result.

We suppress the details of the proof since they are similar to those of Theorem 1. For $\ell \leq 0$ there is nothing to prove. If $\ell > 0$ by Turán's theorem our graph contains at least one triangle. We now use induction from $n-3$ to n and the proof proceeds as the proof of Theorem 1.

REFERENCES

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