# ON WEYL'S CRITERION FOR UNIFORM DISTRIBUTION

H. Davenport, P. Erdös and W. J. LeVeque

1. In his famous memoir [1] of 1916, Weyl gave a necessary and sufficient condition for a sequence $s_1, s_2, \cdots$ of real numbers to be uniformly distributed modulo 1, namely that for each integer $m \neq 0$,

$$S(N) = \frac{1}{N}\sum_{n=1}^{N} e(ms_n) \to 0$$

as $N \to \infty$. (Here $e(\alpha) = e^{2\pi i\alpha}$.) This criterion has been fundamental for much subsequent work on Diophantine approximation.

Now suppose that the sequence $s_n$ is replaced by a sequence $s_n(x)$ depending on a real parameter x, each $s_n(x)$ being bounded and integrable for $a \leq x \leq b$. Let

$$S(N, x) = \frac{1}{N}\sum_{n=1}^{N} e(ms_n(x)).$$

It is natural to ask: what condition on

$$I(N) = \int_a^b |S(N, x)|^2\,dx$$

will ensure that the sequence $s_n(x)$ is uniformly distributed modulo 1 for almost all x, in the sense of Lebesgue measure? We answer this question in the following theorem.

THEOREM. *If the series*

$$\sum N^{-1} I(N)$$

*converges for each integer* $m \neq 0$, *then the sequence* $s_n(x)$ *is uniformly distributed modulo* 1 *for almost all* x *in* $a \leq x \leq b$. *On the other hand, given any increasing function* $\Phi(M)$ *which tends to infinity with* M *(however slowly), there exists a sequence* $s_n(x)$ *which is not uniformly distributed modulo* 1 *for any* x, *and which satisfies the inequality*

$$\sum_{N=1}^{M} N^{-1} I(N) < \Phi(M).$$

2. The proof of the first half of the theorem is based on a principle of interpolation which was used in a particular case by Weyl himself [1; Section 7].

Since $\sum N^{-1} I(N)$ converges, there exists an increasing sequence $\lambda(N)$, with $\lambda(N) \to \infty$, such that

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$$\sum N^{-1} I(N)\lambda(N)$$

converges. (If $r(N) = \Sigma_{N_1 \geq N} N_1^{-1} I(N_1)$, we can take

$$\lambda(n) = \{r^{1/2}(N) + r^{1/2}(N+1)\}^{-1}.)$$

Let $M_1 < M_2 < \cdots$ be positive integers such that

$$M_{r+1} = \left[\frac{\lambda(M_r)}{\lambda(M_r) - 1} M_r\right] + 1.$$

Let $N_r$ be an integer in the range $M_r < N \leq M_{r+1}$ for which $I(N)$ attains its least value. Then

$$I(N_r) \leq \frac{1}{M_{r+1} - M_r} \sum_{N=M_r+1}^{M_{r+1}} I(N) \leq \frac{M_{r+1}}{M_{r+1} - M_r} \sum_{N=M_r+1}^{M_{r+1}} N^{-1} I(N).$$

Since

$$\frac{M_{r+1}}{M_{r+1} - M_r} < \lambda(M_r),$$

we see that

$$I(N_r) \leq \sum_{N=M_r+1}^{M_{r+1}} N^{-1} I(N)\lambda(N).$$

It follows that

$$\sum_r I(N_r)$$

converges. Since $M_{r+1}/M_r \to 1$, it is also true that $N_{r+1}/N_r \to 1$.

By a well known principle (see, for example, [1; Section 7]), it follows that

$$\sum_r |S(N_r, x)|^2$$

converges for almost all x, and *a fortiori* that

$$S(N_r, x) \to 0$$

as $r \to \infty$, for almost all x. Now, if $N_r < N \leq N_{r+1}$, then

$$|NS(N, x) - N_r S(N_r, x)| \leq \sum_{N=N_r+1}^{N_{r+1}} 1 = N_{r+1} - N_r,$$

whence

$$S(N, x) \to 0$$

as $N \to \infty$, for almost all $x$.

The above argument relates to a single value of $m$. But since the union of an enumerable infinity of sets of measure 0 is itself of measure 0, it follows that the result holds for all $m \neq 0$ except in a set of measure 0. Hence, by Weyl's criterion, $s_n(x)$ is uniformly distributed modulo 1 for almost all $x$.

3. For the second half of the theorem, an example suffices. Let $F(x)$ be a rapidly increasing function, defined for $x > 0$, and let $G$ be the function inverse to $F$. Define a sequence $s_n(x)$ by

$$s_n(x) = \begin{cases} 0 & \text{if } F(kx) < n < 2F(kx) \text{ for some } k \\ nx & \text{otherwise.} \end{cases}$$

Then the sequence $s_n(x)$ is not uniformly distributed modulo 1 for any $x$ in $0 < a < x < b$ if $F(x)$ grows at least exponentially; for if $N = [2F(kx)]$, then $s_n(x) = 0$ for roughly half the values of $n \leq N$.

Now,

$$S(N, x) = \frac{1}{N}\sum_{n=1}^{N} e(mnx) + \frac{1}{N}\sum_{n=1}^{N} \sum_{\substack{k \\ F(kx)<n<2F(kx)}} \{1 - e(mnx)\}.$$

The absolute value of the second sum is not greater than

$$2 \sum_{\substack{k \\ F(kx)<N}} F(kx) << F(k_1 x),$$

where $k_1 = k_1(x, N)$ is defined by the condition

$$F(k_1 x) < N \leq F((k_1 + 1)x).$$

(The notation $A(N) << B(N)$ means that there is a constant $c$, independent of $N$, such that $A(N) < cB(N)$ for all relevant $N$.) Hence, for $b > a > 0$ and $m$ a nonzero integer,

$$I(N) = \int_a^b |S(N, x)|^2\, dx << N^{-1} + N^{-2}\int_a^b (F(k_1 x))^2\, dx.$$

All values of $k_1$ that occur satisfy the inequalities

$$k_1 a < G(N), \quad (k_1 + 1)b \geq G(N).$$

A particular value $k$ of $k_1$ in this range occurs if $x$ has the property that

$$\frac{G(N)}{k+1} \leq x < \frac{G(N)}{k}.$$

Hence

$$\int_a^b (F(k_1 x))^2\,dx = \sum_{\frac{G(N)}{b}-1 \le k < \frac{G(N)}{a}} \int_{\frac{G(N)}{k+1}}^{\frac{G(N)}{k}} (F(kx))^2\,dx$$

$$= \sum_{\frac{G(N)}{b}-1 \le k < \frac{G(N)}{a}} \frac{1}{k}\int_{N_1}^{N} u^2\,G'(u)du\,,$$

on putting $kx = G(u)$. Here

$$N_1 = F\left(\frac{k}{k+1}\,G(N)\right).$$

Thus

$$\int_a^b (F(k_1 x))^2\,dx << \int_0^N u^2\,G'(u)du\,.$$

It follows that

(1) $$I(N) << N^{-1} + N^{-2}\int_0^N u^2\,G'(u)du\,.$$

We now conclude that

$$\sum_{N=1}^{M} N^{-1}\,I(N) << 1 + \int_0^M u^2\,G'(u) \sum_{N \ge u} \frac{1}{N^3}\,du << G(M)\,.$$

Thus, by suitable choice of the function G, we can ensure that $\Sigma N^{-1} I(N)$ diverges arbitrarily slowly.

It may be remarked that if we choose G to be a 'smooth' slowly increasing function, it will follow from (1) that $I(N) \to 0$ as $N \to \infty$. For example, if $G(u) = \log\log\log u$, we find that

$$I(N) << \frac{1}{(\log N)(\log\log N)}\,.$$

In particular, therefore, a condition of this type is compatible with $s_n(x)$ being not uniformly distributed for any x in (a, b).

## REFERENCES

1. H. Weyl, *Uber die Gleichverteilung von Zahlen mod. Eins,* Math. Ann. 77 (1916), 313-352.

Trinity College, Cambridge
University College, London
The University of Michigan