A PROBLEM ON TOURNAMENTS

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(received November 14, 1963)

By a tournament we mean the outcome of a round-robin tournament in which there are no draws. Such a tournament may be represented by a graph in which the n players are represented by vertices labelled $1, 2, \ldots, n$, and the outcomes of the games are represented by directed edges so that every pair of vertices is joined by one directed edge. We call such a graph a complete directed graph. One can also represent such a tournament by an $n \times n$ matrix $T = (t_{ij})$ in which t_{ij} is 1 if i beats j, and 0 otherwise, so that T is a $(0, 1)$ matrix with $t_{ij} + t_{ji} = 1$ for $i \neq j$ and (by definition) $t_{ii} = 0$.

In the summer of 1962 K. Schütte asked P. Erdős the following question: Does there exist for every k, a complete directed graph such that for every k vertices $x_1, x_2, \ldots, x_k$ there is one vertex y such that the edges $(x_i y)$, $i = 1, 2, \ldots, k$, are all directed away from y? Erdős [1] proved that, provided $n > (\log 2 + \varepsilon) k^2 2^k$ (ε a positive constant which can be taken arbitrarily close to 0 if k is large enough), there do exist complete directed graphs with this property. He also proved that such graphs do not exist with $n < 2^{k+1} - 1$. It is not obvious, and as far as we know it has never been proved, that if such graphs exist for a given n then they must also exist for every $m > n$.

At the seminar of the Canadian Mathematical Congress in Saskatoon in August, 1963, H. Ryser asked the following:

Canad. Math. Bull. vol. 7, no. 3, July 1964.

Is it true that in every tournament matrix, there is a set of 4 or fewer columns, such that every row has at least one 1 in at least one of these columns. L. Moser showed that the answer is no and in fact showed that for every large n, there are tournament matrices in which for every set of $[\log_2 n - 2\log_2 \log_2 n]$ columns there is some row which has no 1 in any of these columns. He also showed that there does exist, in every $n \times n$ tournament matrix, $[\log_2 (n+1)]$ columns such that every row has a 1 in at least one of these columns. He further observed that for $n > n_o(k, \ell)$ there are $n \times n$ tournament matrices in which for every k columns there are ℓ rows such that the $k \times \ell$ submatrix determined by these columns and rows consists entirely of zeros. It is easy to see that our results, which were obtained independently, are closely related. By our methods we can obtain, almost without any essentially new ideas, somewhat stronger results.

Consider a tournament on n players $1, 2, \ldots, n$. Pick k of them, say $x_1, x_2, \ldots, x_k$. Clearly one of the other players, y, can obtain 2^k different sets of results with the players $x_1, x_2, \ldots, x_k$. Now we prove

THEOREM 1. Let $n > (\log 2 + \varepsilon) k^2 2^k$. Then there exists a positive $\alpha = \alpha(\varepsilon)$ so that for each $\ell \leq k$ and every choice of ℓ players $x_1, x_2, \ldots, x_\ell$, each of the 2^k classes in which the remaining $n-\ell$ players are divided (two players are in the same class if they perform in an identical way against the players $x_1, x_2, \ldots, x_\ell$) contains more than $\alpha n/2^\ell$ players, for all but $o(2^{n(n-1)/2})$ of the tournaments.

By a slightly more complicated calculation we can prove

THEOREM 2. For every $\eta > 0$ there is a $c_1 = c_1(\eta)$ such that for $n > c_1 k^2 2^k$ and any $\ell \leq k$ players $x_1, x_2, \ldots, x_\ell$, each of the 2^ℓ classes contains $(1 + \delta)n/2^\ell$ players, where $|\delta| < \eta$, for all but $o(2^{n(n-1)/2})$ tournaments.

Theorem 2 can also be stated as follows. For every $\eta > 0$ there is a $c_2 = c_2(\eta)$ such that in almost all tournaments on n players, for every set of ℓ players $x_1, x_2, \ldots, x_\ell$, each of the 2^ℓ classes will contain $(1+\delta)n/2^\ell$ players, $|\delta| < \eta$, provided $\ell < \log_2 n - 2\log_2(\log_2 n) - c_2$.

Proof of theorem 1. The total number of tournaments of n players is $2^{n(n-1)/2}$. Thus it will suffice to show that the number of tournaments which do not satisfy the conditions of theorem 1 is $o(2^{n(n-1)/2})$. Further, a simple argument shows that it will suffice to prove the theorem for $\ell = k$. The k players $x_1, x_2, \ldots, x_k$ can be chosen in $\binom{n}{k}$ ways and, as already stated, there are 2^k classes into which the remaining n-k players are decomposed. Let us fix our attention on a particular set of k players $x_1, x_2, \ldots, x_k$ and a particular class (i.e., y is a member of the class if he wins against a fixed subset of the x's and loses against the complementary subset). Let us determine an upper bound for the number R(t) of tournaments in which our class contains exactly t players.

First of all, only the games between $x_1, x_2, \ldots, x_k$ and the remaining n-k players are restricted by our conditions so we have $\binom{n}{2} - k(n-k)$ unrestricted games and these yield for R(t) a factor $2^{\binom{n}{2}-k(n-k)}$. Next, the t players may be chosen from the n-k players in $\binom{n-k}{t}$ ways, and for the games between the t players and $x_1, x_2, \ldots, x_k$ the outcomes are determined. Finally, the games between $x_1, x_2, \ldots, x_k$ and any one of the remaining n-k-t players can go in $2^k - 1$ ways, since the only excluded case is if such a player is in the given class with respect to $x_1, x_2, \ldots, x_k$. Hence

$$(1) \qquad R(t) \leq 2^{\binom{n}{2}-k(n-k)} \binom{n-k}{t} (2^k - 1)^{n-k-t} .$$

Since we are assuming $t \leqq [\alpha n/2^k] = L$, and since the k players can be chosen in $\binom{n}{k}$ ways and there are 2^k classes, the total number of tournaments S which do not satisfy the conditions of theorem 1 fulfills the inequality

$$(2) \qquad S \leq \binom{n}{k} 2^k \sum_{t=0}^{L} R(t) .$$

To obtain an upper bound for S we note first that for k large, $\binom{n}{k} 2^k < n^k$ and that in the range $0 \leq t \leq L$, $R(t)$ is increasing with t. Hence using (1) and (2) we obtain

$$(3) \qquad S < n^k (L+1)\, 2^{\binom{n}{2}} \binom{n}{L} \left(1 - \frac{1}{2^k}\right)^{n-L} 2^{-kL} ,$$

and

$$(4) \qquad S < n^{k+1}\, 2^{\binom{n}{2}} \binom{n}{L} e^{-(n-L)/2^k}\, 2^{-kL} .$$

Our theorem will be established if we can show that $S = o(2^{n(n-1)/2})$ or

$$(5) \qquad n^{k+1} \binom{n}{L} 2^{-kL} e^{-\left(\frac{n-L}{2^k}\right)} = o(1) .$$

Now, note that $\binom{n}{L} 2^{-kL} < (ne/L2^k)^L \leqq (e/\alpha)^L$ so we must still prove only

$$(6) \qquad n^{k+1} \left(\frac{e}{\alpha}\right)^L e^{\left(\frac{n-L}{2^k}\right)} = o(1) .$$

From

$$n > (\log 2 + \varepsilon) k^2 2^k$$

we find

$$\frac{n}{\log n} > (1+\varepsilon_1)\, k\, 2^k$$

and

$$(k+1)\log n < \frac{n}{2^k}(1-\varepsilon_2)\ , \tag{7}$$

where ε_1 and ε_2 are positive numbers depending on ε. Taking logarithm of the left hand side of (6) and using (7) it is seen that it only remains to prove that

$$\frac{n-L}{2^k} - L(1-\log\alpha) - (1-\varepsilon_2)\frac{n}{2^k} \to \infty\ . \tag{8}$$

Since $L = n\alpha/2^k$ and $\alpha(1-\log\alpha) \to 0$ as $\alpha \to 0$ the required result follows.

We suppress the proof of theorem 2 since it is similar to that of theorem 1.

By the method used in the proof of theorem 1 we can also prove

THEOREM 3. Let $\varepsilon < \frac{1}{k}$, $n > n_o(\varepsilon, k)$. Consider all incomplete tournaments on n players who play $[n^{2-\varepsilon}]$ games. The number of tournaments is $\binom{n(n-1)/2}{[n^{2-\varepsilon}]}$. Almost all of these tournaments contain, for each k players, at least one player in each of the 2^k classes.

Theorem 3 is not very far from being best possible since if the number of games is $cn^{2-1/k}$ then we can show that for almost all tournaments there are k players for which there is no player who plays with all of them.

We conclude with two problems:

Problem 1. What is the minimum number of edges in a graph of n vertices so that it can be directed in such a way that to any k vertices $x_1, x_2, \ldots, x_k$ there is a vertex y such that all edges (x_i, y), $i = 1, 2, \ldots, k$ are directed from x_i to y? Of course we must assume here that n is large enough that some complete directed graph has the required property.

Problem 2. Let $n \geq k$. What is the smallest number $E(n;k)$ for which there is an ordinary graph of n vertices and E edges in which for every set of k vertices, there is some vertex, joined to each of these k.

We have solved this problem and hope to return to it.

REFERENCE

1. P. Erdős, Mathematical Gazette, 47 (1963) pp. 220-223.

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