ON A COMBINATORIAL PROBLEM. II*

By

P. ERDŐS (Budapest), member of the Academy

Let M be a set and F a family of its subsets. F is said by E. W. MILLER [5] to possess property B if there exists a subset K of M so that no set of the family F is contained either in K or in $\bar{K}$ ($\bar{K}$ is the complement of K in M).

HAJNAL and I [2] recently published a paper on the property B and its generalisations. One of the unsolved problems we state asks: What is the smallest integer $m(n)$ for which there exists a family F of sets $A_1, \ldots, A_{m(n)}$ each having n elements which does not possess property B? Throughout this paper A_i will denote sets having n elements.

We observed $m(n) \leqq \binom{2n-1}{n}$, $m(1)=1$, $m(2)=3$, $m(3)=7$. As far as I know the value of $m(4)$ is not yet known.

Recently I [3] showed that $m(n) > 2^{n-1}$ for all n and thatfor $n > n_0(\varepsilon)$ $m(n) > (1-\varepsilon)2^n \log 2$. W. M. SCHMIDT [6] proved $m(n) > 2^n(1+4n^{-1})^{-1}$ and up to date this is the best lower bound known for $m(n)$.

Recently ABBOTT and MOSER [1] proved that

$$m(a \cdot b) \leqq m(a)\, m(b)^a. \tag{1}$$

From (1) they deduced that for $n > n_0$, $m(n) < (\sqrt{7}+\varepsilon)^n$ and that $\lim_{n\to\infty} m(n)^{1/n}$ exists. Their method is constructive. By non-constructive methods I now prove

THEOREM 1. $m(n) < n^2 2^{n+1}$.

Theorem 1 thus implies $\lim_{n=\infty} m(n)^{1/n} = 2$. Theorem 1 and the result of SCHMIDT gives

$$2^n(1+4n^{-1})^{-1} < m(n) < n^2 2^{n+1}. \tag{2}$$

It would be interesting to improve the bounds for $m(n)$. A reasonable guess seems to be that $m(n)$ is of the order $n\ 2^n$.

A family of sets F is said to have property $B(s)$ if there exists a set S which has a non-empty intersection with each set of the family, but the cardinal number of the intersection is $< s$. HAJNAL and I asked what is the smallest integer $m(n, s)$ for which there exist sets $\{A_i\}$, $1 \leqq i \leqq m(n, s)$ which does not possess property $B(s)$? Clearly $m(n, n) = m(n)$.

* This paper was written while the author was visiting at the university of Alberta in Edmonton.

Mr. H. L. Abbott pointed it out to me that $m(2k, 2)=3$, $m(2k+1, 2)=4$.

Now we prove Theorem 1. We shall construct our $n^2 2^{n+1}$ sets of n elements not having property B as subsets of a set M of $2n^2$ elements. Suppose I have chosen already k of the sets $(k<n^2 2^{n+1})$ $A_1, \dots, A_k$ and suppose that there are u_k pairs of subsets $\{K_i, \bar{K}_i\}$, $1 \leqq i \leqq u_k$ of M so that no set A_i, $1 \leqq i \leqq k$ is contained either in K or in $\bar{K}$. If $u_k=0$ our Theorem is proved. Assume henceforth $u_k>0$. We shall prove that we can find a set A_{k+1} so that

$$u_{k+1} \leqq u_k \left(1-\frac{1}{2^n}\right). \tag{3}$$

(For each i, $1 \leqq i \leqq u_k$, consider all subsets of n elements of K_i and $\bar{K}_i$.) For fixed i the number of these subsets is clearly ($|B|$ denotes the number of elements of B)

$$\binom{|K_i|}{n}+\binom{|\bar{K}_i|}{n} \geqq 2\binom{n^2}{n} \qquad (|K_i|+|\bar{K}_i| = |M| = 2n^2).$$

Thus the total number of subsets of n elements under consideration $(1 \leqq i \leqq u_k)$ is at least $2u_k\binom{n^2}{n}$.

The total number of subsets of M taken n at a time is $\binom{2n^2}{n}$. Hence at least one of these sets, say A_{k+1}, occurs either in K_i or in $\bar{K}_i$ for at least

$$\frac{2u_k\binom{n^2}{n}}{\binom{2n^2}{n}} = 2u_k \prod_{i=0}^{n-1}(n^2-i)\left(\prod_{i=0}^{n-1}(2n^2-i)\right)^{-1} = \tag{4)}$$

$$= \frac{u_k}{2^{n-1}} \prod_{i=1}^{n-1}\left(1-\frac{i}{2n^2-i}\right) > \frac{u_k}{2^n}$$

values of i. Hence from (4) $u_{k+1} < u_k\left(1-\frac{1}{2^n}\right)$ and (3) is proved.

Clearly $u_0=2^{2n^2-1}$ (since M has 2^{2n^2} subsets). Hence from (3)

$$u_r \leqq 2^{2n^2-1}\left(1-\frac{1}{2^n}\right)^r. \tag{5}$$

Hence from (5) if $r=n^2 2^{n+1}$, $u_r<1$, thus $u_r=0$ and our sets A_i, $1 \leqq i \leqq n^2 2^{n+1}$ do not have property B and the proof of Theorem 1 is complete.

By taking M to have $\left[\frac{n^2}{2}\right]$ elements we could show by slightly more careful calculation that for every $\varepsilon>0$ and $n>n_0(\varepsilon)$

$$m(n)<(1+\varepsilon)e \log 2\, n^2 2^{n-2}. \tag{6}$$

It seems unlikely that (6) can be improved to any great without some new idea.

By methods used in a paper of Rényi and myself [4] I can prove the following

Theorem 2. Let M be a set of N elements. Put

$$k = CN2^n \prod_{i=1}^{n-1} \left(1 - \frac{i}{N-i}\right)^{-1} \tag{7}$$

where C is a sufficiently large absolute constant. Then for all but

$$O\left(\binom{\binom{N}{n}}{k}\right)$$

choices of k subsets A_i, $1 \leqq i \leqq k$ of M, the A's will not have property **B**.

I can show that the order of magnitude in (7) cannot be improved, but I can not determine the correct value of C.

Let M be a set of N elements. Denote by $m_N(n)$ the smallest integer for which there exist subsets A_i, $1 \leqq i \leqq m_N(n)$ of M which do not have property B. The problem makes sense only for $N \geqq 2n-1$ and clearly $m_{2n-1}(n) = \binom{2n-1}{n}$. For $N \geqq 2n-1$, $m_N(n)$ is a non-increasing function of N and for sufficiently large N, $m_N(n) = m(n)$. Let N_0 be the smallest integer for which $m_{N_0}(n) = m(n)$, probably $N_0 = Cn^2$. It seems to me that perhaps the order of magnitude of $m_N(n)$ is

$$N2^n \prod_{i=1}^{n-1} \left(1 - \frac{i}{N-i}\right)^{-1}.$$

This would in particular imply that if $N < c_1 n$, $m_N(n) > (2+c_2)^n$. I have been unable to throw any light on any of these questions.

MATHEMATICAL INSTITUTE,
EÖTVÖS LORÁND UNIVERSITY,
BUDAPEST

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