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ON EXTREMAL PROBLEMS OF GRAPHS AND GENERALIZED GRAPHS

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ABSTRACT

An r-graph is a graph whose basic elements are its vertices and r-tuples. It is proved that to every l and r there is an $\varepsilon(l, r)$ so that for $n > n_0$ every r-graph of n vertices and $n^{r-\varepsilon(l,r)}$ r-tuples contains $r \cdot l$ vertices $x^{(j)}, 1 \leqq j \leqq r, 1 \leqq i \leqq l$, so that all the r-tuples $(x_{i_1}{}^{(1)}, x_{i_2}{}^{(2)}, \cdots, x_{i_r}{}^{(r)})$ occur in the r-graph.

By an r-graph $G^{(r)}(r \geqq 2)$ we shall mean a graph whose basic elements are its vertices and r-tuples; for $r = 2$ we obtain the ordinary graphs. These generalised graphs have not yet been investigated very much. $G^{(r)}(n; m)$ will denote an r-graph of n vertices and m r-tuples; $G^{(r)}\left(n; \binom{n}{r}\right)$, the complete r-graph of n vertices, will be denoted by $K^{(r)}(n)$, i.e., $K^{(r)}(n)$ contains all the r-tuples formed from n elements. $K^{(r)}(n_1, \cdots, n_r)$ will denote the r-graph of $\sum_{j=1}^r n_j$ vertices and $\prod_{j=1}^r n_j$ r-tuples defined as follows: The vertices are

$$x_i^{(j)},\ 1 \leqq j \leqq r,\ 1 \leqq i \leqq n_j$$

and the r-tuples of our r-graph are the $\prod_{j=1}^r n_j$ r-tuples

$$(x_{i_1}^{(1)},\ x_{i_2}^{(2)}, \cdots, x_{i}^{(r)}),\ \ 1 \leqq i_j \leqq n_j, \quad 1 \leqq j \leqq r.$$

Thus $K^{(2)}(2,2)$ is simply a rectangle.

Denote by $f_l^{(r)}(n)$ the smallest integer so that every $G^{(r)}(n; f_l^{(r)})$ contains a complete r-graph of l vertices.

As is well known Turán [5] determined $f_l^{(2)}(n)$ for every l and n and also proved that there is a unique $G^{(2)}(n; f_l^{(2)}(n) - 1)$ which contains no complete 2-graph of l vertices (ordinary graphs have to be denoted as 2-graphs here). In particular $f_3^{(2}(n) = [n^2/4] + 1$.

For $r > 2$ the determination of $f_l^{(r)}(n)$ seems to be a very difficult question which is unsolved for all $r > 2$, $l > r$. (This question was also posed by Turán. Turán in particular conjectured that

$$f_5^{(3)}(n) = n^2(n-1). \tag{1}$$

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Vera T. Sós observed that if (1) is true then the extreme graphs are certainly not unique, and this may be one reason why the proof of (1) is difficult.

It is easy to see that

$$\lim_{n=\infty} f_l^{(r)}(n) \Big/ \binom{n}{r} = c_l^{(r)}$$

exists, but the value of $c_l^{(r)}$ is not known for any $r > 2, l > r$.

Denote by $f(n; K^{(r)}(l_1, \ldots, l_r))$ the smallest integer so that every

$$G^{(r)}(n; f(n; K^{(r)}(l_1, \cdots, l_r))$$

contains a $K^{(r)}(l_1, \cdots, l_r)$. If $\sum_{i=1}^r l_i > n$ we define: $f(n; K^{(r)}(l_1, \cdots, l_r)) = \binom{n}{r} + 1$. In particular, $f(n; K^{(2)}(2, 2))$ is the smallest integer so that every $G^{(2)}(n; f(n; K^{(2)}(2,2))$ contains a rectangle. E. Klein and I [1] proved that

$$c_1 n^{3/2} < f(n; K^{(2)}(2,2)) < c_2 n^{3/2}. \tag{2}$$

Very likely

$$\lim_{n=\infty} f(n; K^{(2)}(2,2))/n^{3/2} = \frac{1}{2\sqrt{2}}. \tag{3}$$

but it is not even known that the limit in (3) exists.

Kövári, the Turáns [4] and I showed that

$$f(n; K^{(2)}(l, l)) < cn^{2-1/l}.$$

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$$f(n; K^{(2)}(l, l)) > c' n^{2-(1/l)}, \tag{4}$$

but we are unable to prove (4) for $l > 2$.

Stone and I [2] proved that for every $\varepsilon > 0$ and a sufficiently small c_ε and $n > n_0(\varepsilon)$

$$f(n; K^{(2)}([c_\varepsilon \log n], [c_\varepsilon \log n])) < \varepsilon n^2. \tag{4'}$$

It can be shown by probabilistic methods (similar to those used in [4] that for sufficiently large $\tilde{c}_\varepsilon$

$$f(n; K^{(2)}([\tilde{c}_\varepsilon \log n], [\tilde{c}_\varepsilon \log n])) > (1-\varepsilon)\binom{n}{2}. \tag{4''}$$

In the present paper we first of all shall prove the following

THEOREM 1. *Let $n > n_0(r, l), l > 1$. Then for sufficiently large $C=(C$ is independent of $n, r, l)$*

$$(5) \qquad n^{r-C(/l^{r-1})} < f(n; K^{(r)}(l,\cdots,l)) \leqq n^{r-(1/l^{r-1})}.$$

We only prove the upper bound and will discuss the lower bound later. We use induction with respect to r. First we prove the right side inequality of (5) for $r=2$, (this is substantially contained in [6], then we use induction with respect to r.

Consider now the case $r=2$. Denote the vertices of our graph $G^{(2)}(n;t)$, $t \geqq n^{2-1/l}$ by $x_1,\cdots,x_n$, and by $v(x_i)$ we denote the valence of x_i (i.e. $v(x_i)$ denotes the number of edges incident to x_i). Clearly

$$(6) \qquad \sum_{i=1}^{n} v(x_i) \geqq 2n^{2-1/l} \ .$$

Let $x_1^{(i)},\cdots,x_{v(x_i)}^{(i)}$ be those x_j's which are joined to x_i. Form all the l-tuples from these vertices for all i, $1 \leqq i \leqq n$. The number of these l-tuples (each counted with the proper multiplicity) clearly equals

$$(7) \qquad \sum_{i=1}^{n} \binom{v(x_i)}{l}.$$

An elementary inequality states that the sum (7) is a minimum if all the $v(x_i)$ are equal ($\sum_{i=1}^{n} v(x_i)$ satisfies (6)$\left(\binom{y}{l}=0 \text{ if } y<l\right)$. Thus by a simple computation for $n > n_0(l)$

$$\sum_{i=1}^{n} \binom{v(x_i)}{l} \geqq n\binom{2n^{1-1/l}}{l} > l\binom{n}{l}.$$

Hence there are l vertices $y_1,\cdots,y_l$ which are all joined to the same l vertices $x_{j_1},\cdots x_{j_l}$, which means that our graph contains a $K^{(2)}(l,l)$ as stated.

Assume now that the right side inequality of (5) holds for $r-1$, we shall prove it for r. First we need the following

LEMMA. *Let S be a set of N elements $y_1,\cdots,y_N$ and let $A_i, 1 \leqq i \leqq n$, be subsets of $\int$. (Assume that $\Omega(\bar{\bar{A}}_i$ denotes the number of elements of A_i)*

$$(8) \qquad \sum_{i=1} \bar{\bar{A}}_i \geqq \frac{nN}{w}.$$

Then if $n \geqq 2l^2w^l$ *there are* l *distinct* A's, $A_{i_1}, \cdots, A_{i_l}$, *so that*

(9) $$\overline{\overline{\bigcap_{j=1}^{l} A_{i_j}}} \geqq \frac{N}{2w^l}.$$

Denote by $f_i(y)$ the characteristic function of A_i (i.e., $f_i(y_j) = 1$ if y_j is in A_i, and is 0 otherwise). Put

$$F(y) = \sum_{i=1}^{n} f_i(y).$$

Clearly by (8)

(10) $$\sum_{j=1}^{N} F(y_j) \geqq \frac{nN}{w}.$$

Thus from (10) we obtain by an elementary inequality that

$$\sum_{j=1}^{N} F(y_j)^l$$

is minimal if for all j $F(y_j) = n/w$, or

(11) $$\sum_{j=1}^{N} F(y_j)^l \geqq N \left(\frac{n}{w}\right)^l.$$

On the other hand we obtain by a simple argument

(12) $$\sum_{j=1}^{N} F(y_j)^l = \sum \overline{\overline{A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_l}}}$$

where the summation in (12) is extended over all the choices of $i_1, \cdots, i_l (1 \leqq i_r \leqq n)$. There are $\prod_{i=0}^{l-1}(n-i) \leqq n^l$ choices of $i_1, \cdots, i_l$ where all the indices are distinct, and if (9) would be false the contribution of these terms to the sum (12) would be less than

(13) $$\frac{Nn^l}{2w^l}.$$

The number of the summands in (12) where not all the indices are distinct is easily seen to be less than l^2n^{l-1}. The contribution of each of these terms to the right side of (12) is clearly at most N. Thus finally from (12) and (13)

(14) $$\sum_{j=1}^{N} F(y_j)^l < \frac{Nn^l}{2w^l} + l^2 n^{l-1} N.$$

Now since $n \geqq 2l^2 w^l$ (14) contradicts (11). Thus (9) must hold for at least one choice of distinct A_i's, $1 \leqq i \leqq l$ which completes the proof of the Lemma.

The Lemma is clearly not best possible, but is good enough for our purpose.

Now we are ready to prove the right hand inequality of (5) for $r > 2$. Assume that it has already been proved for $r-1$ if $n > n_0(r-1, l)$, and we are going to prove it for r if $n > n_0(r, l)$. Suppose then that we have a $G^{(r)}(n;t)$ with $t \geqq n^{r-(1/l^{r-1})}$. Denote by $x_1, \cdots, x_n$ the vertices of our $G^{(r)}$ and by $y_1, \cdots, y_N$, $N = \binom{n}{r-1}$, the set of all $(r-1)$-tuples formed from the x_i, $1 \leqq i \leqq n$. $P_1^{(r)}, \cdots, P_t^{(r)}$ denotes the t r-tuples of our $G^{(r)}(n;t)$. To apply our Lemma denote by A_i the set of all $(r-1)$-tuples y_j such that $y_j U x_i = P_k^{(r)}$ for some $1 \leqq k \leqq t$. We evidently have

$$\sum_{i=1}^{n} \bar{\bar{A_i}} = rt \geqq rn^{r-1/l^{r-1}} > nNr!\ n^{-1/l^{r-1}}.$$

Thus our Lemma applies with $N = \binom{n}{r-1}$, $w = n^{1/l^{r-1}}/r!$, since for $n > n_0(r, l)$ $n \geqq 2l^2 w^l$ is clearly satisfied. We thus obtain that there are l distinct A's $A_{i_1}, \cdots, A_{i_l}$ for which

(15) $$\overline{\overline{\bigcap_{j=1}^{l} A_{i_j}}} \geqq \frac{1}{2}\binom{n}{r-1}(r!\ n^{-1/l^{r-1}})^l > n^{r-1-(1/l^{r-2})}.$$

By (15) there are more than $n^{r-1-1/l^{r-2}}$ $(r-1)$-tuples

(16) $$P_1^{(r-1)}, \cdots P_{t_1}^{(r-1)},\ t_1 > n^{r-1-1/l^{r-2}},$$

so that all the r-tuples

(17) $$(x_{i_s} \notin P_j^{(r-1)}\ x_{i_s} \cup P_j^{(r-1)},\ 1 \leqq s \leqq l,\ 1 \leqq j \leqq t_1$$

is clearly satisfied by our construction) are one of the $P_j^{(r)}$'s of our $G^{(r)}(n;t)$. These $(r-1)$-tuples define a

$$G^{(r-1)}(n-l; t_1),\ t_1 > n^{-1-(1/l^{r-2})}$$

which by our induction hypothesis contains a $K^{(r-1)}(l, \cdots, l)$ if $n > l + n_0(r-1, l)$. By (17) this implies that our $G^{(r)}(n;t)$ contains a $K^{(r)}(l, \cdots, l)$ which proves the right side inequality of (5).

Theorem 1 easily implies the following

COROLLARY. *Let $n > n_0(r,l)$, $t_i \geqq n$, $i = 1, \cdots, r$. Let*

$$G^{(r)}\left(\sum_{=1}^{r} t_i ; U\right),\ U > \frac{3^r r^r}{n^{1/l^{r-1}}} \prod_{i=1} t_i$$

be s subgraph of $K^{(r)}(t_1, \cdots, t_r)$. Then $G^{(r)}(\sum_{i=1}^{r} t_i ; U)$ contains a $K^{(r)}(l, \cdots, l)$

A set of t_i elements can be decomposed into the (non-disjoint) union of $[t_i/[(n/r)]] + 1$ sets having $[n/r]$ elements. Hence clearly a $K^{(r)}(t_1, \cdots, t_r)$ can be decomposed into the union of at most

$$\prod_{i=1} \left\{ \left[\frac{t}{\left[\frac{n}{r}\right]} \right] + 1 \right\} < \frac{3^r r^r}{n^r} \prod_{i=1}^{r} t_i$$

$K^{(r}([n/r], \cdots, [n/r])$'s (the union is non-disjoint but every r-tuple of $K^{(r)}(t_1, \cdots, t_r)$ occurs in at least one of the $K^{(r)}([n/r], \cdots, [n/r])$'s). Thus at least one of these $K^{(r)}([n/r], \cdots, [n/r]$'s say $K_1^{(r)}$ contains at least $n^{r-1/l^{r-1}}$ r-tuples. Our $K_1^{(r)}$ has $r[n/r] \leqq n$ vertices, hence the corollary follows from theorem 1 (the right side inequality of (5)).

The corollary has applications in number theory, this will be discussed in a subsequent paper.

Without much change in the proof of Theorem 1 we could show that the right side inequality of (5) holds for every $n \geqq rl$. But in fact the right side of (5) is trivial if $l > 2(\log n)^{1/r-1}$, for then

$$\binom{n}{r} < n^{r-(1/l^{r-1})},$$

Further we can prove the following

THEOREM 2. *Let $\alpha > 0$ be any number, $n > n_0(\alpha, l, r)$, $2 \leqq l < \alpha(\log n)^{1/(r-1)}$. Then we have for a sufficiently large absolute constant C_1*

$$\binom{n}{r} n^{-C_1/l^{1-1}} < f(n; K^{(r)}(l, \cdots, l)) < \binom{n}{r} n^{-1/l^{n-1}}. \tag{18}$$

We do not prove the upper bound of (18) since it is similar to that of (5), we have only to carry out the estimations and the induction with respect to r a little more carefully. The most interesting special cases are those which correspond to (4′) and (4″). For every $\varepsilon > 0$ and a sufficiently small $c_\varepsilon^{(r)}$

$$f(n; K^{(r)}([c_\varepsilon^{(r)}(\log n)^{1/(r-1)}], \cdots, [c_\varepsilon^{(r)}(\log n)^{1/(r-1)}] < \varepsilon n^r. \tag{18′}$$

(18′) in fact follows from the fact that the right side inequality of (5) holds for every $n \geqq lr$. Further we have for a sufficiently large $\tilde{c}_{\varepsilon}^{(r)}$

$$(18'') \quad f(n; K^{(r)}([\tilde{c}_{\varepsilon}^{(r)}(\log n)^{1/(r-1)}], \cdots, [\tilde{c}_{\varepsilon}^{(r)}(\log n)^{1/(r-1)}])) > (1-\varepsilon)\binom{n}{r} .$$

To give the reader an illustration how to prove the lower bound of (5) and (18) we prove in full detail (18″) for $\varepsilon = \frac{1}{2}$. In fact we prove a stronger result. If $G^{(r)}(n;m)$ is an r-graph then $\bar{G}^{(r)}(n;m)$ will denote its complementary graph i.e. the $G^{(r)}(n;\binom{n}{r} - m)$ whose r-tuples are precisely those which do not occur in $G^{(r)}(n; m)$.

THEOREM 3. *Put* $t = [4(\log n)^{1/(r-1)}] + 1$. *For every n there is a* $G^{(r)}(n)$* *so that neither* $G^{(r)}(n)$ *nor* $\bar{G}^{(r)}(n)$ *contains a* $K^{(r)}(t, \cdots, t)$.

The proof will follow very closely the method used in [3]. The total number of r-graphs $G^{(r)}(n)$ is clearly $2^{(n/r)}$. The number of those r-graphs for which either $G^{(r)}(n)$ or $\bar{G}^{(r)}(n)$ contains a $K^{(r)}(t, \cdots, t)$ having the vertices $x_i^{(j)}$, $1 \leqq i \leqq t; 1 \leqq j \leqq r$ clearly equals

$$2 \cdot 2^{(n/r) - t^r},$$

since the t^r r-tuples of our $K^{(r)}(t, \cdots, t)$ either all have to belong to our $G^{(r)}(n)$, or none of them belong to our $G^{(r)}(n)$. The number of choices for our $K^{(r)}(t, \cdots, t)$ is clearly less than $n^{rt}/r! \leqq \frac{1}{2} n^{rt}$. Therefore the number of graphs $G^{(r)}(n)$ for which $G^{(r)}(n)$ or $\bar{G}^{(r)}(n)$ contains a $K^{(r)}(t, \cdots, t)$ is clearly less than

$$n^{rt} \cdot 2^{(n/r) - t^r} < 2^{(n/r)} .$$

Thus there is a $G^{(r)}(n)$ so that neither $G^{(r)}(n)$ nor $\bar{G}^{(r)}(n)$ contains *a* $K^{(r)}(t, \cdots, t)$, as stated.

The proof of the lower bound of (5) and (18) uses the same methods combined with the methods of [4].

It is possible that

$$\lim_{n=\infty} f(n; K^{(r)}(l, \cdots, l))/n^{r-(1/l^{r-1})}$$

exists and is different from 0 (by (5) it is $< \Delta 1$), but as stated in (3) this is not even known for $r = l = 2$.

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