ON THE MAXIMAL NUMBER OF INDEPENDENT CIRCUITS IN A GRAPH

By

G. DIRAC (Hamburg) and P. ERDŐS (Budapest), member of the Academy

1. Introduction

In a recent paper [1] K. CORRÁDI and A. HAJNAL proved that if a finite graph without multiple edges contains at least $3k$ vertices and the valency of every vertex is at least $2k$, where k is a positive integer, then the graph contains k independent circuits, i. e. the graph contains as a subgraph a set of k circuits no two of which have a vertex in common. The present paper contains extensions of this theorem. In a recent paper [2] P. ERDŐS and L. PÓSA proved, among other things, that if a finite graph with or without loops and multiple edges contains n vertices and at least $n+4$ edges, then the graph contains two circuits without an edge in common. The present paper contains analogous results for planar graphs.

We adopt the following notation: $\bigcirc^k$ denotes a graph consisting of k independent circuits, $k\bigcirc$ denotes a graph consisting of k or more circuits no two of which have an edge in common. If $\mathcal{G}$ is a graph then $\mathcal{V}(\mathcal{G})$ denotes the set of vertices of $\mathcal{G}$, $\mathcal{V}_i(\mathcal{G})$ denotes the set of vertices of $\mathcal{G}$ having valency i in $\mathcal{G}$ (i being a non-negative integer), $\mathcal{V}_{\leqq i}(\mathcal{G})$, $\mathcal{V}_{\geqq i}(\mathcal{G})$ denote the set of vertices of $\mathcal{G}$ having valency $\leqq i$ and $\geqq i$, respectively, and $\mathcal{E}(\mathcal{G})$ denotes the set of edges of $\mathcal{G}$. The valency of the vertex x in the graph $\mathcal{G}$ will be denoted by $v(x, \mathcal{G})$. $|\mathcal{V}(\mathcal{G})|$ will be denoted by $V(\mathcal{G})$, $|\mathcal{E}(\mathcal{G})|$ by $E(\mathcal{G})$ etc.

In this notation the theorem of CORRÁDI and HAJNAL quoted above states that if $\mathcal{G}$ is a finite graph without multiple edges and if $V(\mathcal{G})\geqq 3k$ and $\mathcal{V}_{\leqq 2k-1}(\mathcal{G})=0$, then $\mathcal{G}\supset\bigcirc^k$; and the theorem of ERDŐS and PÓSA quoted above states that if $\mathcal{G}$ is a finite graph and $E(\mathcal{G})\geqq V(\mathcal{G})+4$, then $\mathcal{G}\supset 2\bigcirc$.

2. Concerning the existence of two independent circuits in finite graphs without multiple edges

THEOREM 1. *Let $\mathcal{G}$ denote a finite graph without loops or multiple edges.*

I. *If $V(\mathcal{G})\geqq 6$, $V_{\leqq 2}(\mathcal{G})=0$, and $V_{\geqq 4}(\mathcal{G})\geqq 4$, then $\mathcal{G}\supset\bigcirc^2$.*

II. *If $V(\mathcal{G})=7$ and $V_{\geqq 4}(\mathcal{G})\geqq 6$, then $\mathcal{G}\supset\bigcirc^2$.*

III. *If $V(\mathcal{G})=8$, $V_{\geqq 4}(\mathcal{G})\geqq 6$, and if $\mathcal{G}$ does not contain a vertex having valency 4 joined to two vertices having valency 1 then $\mathcal{G}\supset\bigcirc^2$.*

IV. *If $V(\mathcal{G})\geqq 9$ and $V_{\geqq 4}(\mathcal{G})\geqq \frac{1}{2}V(\mathcal{G})+2$, then $\mathcal{G}\supset\bigcirc^2$.*

V. *If $V(\mathcal{G})\geqq 9$ and $V_{\geqq 4}(\mathcal{G})-V_{\leqq 2}(\mathcal{G})\geqq 4$, then $\mathcal{G}\supset\bigcirc^2$.*

VI. *If* $V(\mathcal{G})=7$, $V_{\geqq 4}(\mathcal{G})=5$, *and* $V_2(\mathcal{G})=V_3(\mathcal{G})=1$, *then* $\mathcal{G}\supset \bigcirc^2$.

VII. *If* $V_{\geqq 4}(\mathcal{G})=5$, $V_3(\mathcal{G})=2$, *and* $V_{\leqq 2}(\mathcal{G})=1$, *then* $\mathcal{G}\supset \bigcirc^2$.

DEFINITION. A graph *barely satisfies* the conditions of I, III, IV, or Theorem 3, if it satisfies the conditions of I, III, IV, or Theorem 3, respectively, but when any one of its edges is deleted the remaining graph no longer does so.

It is easy to see that

(1) *If a graph satisfies the conditions of I or III or IV then it contains a subgraph having the same vertices which barely satisfies them.*

PROOF OF I. Suppose first that $V(\mathcal{G})=6$. Let the vertices of $\mathcal{G}$ be denoted by $g_1, g_2, \ldots, g_6$. The following two alternatives will be considered separately: (i) $\mathcal{G}$ contains two vertices of valency $\geqq 4$ not joined by an edge. (ii) Each pair of vertices of valency $\geqq 4$ are joined by an edge.

Assuming that (i) holds, it may be supposed that $v(g_1, \mathcal{G})\geqq 4$, $v(g_2, \mathcal{G})\geqq 4$ and $(g_1, g_2)\notin\mathcal{G}$. Then g_1 and g_2 are both joined to each of g_3, g_4, g_5, g_6. $V_0(\mathcal{G}-g_1-g_2)=0$ because $V_{\leqq 2}(\mathcal{G})=0$, and $V_{\geqq 2}(\mathcal{G}-g_1-g_2)\geqq 2$ because $V_{\geqq 4}(\mathcal{G})\geqq 4$; it follows that $\mathcal{G}-g_1-g_2$ contains two edges without a common vertex, and hence $\mathcal{G}\supset \bigcirc^2$.

Assuming that (ii) holds, it may be supposed that g_1, g_2, g_3, g_4 each have valency $\geqq 4$. Each of them is joined to at least one of g_5 and g_6 and g_5 and g_6 are each joined to at least two of g_1, g_2, g_3, g_4. The three alternatives that g_5 is joined to exactly two, three or four of g_1, g_2, g_3, g_4 will be considered in turn: If $(g_5, g_1)\in\mathcal{G}$, $(g_5, g_2)\in\mathcal{G}$, $(g_5, g_3)\notin\mathcal{G}$ and $(g_5, g_4)\notin\mathcal{G}$ then $(g_6, g_3)\in\mathcal{G}$ and $(g_6, g_4)\in\mathcal{G}$, so in this case $\mathcal{G}$ contains the two independent circuits $[g_1, g_2, g_5]$ and $[g_3, g_4, g_6]$. If $(g_5, g_1)\in\mathcal{G}$, $(g_5, g_2)\in\mathcal{G}$, $(g_5, g_3)\in\mathcal{G}$ and $(g_5, g_4)\notin\mathcal{G}$, then $(g_6, g_4)\in\mathcal{G}$, and it may be supposed that $(g_6, g_1)\in\mathcal{G}$ (for $v(g_6, \mathcal{G})\geqq 3$), in this case $\mathcal{G}$ contains the two independent circuits $[g_2, g_3, g_5]$ and $[g_1, g_4, g_6]$. If $(g_5, g_i)\in\mathcal{G}$ for $i=1, 2, 3, 4$, then it may be supposed that $(g_6, g_1)\in\mathcal{G}$ and $(g_6, g_2)\in\mathcal{G}$ (for $v(g_6, \mathcal{G})\geqq 3$) and in this case $\mathcal{G}$ contains the two independent circuits $[g_3, g_4, g_5]$ and $[g_1, g_2, g_6]$. Hence $\mathcal{G}\supset \bigcirc^2$ if (ii) holds. So I is true in the case $V(\mathcal{G})=6$.

The proof of I will now be completed by *reductio ad absurdum*. Assume that I is untrue. Then by (1) there exists a graph $\mathcal{G}_0$ with the following properties: $\mathcal{G}_0$ barely satisfies the conditions of I and $\mathcal{G}_0 \not\supset \bigcirc^2$, and all graphs which satisfy the conditions of I and contain fewer vertices then $\mathcal{G}_0$ contain two independent circuits. $V(\mathcal{G}_0)\geqq 7$ because I is true for graphs with six vertices.

It is easy to see that

(2) *If a graph barely satisfies the conditions of* I *then at least one of the end-vertices of every edge has valency* 3 *or* 4, *and at least one vertex has valency* 3.

Let d denote a vertex of $\mathcal{G}_0$ having valency 3, and let d_1, d_2, d_3 denote the three vertices of $\mathcal{G}_0$ to which d is joined. The three alternatives $E(\mathcal{G}_0(d_1, d_2, d_3)) \leqq 1$, $E(\mathcal{G}_0(d_1, d_2, d_3)) = 2$, $E(\mathcal{G}_0(d_1, d_2, d_3)) = 3$ will be considered separately.

Assuming that $E(\mathcal{G}_0(d_1, d_2, d_3)) \leqq 1$, it may be supposed that $(d_1, d_3) \notin \mathcal{G}_0$ and $(d_2, d_3) \notin \mathcal{G}_0$. Let $\mathcal{G}' = (\mathcal{G}_0 - d) \cup (d_1, d_3) \cup (d_2, d_3)$. $\mathcal{G}'$ satisfies the conditions of I. Hence $\mathcal{G}' \supset \bigcirc^2$ because of the minimal nature of $\mathcal{G}_0$. It follows that $\mathcal{G}_0 \supset \bigcirc^2$. ($(d_1, d_3)$ can be replaced by $d \cup (d, d_1) \cup (d, d_3)$ if necessary, alternatively $d_3 \cup \cup (d_1, d_3) \cup (d_2, d_3)$ can be replaced by $d \cup (d, d_1) \cup (d, d_2)$ if necessary. This argument is used in [2].) This contradicts $\mathcal{G}_0 \not\supset \bigcirc^2$.

Assuming that $E(\mathcal{G}_0(d_1, d_2, d_3)) = 2$, it may be supposed that $(d_1, d_2) \in \mathcal{G}_0$, $(d_1, d_3) \in \mathcal{G}_0$ and $(d_2, d_3) \notin \mathcal{G}_0$. There are two alternatives: $v(d_1, \mathcal{G}_0) \geqq 5$ and $v(d_1, \mathcal{G}_0) \leqq 4$. If $v(d_1, \mathcal{G}_0) \geqq 5$ then $(\mathcal{G}_0 - d) \cup (d_2, d_3)$ satisfies the conditions of I, and therefore contains two independent circuits because of the minimal nature of $\mathcal{G}_0$; it follows that $\mathcal{G}_0 \supset \bigcirc^2$ (replacing (d_2, d_3) by $d \cup (d, d_2) \cup (d, d_3)$ if necessary; this argument is used in [2]), in contradiction to $\mathcal{G}_0 \not\supset \bigcirc^2$. If $v(d_1, \mathcal{G}_0) \leqq 4$ then a contradiction is arrived at as follows: $\mathcal{G}_0 - d - d_1 - d_2 - d_3 \not\supset \bigcirc$, so if it has ν connected components then it contains at most $V(\mathcal{G}_0) - 4 - \nu$ edges. Hence $E(\mathcal{G}_0) \leqq \leqq V(\mathcal{G}_0) - 4 - \nu + v(d_2, \mathcal{G}_0) + v(d_3, \mathcal{G}_0) + 2$ (since $v(d_1, \mathcal{G}_0) \leqq 4$). On the other hand, summing the valencies of all the vertices of $\mathcal{G}_0$, $2E(\mathcal{G}_0) \geqq v(d_2, \mathcal{G}_0) + v(d_3, \mathcal{G}_0) + + 8 + 3(V(\mathcal{G}_0) - 4)$. From the two inequalities it follows that $V(\mathcal{G}_0) \leqq v(d_2, \mathcal{G}_0) + + v(d_3, \mathcal{G}_0) - 2\nu$. But d_2 and d_3 are each joined to at most one vertex of each connected component of $\mathcal{G}_0 - d - d_1 - d_2 - d_3$ because $\mathcal{G}_0 \not\supset \bigcirc^2$, so $v(d_2, \mathcal{G}_0) \leqq \nu + 2$ and $v(d_3, \mathcal{G}_0) \leqq \nu + 2$. Hence $V(\mathcal{G}_0) \leqq 4$ which is a contradiction.

The only remaining alternative is that $E(\mathcal{G}_0(d_1, d_2, d_3)) = 3$. By (2) it may be assumed that $v(d_1, \mathcal{G}_0) \leqq 4$ and $v(d_2, \mathcal{G}_0) \leqq 4$. $\mathcal{G}_0 - d - d_1 - d_2 - d_3 \not\supset \bigcirc$ so $E(\mathcal{G}_0 - d - d_1 - d_2 - d_3) \leqq V(\mathcal{G}_0) - 5$. Consequently $E(\mathcal{G}_0) \leqq V(\mathcal{G}_0) + v(d_3, \mathcal{G}_0)$. On the other hand, summing the valencies of all the vertices of $\mathcal{G}_0$, $2E(\mathcal{G}_0) \geqq v(d_3, \mathcal{G}_0) + + 12 + 3(V(\mathcal{G}_0) - 4) = 3V(\mathcal{G}_0) + v(d_3, \mathcal{G}_0)$. From the two inequalities it follows that $v(d_3, \mathcal{G}_0) \geqq V(\mathcal{G}_0)$, which is absurd because $\mathcal{G}_0$ contains no loops or multiple edges.

The hypothesis that I is untrue leads to a contradiction, therefore I is true.

Proof of II. If $V_{\leqq 2}(\mathcal{G}) = 0$ then $\mathcal{G} \supset \bigcirc^2$ by I. If $V_{\leqq 2}(\mathcal{G}) \neq 0$ then $V_{\leqq 2}(\mathcal{G}) = 1$. Suppose that $V_{\leqq 2}(\mathcal{G}) = 1$ and let v denote a vertex having valency $\leqq 2$. Then $V(\mathcal{G} - v) = 6$, $V_{\leqq 2}(\mathcal{G} - v) = 0$ and $V_{\geqq 4}(\mathcal{G} - v) \geqq 4$. Therefore $\mathcal{G} - v \supset \bigcirc^2$ by I.

Proof of III. By (1) in order to prove III it is sufficient to prove it for graphs which barely satisfy its conditions. Let $\mathcal{G}$ denote a graph which barely satisfies the conditions of III.

(3) $V_{\leqq 3}(\mathcal{G}) \geqq 1$.

For if $V_{\leq 3}(\mathcal{G})=0$ then the graph obtained by deleting any one edge from $\mathcal{G}$ satisfies the conditions of III.

If $V_0(\mathcal{G})\geqq 1$ *then* $\mathcal{G}\supset \bigcirc^2$.

For if a is a vertex of $\mathcal{G}$ having valency 0 then $V(\mathcal{G}-a)=7$ and $V_{\geqq 4}(\mathcal{G}-a)\geqq 6$, so $\mathcal{G}-a\supset \bigcirc^2$ by II. *In the remainder of the proof of* III *suppose that* $V_0(\mathcal{G})=0$.

If $V_2(\mathcal{G})\geqq 1$ *then it follows that* $\mathcal{G}\supset \bigcirc^2$.

For suppose that c is a vertex of $\mathcal{G}$ having valency 2 and is joined to the vertices c_1 and c_2. Either $(c_1, c_2)\notin \mathcal{G}$ or $(c_1, c_2)\in \mathcal{G}$. If $(c_1, c_2)\notin\mathcal{G}$ then let $\mathcal{G}'=(\mathcal{G}-c)\cup \cup(c_1, c_2)$. $V(\mathcal{G}')=7$ and $V_{\geqq 4}(\mathcal{G})\geqq 6$, consequently $\mathcal{G}'\supset \bigcirc^2$ by II. It follows that $\mathcal{G}\supset \bigcirc^2$. ($(c_1, c_2)$ can be replaced by $c\cup(c, c_1)\cup(c, c_2)$ if necessary. This argument is used in [2].)

Suppose that $(c_1, c_2)\in\mathcal{G}$. If $v(c_1, \mathcal{G})\geqq 5$ and $v(c_2, \mathcal{G})\geqq 5$, then $V_{\geqq 4}(\mathcal{G}-c)= =V_{\geqq 4}(\mathcal{G})\geqq 6$ and $V(\mathcal{G}-c)=7$, so $\mathcal{G}-c\supset \bigcirc^2$ by II. There remains the alternative that $v(c_1, \mathcal{G})\leqq 4$ or $v(c_2, \mathcal{G})\leqq 4$, let it be assumed that $v(c_1, \mathcal{G})\leqq 4$. It follows that $\mathcal{G}\supset \bigcirc^2$. For suppose on the contrary that $\mathcal{G}\not\supset \bigcirc^2$. Then $E(\mathcal{G}-c-c_1-c_2)\leqq 4$ because $\mathcal{G}-c-c_1-c_2\not\supset \bigcirc$. Hence $E(\mathcal{G})\leqq 4+v(c_1, \mathcal{G})-2+v(c_2, \mathcal{G})-2+3\leqq \leqq 7+v(c_2, \mathcal{G})$. On the other hand, summing the valencies of all the vertices of $\mathcal{G}$, $2E(\mathcal{G})\geqq 5\cdot 4+v(c_2, \mathcal{G})+2 = 22+v(c_2, \mathcal{G})$ since $v(c, \mathcal{G})=2$ and $V_0(\mathcal{G})=0$. From the two inequalities it follows that $v(c_2, \mathcal{G})\geqq 8$ which is absurd because $\mathcal{G}$ contains no loops or multiple edges. Hence $\mathcal{G}\supset \bigcirc^2$ if $V_2(\mathcal{G})\geqq 1$. *In the remainder of the proof of* III *suppose that* $V_2(\mathcal{G})=0$.

If $V_3(\mathcal{G})\geqq 1$ *then it follows that* $\mathcal{G}\supset \bigcirc^2$.

For suppose that d is a vertex of $\mathcal{G}$ having valency 3 and is joined to the vertices d_1, d_2 and d_3. The two alternatives $E(\mathcal{G}(d_1, d_2, d_3))\leqq 1$ and $E(\mathcal{G}(d_1, d_2, d_3))\geqq 2$ will be considered separately.

Assuming that $E(\mathcal{G}(d_1, d_2, d_3))\leqq 1$, it may be supposed that $(d_1, d_3)\notin\mathcal{G}$ and $(d_2, d_3)\notin G$. Let $\mathcal{G}'=(\mathcal{G}-d)\cup(d_1, d_3)\cup(d_2, d_3)$. Clearly $V(\mathcal{G}')=7$ and $V_{\geqq 4}(\mathcal{G}')\geqq \geqq V_{\geqq 4}(\mathcal{G})\geqq 6$, so $\mathcal{G}'\supset \bigcirc^2$ by II. It follows as in the proof of I that $\mathcal{G}\supset \bigcirc^2$.

Assuming that $E(\mathcal{G}(d_1, d_2, d_3))\geqq 2$, it may be supposed that $(d_1, d_2)\in\mathcal{G}$ and $(d_1, d_3)\in\mathcal{G}$. If any edge incident with d is deleted from $\mathcal{G}$ then the remaining graph does not contain two vertices of valency 1, therefore the remaining graph contains only five vertices of valency $\geqq 4$, since $\mathcal{G}$ barely satisfies the conditions of III. Consequently $v(d_1, \mathcal{G})=v(d_2, \mathcal{G})=v(d_3, \mathcal{G})=4$. It follows that $\mathcal{G}\supset \bigcirc^2$. For suppose on the contrary that $\mathcal{G}\not\supset \bigcirc^2$. Then $E(\mathcal{G}-d-d_1-d_2-d_3)\leqq 3$ because $\mathcal{G}-d-d_1- -d_2-d_3\not\supset \bigcirc$. Hence $E(\mathcal{G})\leqq 3+5+5 = 13$. On the other hand, summing the valencies of all the vertices of $\mathcal{G}$, $2E(\mathcal{G})\geqq 6\cdot 4+3 = 27$, since $v(d, \mathcal{G})=3$. This contradiction proves that $\mathcal{G}\supset \bigcirc^2$ if $V_3(\mathcal{G})\geqq 1$.

It thus remains only to consider the case in which $V_i(\mathcal{G})=0$ *for* $i=0, 2$ *and* 3. By (3) there are then two alternatives: $V_{\geqq 4}(\mathcal{G})=7$ and $V_1(\mathcal{G})=1$, or else $V_{\geqq 4}(\mathcal{G})=6$ and $V_1(\mathcal{G})=2$.

Assume first that $V_{\geqq 4}(\mathcal{G})=7$ and $V_1(\mathcal{G})=1$. Let b denote the vertex of $\mathcal{G}$ having valency 1. Then $V(\mathcal{G}-b)=7$ and $V_{\geqq 4}(\mathcal{G}-b)\geqq 6$. Therefore by II $\mathcal{G}-b\supset\bigcirc^2$.

Assume next that $V_{\geqq 4}(\mathcal{G})=6$ and $V_1(\mathcal{G})=2$. Let b and b' denote the vertices of $\mathcal{G}$ having valency 1. Either b and b' are both joined to the same vertex, or they are not. If b and b' are both joined to the same vertex b_1 say, then it follows from the conditions of III that $v(b_1, \mathcal{G})\geqq 5$. Therefore $\mathcal{G}-(b, b_1)$ satisfies the conditions of III, in contradiction to the definition of $\mathcal{G}$. Hence b and b' are not both joined to the same vertex. Therefore $V(\mathcal{G}-b-b')=6$, $V_{\leqq 2}(\mathcal{G}-b-b')=0$ and $V_{\geqq 4}(\mathcal{G}-b-b')\geqq 4$. Consequently by I $\mathcal{G}-b-b'\supset\bigcirc^2$. III is now proved.

PROOF OF IV by *reductio ad absurdum*. Assume that IV is untrue. Then by (1) there exists a graph $\mathcal{G}_0$ with the following properties: $\mathcal{G}_0$ barely satisfies the conditions of IV and $\mathcal{G}_0\not\supset\bigcirc^2$, and all graphs which satisfy the conditions of IV and have fewer vertices than $\mathcal{G}_0$ contain two independent circuits.

(4) *At least one end-vertex of every edge contained in* $\mathcal{G}_0$ *has valency* 4, *and* $V_{\leqq 3}(\mathcal{G}_0)\geqq 1$.

For, since $\mathcal{G}_0$ barely satisfies the conditions of IV, if any edge is deleted from $\mathcal{G}_0$ the number of vertices of valency $\geqq 4$ is decreased; and if $V_{\leqq 3}(\mathcal{G}_0)=0$ then if any edge is deleted from $\mathcal{G}_0$, the remaining graph satisfies the conditions of IV, which contradicts the hypothesis that $\mathcal{G}_0$ barely satisfies the conditions of IV.

By summing the valencies of all the vertices of $\mathcal{G}_0$ we have that

(5) *If* $V_0(\mathcal{G}_0)=0$ *then* $E(\mathcal{G}_0)\geqq 1\frac{1}{4}V(\mathcal{G}_0)+3$, *if* $V_0(\mathcal{G}_0)=0$ *and* $V_2(\mathcal{G}_0)\geqq 1$, *then* $E(\mathcal{G}_0)\geqq 1\frac{1}{4}V(\mathcal{G}_0)+3\frac{1}{2}$, *and if* $V_0(\mathcal{G}_0)=0$ *and* $V_3(\mathcal{G}_0)\geqq 1$, *then* $E(\mathcal{G}_0)\geqq 1\frac{1}{4}V(\mathcal{G}_0)+4$.

It will now be proved that $V_i(\mathcal{G}_0)=0$ for $i=0, 2, 3$.

(6) $V_0(\mathcal{G}_0)=0$.

For suppose on the contrary that the vertex a of $\mathcal{G}_0$ has valency 0. If $V(\mathcal{G}_0)=9$ then $V(\mathcal{G}_0-a)=8$ and $V_{\geqq 4}(\mathcal{G}_0-a)\geqq 7$, so by III $\mathcal{G}_0-a\supset\bigcirc^2$, contrary to $\mathcal{G}_0\not\supset\bigcirc^2$ If $V(\mathcal{G}_0)\geqq 10$, then $\mathcal{G}_0-a$ satisfies the conditions of IV, and so $\mathcal{G}_0-a\supset\bigcirc^2$ because of the minimal property of $\mathcal{G}_0$ whereas $\mathcal{G}_0\not\supset\bigcirc^2$. Hence $V_0(\mathcal{G}_0)=0$.

(7) $V_2(\mathcal{G}_0)=0$.

For suppose on the contrary that the vertex c of $\mathcal{G}_0$ has valency 2 and is joined to the vertices c_1 and c_2. The two alternatives $(c_1, c_2)\notin\mathcal{G}_0$ and $(c_1, c_2)\in\mathcal{G}_0$ will be considered in turn and a contradiction will be derived in each case. If $(c_1, c_2)\notin\mathcal{G}_0$ then let $\mathcal{G}'=(\mathcal{G}_0-c)\cup(c_1, c_2)$. Clearly $V_{\geqq 4}(\mathcal{G}')=V_{\geqq 4}(\mathcal{G}_0)$. If $V(\mathcal{G}_0)=9$ then $V(\mathcal{G}')=8$ and $V_{\geqq 4}(\mathcal{G}')\geqq 7$, consequently $\mathcal{G}'\supset\bigcirc^2$ by III, if $V(\mathcal{G}_0)\geqq 10$ then $\mathcal{G}'$ satisfies the conditions of IV, so $\mathcal{G}'\supset\bigcirc^2$ because of the minimal property of $\mathcal{G}_0$. From $\mathcal{G}'\supset\bigcirc^2$ it follows as in the proof of III that $\mathcal{G}_0\supset\bigcirc^2$, whereas $\mathcal{G}_0\not\supset\bigcirc^2$. Suppose next that $(c_1, c_2)\in\mathcal{G}_0$. By (4) $v(c_1, \mathcal{G}_0)=v(c_2, \mathcal{G}_0)=4$. $E(\mathcal{G}_0-c-c_1-c_2)\leqq$

$\leqq V(\mathcal{G}_0)-4$ because $\mathcal{G}_0-c-c_1-c_2 \not\supset \bigcirc$. It follows that $E(\mathcal{G}) \leqq V(\mathcal{G}_0)+3$, which contradicts (5). Hence $V_2(\mathcal{G}_0)=0$.

(8) $V_3(\mathcal{G}_0)=0$.

For suppose on the contrary that the vertex d of $\mathcal{G}_0$ has valency 3 and is joined to the vertices d_1, d_2 and d_3. The two alternatives $E(\mathcal{G}_0(d_1, d_2, d_3)) \leqq 1$ and $E(\mathcal{G}_0(d_1, d_2, d_3)) \geqq 2$ will be considered in turn and a contradiction derived in each case.

Assuming that $E(\mathcal{G}_0(d_1, d_2, d_3)) \leqq 1$, it may be supposed that $(d_1, d_3) \notin \mathcal{G}_0$ and $(d_2, d_3) \notin \mathcal{G}_0$. Let $\mathcal{G}' = (\mathcal{G}_0-d) \cup (d_1, d_3) \cup (d_2, d_3)$. Clearly $V_{\geqq 4}(\mathcal{G}') \geqq V_{\geqq 4}(\mathcal{G}_0)$. If $V(\mathcal{G}_0)=9$ then $V(\mathcal{G}')=8$ and $V_{\geqq 4}(\mathcal{G}') \geqq 7$, and consequently $\mathcal{G}' \supset \bigcirc^2$ by III; if $V(\mathcal{G}_0) \geqq 10$ then $\mathcal{G}'$ satisfies the conditions of IV, and consequently $\mathcal{G}' \supset \bigcirc^2$ because of the minimal property of $\mathcal{G}_0$. From $\mathcal{G}' \supset \bigcirc^2$ it follows as in the proof of I that $\mathcal{G}_0 \supset \bigcirc^2$, whereas $\mathcal{G}_0 \not\supset \bigcirc^2$. There remains the alternative that $E(\mathcal{G}_0(d_1, d_2, d_3)) \geqq 2$. By (4) $v(d_1, \mathcal{G}_0) = v(d_2, \mathcal{G}_0) = v(d_3, \mathcal{G}_0)=4$. $E(\mathcal{G}_0-d-d_1-$ $-d_2-d_3) \leqq V(\mathcal{G}_0)-5$ because $\mathcal{G}_0-d-d_1-d_2-d_3 \not\supset \bigcirc$. It follows that $E(\mathcal{G}_0) \leqq$ $\leqq V(\mathcal{G}_0)+5$, which contradicts (5). Hence $V_3(\mathcal{G}_0)=0$.

From the conditions of IV it follows that $V_{\geqq 4}(\mathcal{G}_0) \geqq 7$ and $V_{\geqq 4}(\mathcal{G}_0)-V_{\leqq 3}(\mathcal{G}_0) \geqq$ $\geqq 4$. By (6), (7) and (8) $V(\mathcal{G}_0)=V_1(\mathcal{G}_0)+V_{\geqq 4}(\mathcal{G}_0)$ and by (4) each vertex of valency 1 is joined to a vertex of valency 4. Let $\mathcal{G}''$ denote the graph obtained by deleting all vertices of valency 1 from $\mathcal{G}$. Clearly $V(\mathcal{G}'')=V_{\geqq 4}(\mathcal{G}_0) \geqq 7$, $V_{\geqq 4}(\mathcal{G}'')=V_{\geqq 4}(\mathcal{G}_0)-$ $-V_1(\mathcal{G}_0) \geqq 4$ and $V_{\leqq 2}(\mathcal{G}'')=0$. Therefore by I $\mathcal{G}'' \supset \bigcirc^2$, whereas $\mathcal{G}_0 \not\supset \bigcirc^2$. This contradiction proves IV.

Proof of V. If $V_3(\mathcal{G})=0$ then $V(\mathcal{G})=V_{\leqq 2}(\mathcal{G})+V_{\geqq 4}(\mathcal{G})$ and so $V_{\geqq 4}(\mathcal{G}) \geqq$ $\geqq \frac{1}{2} V(\mathcal{G})+2$; hence by IV $\mathcal{G} \supset \bigcirc^2$. Suppose that $V_3(\mathcal{G}) \geqq 1$, and let $x_1, \ldots, x_u$ denote the vertices having valency 3 in $\mathcal{G}$. Let $y_1, \ldots, y_u$ be u distinct vertices none of which belong to $\mathcal{G}$, and let $\mathcal{G}^* = \mathcal{G} \cup \{y_1, \ldots, y_u\} \cup (x_1, y_1) \cup \ldots \cup (x_u, y_u)$. Clearly $V(\mathcal{G}^*)=V(\mathcal{G})+u=V(\mathcal{G})+V_3(\mathcal{G})$; $V_{\leqq 2}(\mathcal{G}^*)=V_{\leqq 2}(\mathcal{G})+V_3(\mathcal{G})$; $V_{\geqq 4}(\mathcal{G}^*)=V_{\geqq 4}(\mathcal{G})+$ $+V_3(\mathcal{G})$. Hence $V_{\geqq 4}(\mathcal{G}^*) \geqq V_{\leqq 2}(\mathcal{G}^*)+4$, since $V_{\geqq 4}(\mathcal{G}) \geqq V_{\leqq 2}(\mathcal{G})+4$. Also $V_3(\mathcal{G}^*)=0$. It follows that $V_{\geqq 4}(\mathcal{G}^*) \geqq \frac{1}{2} V(\mathcal{G}^*)+2$. Hence by IV $\mathcal{G}^* \supset \bigcirc^2$, consequently $\mathcal{G} \supset \bigcirc^2$. This proves V.

Proof of VI. Let the vertices of $\mathcal{G}$ be $g_1, g_2, \ldots, g_7$, where $v(g_1, \mathcal{G})=2$ and $v(g_2, \mathcal{G})=3$. Let g_8, g_9, g_{10} be three distinct vertices not belonging to $\mathcal{G}$ and let $\mathcal{G}^\circ = \mathcal{G} \cup \{g_8 g_9 g_{10}\} \cup (g_1, g_8) \cup (g_2, g_9) \cup (g_1, g_{10})$. Then $V(\mathcal{G}^\circ)=10$ and $V_{\geqq 4}(\mathcal{G}^\circ)=7$, so $\mathcal{G}^\circ \supset \bigcirc^2$ by IV, and consequently $\mathcal{G} \supset \bigcirc^2$.

Proof of VII. Construct $\mathcal{G}^+$ from $\mathcal{G}$ by adding a new vertex not in $\mathcal{G}$ and joining it by an edge to a vertex of $V_3(\mathcal{G})$. Then $V_{\geqq 4}(\mathcal{G}^+)=6$, $V_3(\mathcal{G}^+)=1$, and $V_{\leqq 2}(\mathcal{G}^+)=2$. Hence by V $\mathcal{G}^+ \supset \bigcirc^2$, consequently $\mathcal{G} \supset \bigcirc^2$.

Remarks concerning Theorem 1. 1. IV *and* V *are equivalent.* For V has been deduced from IV, and IV clearly follows from V.

2. *If* $V(\mathcal{G}) \geqq 9$ *then* I *is a consequence of* V.

3. I, ..., VII *are best possible.* In order to demonstrate this we define the graphs $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ as follows: $\mathcal{V}(\mathcal{A}) = \{x_1, x_2, x_3, y_1, \dots, y_u\}$ $(u \geqq 3)$, $\mathcal{E}(\mathcal{A}) = \{(x_1, x_2)(x_2, x_3)(x_3, x_1)(x_i, y_j)\}$ $(i = 1, 2, 3;\ \ j = 1, \dots, u)$. $\mathcal{V}(\mathcal{A}') = \mathcal{V}(\mathcal{A}) \cup \{z_1 \dots z_u\}$, $\mathcal{E}(\mathcal{A}') = \mathcal{E}(\mathcal{A}) \cup \{(y_1, z_1) \dots (y_u, z_u)\}$. $\mathcal{V}(\mathcal{B}) = \{f_1 f_2 \dots f_6\}$, $\mathcal{E}(\mathcal{B}) = \{(f_i, f_j)\} \cup (f_6, f_1) \cup (f_6, f_5) - (f_1, f_5)$ $(i = 1, \dots, 5;\ j = 1, \dots, 5)$. $\mathcal{V}(\mathcal{C}) = \mathcal{V}(\mathcal{B}) + f_7$, $\mathcal{E}(\mathcal{C}) = \mathcal{E}(\mathcal{B}) + (f_6, f_7)$. $\mathcal{V}(\mathcal{D}) = \mathcal{V}(\mathcal{C}) + f_8$, $\mathcal{E}(\mathcal{D}) = \mathcal{E}(\mathcal{C}) + (f_6, f_8)$. $V(\mathcal{A}) \geqq 6$, $V_{\leqq 2}(\mathcal{A}) = 0$, and $V_{\geqq 4}(\mathcal{A}) = 3$ but $\mathcal{A} \not\supset \bigcirc^2$. $V(\mathcal{A}') \geqq 9$, $V_{\geqq 4}(\mathcal{A}') = \frac{1}{2} V(\mathcal{A}') + 1\frac{1}{2}$ but $\mathcal{A}' \not\supset \bigcirc^2$. $V(\mathcal{B}) = 6$, $V_4(\mathcal{B}) = 5$ and $V_2(\mathcal{B}) = 1$, but $\mathcal{B} \not\supset \bigcirc^2$. $V(\mathcal{C}) = 7$, $V_4(\mathcal{C}) = 5$, $V_3(\mathcal{C}) = 1$ and $V_1(\mathcal{C}) = 1$ but $\mathcal{C} \not\supset \bigcirc^2$. $V(\mathcal{D}) = 8$, $V_4(\mathcal{D}) = 6$, $V_1(\mathcal{D}) = 2$ but $\mathcal{D} \not\supset \bigcirc^2$.

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ show that the conditions of I cannot be relaxed. $\mathcal{C}, \mathcal{D}$ and $\mathcal{A}'$, respectively, show that the conditions of II, III and IV cannot be relaxed. $\mathcal{A}$ and $\mathcal{A}'$ show that the conditions of V cannot be relaxed. $\mathcal{C}$ and $\mathcal{D}$, respectively, show that the conditions of VI and VII cannot be relaxed.

For finite graphs without multiple edges which contain at least one loop a much weaker condition ensures the existence of two independent circuits.

THEOREM 2. *If* $\mathcal{G}$ *is a finite graph containing at least one loop but no multiple edges, and if* $V(\mathcal{G}) \geqq 3$ *and* $V_{\geqq 4}(\mathcal{G}) \geqq V_{\leqq 2}(\mathcal{G})$, *then* $\mathcal{G} \supset \bigcirc^2$.

PROOF. Let m denote a vertex of $\mathcal{G}$ which is incident with one or more loops. If $\mathcal{G} - m$ contains a loop, then $\mathcal{G} \supset \bigcirc^2$. In what follows suppose that $\mathcal{G} - m$ contains no loop. Clearly $V_{\geqq 3}(\mathcal{G} - m) \geqq V_{\geqq 4}(\mathcal{G}) - 1$, and $V_{\leqq 1}(\mathcal{G} - m) \leqq V_{\leqq 2}(\mathcal{G})$. Therefore $V_{\geqq 3}(\mathcal{G} - m) + 1 \geqq V_{\geqq 4}(\mathcal{G}) \geqq V_{\leqq 2}(\mathcal{G}) \geqq V_{\leqq 1}(\mathcal{G} - m)$. Also $V(G - m) \geqq 2$. It follows that $\mathcal{G} - m \supset \bigcirc$, for if $\mathcal{H}$ is a finite graph without circuits and with at least two vertices, then $V_{\leqq 1}(\mathcal{H}) \geqq V_{\geqq 3}(\mathcal{H}) + 2$ as may easily be verified. Hence $\mathcal{G} \supset \bigcirc^2$.

REMARKS. 1. *The conditions of Theorem* 2 *cannot be relaxed.* The example of a graph consisting of a vertex incident with four loops joined to a vertex of valency 1 shows that the condition $V(\mathcal{G}) \geqq 3$ is essential. The example of a graph consisting of a vertex incident with at least two loops joined to every vertex of a path shows that the condition $V_{\geqq 4}(\mathcal{G}) \geqq V_{\leqq 2}(\mathcal{G})$ is essential.

2. *No result analogous to Theorems* 1 *and* 2 *can be obtained for graphs which may contain multiple edges.* This is shown by the example of a graph consisting of a vertex incident with any number of loops joined to the vertices of a path by any number of edges.

3. Concerning the existence of three or more independent circuits in finite graphs without multiple edges

THEOREM 3. *If* $\mathcal{G}$ *is a finite graph without loops or multiple edges and* k *is a natural number* $\geqq 3$, *and if* $V_{\geqq 2k}(\mathcal{G}) - V_{\leqq 2k-2}(\mathcal{G}) \geqq k^2 + 2k - 4$, *then* $\mathcal{G} \supset \bigcirc^k$.

N. B. Theorem 1. V states that this is true for $k = 2$ if $V(\mathcal{G}) \leqq 9$.

PROOF OF THEOREM 3 by *reductio ad absurdum*. Assume that Theorem 3 is untrue. Let $\varkappa$ denote the least value of k for which the assertion of Theorem 3 is false, and among the graphs $\mathcal{G}$ such that $V_{\geqq 2\varkappa}(\mathcal{G})-V_{\leqq 2\varkappa-2}(\mathcal{G})\geqq \varkappa^2+2\varkappa-4$ and $\mathcal{G}\not\supset \bigcirc^{\varkappa}$ let $\mathcal{G}_0$ be one with the least possible number of vertices and barely satisfying the conditions of Theorem 3 with $k=\varkappa$. This will be shown to lead to a contradiction by a method amounting to an induction process starting from the particular case of the theorem of CORRÁDI and HAJNAL with $\varkappa^2+2\varkappa-4\leqq V(\mathcal{G})\leqq \varkappa^2+2\varkappa-3$ (note that $\varkappa^2+2\varkappa-4\geqq 3\varkappa+2$ if $\varkappa\geqq 3$).

(1) *At least one of the end-vertices of every edge of* $\mathcal{G}_0$ *has valency* $2\varkappa$ *or* $2\varkappa-1$.

For, since $\mathcal{G}_0$ barely satisfies the conditions of Theorem 3 with $k=\varkappa$, by deleting any edge from $\mathcal{G}_0$ either $V_{\geqq 2\varkappa}(\mathcal{G}_0)$ is decreased or $V_{\leqq 2\varkappa-2}(\mathcal{G}_0)$ is increased.

(2) $V_{\leqq 2\varkappa-1}(\mathcal{G}_0)\geqq 1$.

If $V(\mathcal{G}_0)\geqq \varkappa^2+2\varkappa-2$ then $V_{\leqq 2\varkappa-1}(\mathcal{G}_0)\geqq 1$ because $\mathcal{G}_0$ barely satisfies the conditions of Theorem 3. If $\varkappa^2+2\varkappa-4\leqq V(\mathcal{G}_0)\leqq \varkappa^2+2\varkappa-3$ then $V_{\leqq 2\varkappa-1}(\mathcal{G}_0)\geqq 1$ can be deduced with the help of the theorem of CORRÁDI and HAJNAL: if $V_{\leqq 2\varkappa-1}(\mathcal{G}_0)=0$ then, since $\varkappa^2+2\varkappa-4\leqq 3\varkappa+2$ because $\varkappa\geqq 3$, $\mathcal{G}_0\supset \bigcirc^{\varkappa}$, which contradicts $\mathcal{G}_0\not\supset \bigcirc^{\varkappa}$.

(3) $V_0(\mathcal{G}_0)=0$.

For if a is an isolated vertex of $\mathcal{G}_0$ then $V_{\geqq 2\varkappa}(\mathcal{G}_0-a)-V_{\leqq 2\varkappa-2}(\mathcal{G}_0-a)>$ $>\varkappa^2+2\varkappa-4$ and $\mathcal{G}_0-a\not\supset \bigcirc^{\varkappa}$, which contradicts the minimal property of $\mathcal{G}_0$.

(4) $V_1(\mathcal{G}_0)=0$.

For if b is a vertex of $\mathcal{G}_0$ having valency 1 then $V_{\geqq 2\varkappa}(\mathcal{G}_0-b)-V_{\leqq 2\varkappa-2}(\mathcal{G}_0-b)\geqq$ $\geqq \varkappa^2+2\varkappa-4$ and $\mathcal{G}_0-b\not\supset \bigcirc^{\varkappa}$, which contradicts the minimal property of $\mathcal{G}_0$.

(5) $\mathcal{G}_0$ *contains at least one triangle.*

For suppose on the contrary that $\mathcal{G}_0$ contains no triangle. By (2) $\mathcal{G}_0$ contains a vertex of valency $\leqq 2\varkappa-1$, let e denote such a vertex and let $v(e,\mathcal{G}_0)=v$; by (3) and (4) $2\leqq v\leqq 2\varkappa-1$; let $e_1, \ldots, e_v$ denote the vertices of $\mathcal{G}_0$ to which e is joined. No two of $e_1, \ldots, e_v$ are joined by an edge because $\mathcal{G}_0$ contains no triangle by hypothesis. Let $\mathcal{G}'=(\mathcal{G}_0-e)+(e_v, e_{v-1})+\ldots+(e_v, e_1)$. Clearly $V_{\geqq 2\varkappa}(\mathcal{G}')\geqq V_{\geqq 2\varkappa}(\mathcal{G}_0)$ and $V_{\leqq 2\varkappa-2}(\mathcal{G}')\leqq V_{\leqq 2\varkappa-2}(\mathcal{G}_0)$, so $V_{\geqq 2\varkappa}(\mathcal{G}')-V_{\leqq 2\varkappa-2}(\mathcal{G}')\geqq \varkappa^2+2\varkappa-4$, hence $\mathcal{G}'\supset \bigcirc^{\varkappa}$ by the minimal property of $\mathcal{G}_0$. It follows as in the proof of Theorem 1. I, that $\mathcal{G}_0\supset \bigcirc^{\varkappa}$, whereas $\mathcal{G}_0\not\supset \bigcirc^{\varkappa}$. This contradiction proves (5).

Let x_1, x_2, x_3 be the vertices of a triangle contained in $\mathcal{G}_0$ and let $\mathcal{G}''=\mathcal{G}-x_1-$ $-x_2-x_3$.

(6) $\mathcal{G}''\not\supset \bigcirc^{\varkappa-1}$.

Because $\mathcal{G}_0\not\supset \bigcirc^{\varkappa}$.

Let s denote the set of those vertices of $\mathcal{G}''$ which are in $\mathcal{G}_0$ joined to all three vertices x_1, x_2, x_3.

(7) $|s| \leqq 2\varkappa - 2$.

Because by (1) at least two of x_1, x_2, x_3 have valency $\leqq 2\varkappa$ in $\mathcal{G}_0$.

It is easy to see that

(8) $\mathcal{V}_{\geqq 2\varkappa}(\mathcal{G}_0) \subseteqq \mathcal{V}_{\geqq 2\varkappa-2}(\mathcal{G}'') \cup (s \cap \mathcal{V}_{2\varkappa-3}(\mathcal{G}'')) \cup \{x_1 x_2 x_3\}$

and

(9) $\mathcal{V}_{\leqq 2\varkappa-2}(\mathcal{G}_0) \supseteqq \mathcal{V}_{\leqq 2\varkappa-4}(\mathcal{G}'') - (s \cap \mathcal{V}_{2\varkappa-4}(\mathcal{G}''))$.

From (7), (8) and (9) and the conditions of Theorem 3 it follows that

$$(10)\quad \begin{aligned} &V_{\geqq 2\varkappa-2}(\mathcal{G}'') - V_{\leqq 2\varkappa-4}(\mathcal{G}'') \geqq \\ &\geqq V_{\geqq 2\varkappa}(\mathcal{G}_0) - V_{\leqq 2\varkappa-2}(\mathcal{G}_0) - 2\varkappa - 1 \geqq (\varkappa-1)^2 + 2(\varkappa-1) - 4. \end{aligned}$$

It follows from (10) that $\varkappa = 3$, because if $\varkappa \geqq 4$ then by (10) and by the minimal property of $\varkappa$, $\mathcal{G}'' \supset \bigcirc^{\varkappa-1}$ contrary to (6). If $\varkappa = 3$ then $V(\mathcal{G}_0) \geqq 11$ from the conditions of Theorem 3 with $k = 3$. The two alternatives $V(\mathcal{G}_0) = 11$ and $V(\mathcal{G}_0) \geqq 12$ will be considered in turn: If $V(\mathcal{G}_0) = 11$ then $V_{\leqq 5}(\mathcal{G}_0) = 0$ from the conditions of Theorem 3 with $k = 3$, hence in this case, by the theorem of HAJNAL and CORRÁDI [1], $\mathcal{G}_0 \supset \bigcirc^3$ contrary to hypothesis. If $V(\mathcal{G}_0) \geqq 12$ then $V(\mathcal{G}'') \geqq 9$, and by (10) $V_{\geqq 4}(\mathcal{G}'') - V_{\leqq 2}(\mathcal{G}'') \geqq 4$, so in this case, by Theorem 1. V, $\mathcal{G}'' \supset \bigcirc^2$, which contradicts (6). The assumption that Theorem 3 is false therefore leads to a contradiction.

REMARKS. 1. Theorem 3 is probably not best possible. On the other hand if $V_{\geqq 2k}(\mathcal{G}) - V_{\leqq 2k-2}(\mathcal{G}) = 2k-1$ then it is possible that $\mathcal{G} \not\supset \bigcirc^k$ whatever the values of $k (\geqq 3)$ and of $V(\mathcal{G})$. For example let $\mathcal{V}(\mathcal{G}) = \{x_1, \ldots, x_{2k-1}, y_1, \ldots y_u, z_1 \ldots z_u\}$ $(u \geqq \frac{1}{2}(k+1))$ and $\mathcal{E}(\mathcal{G}) = \{(x_i, x_j), (x_i, y_h), (y_h, z_h)\}$ $(i = 1, \ldots, 2k-1;\ j = 1, \ldots, 2k-1;\ i \neq j;\ h = 1, \ldots, u)$. $V_{\geqq 2k}(\mathcal{G}) = \frac{1}{2} V(\mathcal{G}) + \frac{1}{2}(2k-1)$.

2. It is worth noting, that in the proof of Theorem 3 only the particular case of the theorem of CORRÁDI and HAJNAL in which $k^2 + 2k - 4 \leqq V(\mathcal{G}) \leqq k^2 + 2k - 3$ was used. By a simpler method not using the theorem of CORRÁDI and HAJNAL at all the weaker result can be proved that corresponding to any integers $k \geqq 2$ and $c \geqq 1$ there exists a number $n(k, c)$ such that if the graph $\mathcal{G}$ satisfies the conditions $V_{\leqq 2k-1}(\mathcal{G}) \leqq c$ and $V(\mathcal{G}) \geqq n(k, c)$, then $\mathcal{G} \supset \bigcirc^k$.

In the case of finite graphs without multiple edges which contain loops much weaker conditions than those of Theorem 3 ensure the existence of k independent circuits, as Theorem 4 shows. In any concrete case it is best to delete all vertices incident with loops (together with all edges incident with at least one of them) and apply Theorem 2 or Theorem 3 to the remaining graph however, because Theorem 4 takes into account all the most unfavourable situations.

THEOREM 4. *Let $\mathcal{G}$ be a finite graph without multiple edges and k a natural number $\geqq 3$. If exactly $k-1$ vertices of $\mathcal{G}$ are incident with loops, and $V(\mathcal{G}) \geqq k+1$ and $V_{\geqq k+2}(\mathcal{G}) - V_{\leqq k}(\mathcal{G}) \geqq k-2$, then $\mathcal{G} \supset \bigcirc^k$. If exactly l vertices of $\mathcal{G}$ are incident with loops, where $l \leqq k-2$, and $V(\mathcal{G}) \geqq l+9$ and $V_{\geqq 2k-l}(\mathcal{G}) - V_{\leqq 2k-l-2}(\mathcal{G}) \geqq \geqq (k-l)^2 + 2k - l - 4$, then $\mathcal{G} \supset \bigcirc^k$.*

PROOF. Let L denote the set of those vertices of $\mathcal{G}$ which are indicent with loops.

Suppose first that $|L|=k-1$, $V(\mathcal{G})\geqq k+1$ and $V_{\geqq k+2}(\mathcal{G})-V_{\leqq k}(\mathcal{G})\leqq k-2$. Then clearly $V_{\geqq 3}(\mathcal{G}-L)\geqq V_{\geqq k+2}(\mathcal{G})-k+1$ and $V_{\leqq 1}(\mathcal{G}-L)\leqq V_{\leqq k}(\mathcal{G})$. Hence $V_{\geqq 3}(\mathcal{G}-L)-V_{\leqq 1}(\mathcal{G}-L)\geqq +1$, consequently, since $V_{\leqq 1}(\mathcal{G}-L)\geqq 2$, $\mathcal{G}-L\supset\bigcirc$. Therefore $\mathcal{G}\supset\bigcirc^k$.

Suppose secondly that $|L|=l\leqq k-2$, $V(\mathcal{G})\geqq l+9$ and $V_{\geqq 2k-l}(\mathcal{G})-V_{\leqq 2k-l-2}(\mathcal{G})\geqq(k-2)^2+2k-l-4$. Clearly $V_{\geqq 2k-2l}(\mathcal{G}-L)\geqq V_{\geqq 2k-l}(\mathcal{G})-l$ and $V_{\leqq 2k-2l-2}(\mathcal{G}-L)\leqq V_{\leqq 2k-l-2}(\mathcal{G})$. Hence $V_{\geqq 2(k-l)}(\mathcal{G}-L)-V_{\leqq 2(k-l)-2}(\mathcal{G}-L)\geqq\geqq(k-l)^2+2(k-l)-4$. It follows by Theorem 1. V ($V(\mathcal{G}-L)\geqq 9$) or Theorem 3 that $\mathcal{G}-L\supset\bigcirc^{k-l}$. Therefore $\mathcal{G}\supset\bigcirc^k$.

4. Concerning the existence of independent circuits in finite planar graphs without multiple edges

THEOREM 5. *Let $\mathcal{G}$ denote a finite planar graph without loops or multiple edges.*

I. *If $V(\mathcal{G})\geqq 6$, $V_{\leqq 2}(\mathcal{G})=0$ and $V_{\geqq 4}(\mathcal{G})\geqq 2$, then $\mathcal{G}\supset\bigcirc^2$.*

II. *If $V(\mathcal{G})\geqq 8$ and $V_{\geqq 4}(\mathcal{G})\geqq\frac{1}{2}V(\mathcal{G})+1\frac{1}{2}$, then $\mathcal{G}\supset\bigcirc^2$.*

III. *If $V(\mathcal{G})\geqq 8$ and $V_{\geqq 4}(\mathcal{G})-V_{\leqq 2}(\mathcal{G})\geqq 3$ then $\mathcal{G}\supset\bigcirc^2$.*

If $V_{\geqq 2k}(\mathcal{G})-V_{\leqq 2k-2}(\mathcal{G})\geqq 5k-7$ where k is an integer $\geqq 3$, then $\mathcal{G}\supset\bigcirc^k$.

PROOF OF I. Suppose first that $V(\mathcal{G})=6$ and that $\mathcal{G}$ barely satisfies the conditions of I. Then clearly $V_3(\mathcal{G})\geqq 1$. Let $\mathcal{V}(\mathcal{G})=\{g_1g_2\dots g_6\}$ and suppose that $v(g_1,\mathcal{G})=3$ and that g_1 is joined to g_2, g_3 and g_4 in $\mathcal{G}$.

(1) $(g_5,g_6)\in\mathcal{G}$.

(2) $v(g_5,\mathcal{G})=3$ *or* $v(g_6,\mathcal{G})=3$.

(3) $E(\mathcal{G}(g_2,g_3,g_4))\geqq 1$.

For if (1) or (2) or (3) were untrue then $(g_i,g_j)\in\mathcal{G}$ for $i=2,3,4$ and $j=1,5,6$ and so $\mathcal{G}$ would not be planar.

The three alternatives $E(\mathcal{G}(g_2,g_3,g_4))=1,2,3$ will now be considered in turn.

Assuming that $E(\mathcal{G}(g_2,g_3,g_4))=1$ it will be supposed that $(g_2,g_3)\in\mathcal{G}$. Then $(g_4,g_5)\in\mathcal{G}$ and $(g_4,g_6)\in\mathcal{G}$. In this case by (1) $\mathcal{G}\supset[g_1,g_2,g_3]\cup[g_4,g_5,g_6]$.

Assuming that $E(\mathcal{G}(g_2,g_3,g_4))=2$ it will be supposed that $(g_2,g_3)\in\mathcal{G}$ and $(g_2,g_4)\in\mathcal{G}$. Then either $v(g_3,\mathcal{G})\geqq 4$ or $v(g_4,\mathcal{G})\geqq 4$, because if $v(g_3,\mathcal{G})=v(g_4,\mathcal{G})=3$ then $v(g_5,\mathcal{G})\geqq 4$ or $v(g_6,\mathcal{G})\geqq 4$ since $V_{\geqq 4}(\mathcal{G})\geqq 2$; and if e. g. $v(g_5,\mathcal{G})\geqq 4$ then $(g_3,g_5),(g_4,g_5)\in\mathcal{G}$ so (g_3,g_6), $(g_4,g_6)\notin\mathcal{G}$, therefore $v(g_6,\mathcal{G})\leqq 2$ whereas $V_{\leqq 2}(\mathcal{G})=0$. If e. g. $v(g_4,\mathcal{G})\geqq 4$ then $\mathcal{G}\supset[g_1,g_2,g_3]\cup[g_4,g_5,g_6]$.

Assume finally that $E(\mathcal{G}(g_2,g_3,g_4))=3$. At least one of g_2,g_3,g_4 is joined to both of g_5,g_6 because $v(g_5,\mathcal{G})\geqq 3$ and $v(g_6,\mathcal{G})\geqq 3$; suppose that $(g_4,g_5)\in\mathcal{G}$ and $(g_4,g_6)\in\mathcal{G}$, then $\mathcal{G}\supset[g_1,g_2,g_3]\cup[g_4,g_5,g_6]$. I is thus true if $V(\mathcal{G})=6$.

The proof of I will now be completed by *reductio ad absurdum*. Assume that I is untrue. Then there exists a planar graph $\mathcal{G}_0$ with the following properties: $\mathcal{G}_0$ barely satisfies the conditions of I and $\mathcal{G}_0 \not\supset \bigcirc^2$, and all graphs which satisfy the conditions of I and have fewer vertices than $\mathcal{G}_0$ contain two independent circuits. $V(\mathcal{G}_0) \geqq 7$ because I is true for planar graphs with six vertices.

(4) *At least one of the end-vertices of each edge of* $\mathcal{G}_0$ *has valency* 3 *or* 4, *and at least one vertex of* $\mathcal{G}_0$ *has valency* 3.

Because $\mathcal{G}_0$ barely satisfies the conditions of I and $V(\mathcal{G}_0) \geqq 7$.

Let d denote a vertex of $\mathcal{G}_0$ having valency 3 and let d_1, d_2, d_3 be the three vertices of $\mathcal{G}_0$ to which d is joined. The three alternatives $E(\mathcal{G}_0(d_1, d_2, d_3)) \leqq 1$, $E(\mathcal{G}_0(d_1, d_2, d_3)) = 2$, $E(\mathcal{G}_0(d_1, d_2, d_3)) = 3$ will be considered separately.

$E(\mathcal{G}_0(d_1, d_2, d_3)) \leqq 1$ leads to a contradiction exactly as in the proof of Theorem 1. I.

$E(\mathcal{G}_0(d_1, d_2, d_3)) = 2$ and $(d_1, d_2), (d_1, d_3) \in \mathcal{G}_0$ and $v(d_1, \mathcal{G}_0) \geqq 5$ leads to a contradiction exactly as in the proof of Theorem 1. I. If $v(d_1, \mathcal{G}_0) \leqq 4$ then a contradiction is arrived at as follows: If $\mathcal{G} - d - d_1 - d_2 - d_3$ has v connected components, then $E(\mathcal{G}_0) \leqq V(\mathcal{G}_0) - 2 - v + v(d_2, \mathcal{G}_0) + v(d_3, \mathcal{G}_0)$ as before. On the other hand, summing the valencies of all the vertices of $\mathcal{G}_0$, $2E(\mathcal{G}_0) \geqq v(d_2, \mathcal{G}_0) + v(d_3, \mathcal{G}_0) + 3(V(\mathcal{G}_0) - 2)$. From the two inequalities it follows that $V(\mathcal{G}_0) \leqq v(d_2, \mathcal{G}_0) + v(d_3, \mathcal{G}_0) + 2 - 2v$. $v(d_2, \mathcal{G}_0) \leqq v + 2$ and $v(d_3, \mathcal{G}_0) \leqq v + 2$, since d_2 and d_3 are each joined to at most one vertex of each connected component of $\mathcal{G}_0 - d - d_1 - d_2 - d_3$ because $\mathcal{G}_0 \not\supset \bigcirc^2$. Hence $V(\mathcal{G}_0) \leqq 6$ which contradicts $V(\mathcal{G}_0) \geqq 7$.

The only remaining alternative is that $E(\mathcal{G}_0(d_1, d_2, d_3)) = 3$. By (4) it may be assumed that $v(d_1, \mathcal{G}_0) \leqq 4$ and $v(d_2, \mathcal{G}_0) \leqq 4$. $\mathcal{G}_0 - d - d_1 - d_2 - d_3 \not\supset \bigcirc$ so $E(\mathcal{G}_0 - d - d_1 - d_2 - d_3) \leqq V(\mathcal{G}_0) - 4 - v$ where v is the number of connected components of $\mathcal{G}_0 - d - d_1 - d_2 - d_3$. Consequently $E(\mathcal{G}_0) \leqq V(\mathcal{G}_0) + v(d_3, \mathcal{G}_0) + 1 - v$. On the other hand, summing the valencies of all the vertices of $\mathcal{G}_0$, $2E(\mathcal{G}_0) \geqq v(d_3, \mathcal{G}_0) + 4 + 3(V(\mathcal{G}_0) - 2)$. From the two inequalities it follows that $V(\mathcal{G}_0) \leqq v(d_3, \mathcal{G}_0) + 4 - 2v$. But d_3 is joined to at most one vertex of each connected component of $\mathcal{G}_0 - d - d_1 - d_2 - d_3$ because $\mathcal{G}_0 \not\supset \bigcirc^2$, so $v(d_3, \mathcal{G}_0) \leqq v + 3$. Hence $V(\mathcal{G}_0) \leqq 7 - v \leqq 6$. But $V(\mathcal{G}_0) \geqq 7$. With this contradiction the proof of I is complete.

PROOF OF II.

(5) *If* $\mathcal{H}$ *is a graph without loops or multiple edges and* $V(\mathcal{H}) = 6$, $V_{\geqq 4}(\mathcal{H}) = 5$ *and* $V_2(\mathcal{H}) = 1$, *then* $\mathcal{H}$ *is not planar.*

For suppose on the contrary that $\mathcal{H}$ is planar. Let b denote the vertex of $\mathcal{H}$ having valency 2 and let b_1, b_2 denote the two vertices of H to which b is joined. In $\mathcal{H} - b$ every vertex other than b_1 and b_2 has valency 4. Consequently $(b_1, b_2) \notin \mathcal{H}$ since $\mathcal{H}$ is planar. Therefore $(\mathcal{H} - b) \cup (b_1, b_2)$ is a planar graph without loops or multiple edges containing five vertices each having valency 4. This contradiction proves (5).

Now suppose first that $V(\mathcal{G})=8$. Then $V_{\geqq 4}(\mathcal{G})\geqq 6$. $\mathcal{G}$ does not contain a vertex of valency 4 joined to two vertices of valency 1, because if it did then the graph obtained from $\mathcal{G}$ by deleting the two vertices of valency 1 together with the edges incident with them would contradict (5). Therefore $\mathcal{G}\supset \bigcirc^2$ by Theorem 1. III.

The proof of II will now be completed by *reductio ad absurdum*. Assume that II is untrue. Then there exists a planar graph $\mathcal{G}_0$ with the following properties: $\mathcal{G}_0$ barely satisfies the conditions of II and $\mathcal{G}_0 \not\supset \bigcirc^2$, and all planar graphs which satisfy the conditions of II and have fewer vertices than $\mathcal{G}_0$ contain two independent circuits. It has just been proved that

(6) $V(\mathcal{G}_0)\geqq 9$.

As in the proof of Theorem 1. IV

(7) *At least one end-vertex of every edge contained in* $\mathcal{G}_0$ *has valency* 4, *and* $V_{\leqq 3}(\mathcal{G}_0)\geqq 1$.

By summing the valencies of all the vertices of $\mathcal{G}_0$ we have that

(8) *If* $V_0(\mathcal{G}_0)=0$ *then* $E(\mathcal{G}_0)\geqq 1\frac{1}{4}V(\mathcal{G}_0)+2\frac{1}{4}$, *if* $V_0(\mathcal{G}_0)=0$ *and* $V_2(\mathcal{G}_0)\geqq 1$ *then* $E(\mathcal{G}_0)\geqq 1\frac{1}{4}V(\mathcal{G}_0)+2\frac{3}{4}$, *and if* $V_0(\mathcal{G}_0)=0$ *and* $V_3(\mathcal{G}_0)\geqq 1$ *then* $E(\mathcal{G}_0)\geqq 1\frac{1}{4}V(\mathcal{G}_0)+3\frac{1}{4}$.

It will now be proved that $V_i(\mathcal{G}_0)=0$ for $i=0, 2, 3$.

(9) $V_0(\mathcal{G}_0)=0$.

For suppose on the contrary that the vertex a of $\mathcal{G}_0$ has valency 0. Then $(\mathcal{G}_0-a)$ satisfies the conditions of II (by (6)) and therefore $\mathcal{G}_0-a\supset \bigcirc^2$ by the minimal property of $\mathcal{G}_0$.

(10) $V_2(\mathcal{G}_0)=0$.

For suppose on the contrary that the vertex c of $\mathcal{G}_0$ has valency 2 and is joined to the vertices c_1 and c_2. The two alternatives $(c_1, c_2)\notin\mathcal{G}_0$ and $(c_1, c_2)\in\mathcal{G}_0$ will be considered in turn and a contradiction will be derived in both cases. If $(c_1, c_2)\notin\mathcal{G}$ then $(\mathcal{G}_0-c)\cup(c_1, c_2)$ satisfies the conditions of II (by (6)) and therefore contains two independent circuits (because of the minimal property of $\mathcal{G}_0$), it follows as in the proof of Theorem 1. III that $\mathcal{G}_0\supset\bigcirc^2$, whereas $\mathcal{G}_0\not\supset\bigcirc^2$. Suppose next that $(c_1, c_2)\in\mathcal{G}_0$. By (7) $v(c_1, \mathcal{G}_0)=v(c_2, \mathcal{G}_0)=4$. $E(\mathcal{G}_0-c-c_1-c_2)\leqq V(\mathcal{G}_0)-4$ because $\mathcal{G}_0-c-c_1-c_2\not\supset\bigcirc$. It follows that $E(\mathcal{G}_0)\leqq V(\mathcal{G}_0)+3$. By (8) $E(\mathcal{G}_0)\geqq$ $\geqq 1\frac{1}{4}V(\mathcal{G}_0)+2\frac{3}{4}$. From the two inequalities it follows that $V(\mathcal{G}_0)\leqq 1$, which contradicts (6). Hence $V_2(\mathcal{G}_0)=0$.

(11) $V_3(\mathcal{G}_0)=0$.

For suppose on the contrary that the vertex d of $\mathcal{G}_0$ has valency 3 and is joined to the vertices d_1, d_2 and d_3. The two alternatives $E(\mathcal{G}_0(d_1, d_2, d_3))\leqq 1$ and

$E(\mathcal{G}_0(d_1, d_2, d_3)) \geqq 2$ will be considered in turn and a contradiction derived in both cases.

Assuming that $E(\mathcal{G}_0(d_1, d_2, d_3)) \leqq 1$, let it be supposed that $(d_1, d_3) \notin \mathcal{G}_0$ and $(d_2, d_3) \notin \mathcal{G}_0$. Then $(\mathcal{G}_0 - d) \cup (d_1, d_3) \cup (d_2, d_3)$ satisfies the conditions of II (by (6)) and therefore contains two independent circuits (because of the minimal property of $\mathcal{G}_0$), it follows as in the proof of Theorem 1. I that $\mathcal{G}_0 \supset \bigcirc^2$, whereas $\mathcal{G}_0 \not\supset \bigcirc^2$. There remains the alternative that $E(\mathcal{G}_0(d_1, d_2, d_3)) \geqq 2$. By (7) $v(d_1, \mathcal{G}_0) = v(d_2, \mathcal{G}_0) = v(d_3, \mathcal{G}_0) = 4$. $E(\mathcal{G}_0 - d - d_1 - d_2 - d_3) \leqq V(\mathcal{G}_0) - 5$ because $\mathcal{G}_0 - d - d_1 - d_2 - d_3 \not\supset \bigcirc$. It follows that $E(\mathcal{G}_0) \leqq V(\mathcal{G}_0) + 5$. By (8) $E(\mathcal{G}_0) \geqq 1\frac{1}{4}(V(\mathcal{G}_0) + 3\frac{1}{4}$. From the two inequalities it follows that $V(\mathcal{G}_0) \leqq 7$, which contradicts (6). Hence $V_3(\mathcal{G}_0) = 0$.

From the conditions of II it follows that $V_{\geqq 4}(\mathcal{G}_0) \geqq 6$ and $V_{\geqq 4}(\mathcal{G}_0) - V_{\leqq 3}(\mathcal{G}_0) \geqq 3$. By (9), (10) and (11) $V(\mathcal{G}_0) = V_1(\mathcal{G}_0) + V_{\geqq 4}(\mathcal{G}_0)$ and by (7) each vertex of valency 1 is joined to a vertex of valency 4. If $\mathcal{G}_0$ does not contain a vertex having valency 4 joined to at least two vertices having valency 1, then let $\mathcal{G}''$ denote the graph obtained by deleting all vertices of valency 1 from $\mathcal{G}_0$. Clearly $V(\mathcal{G}'') \geqq 6$, $V_{\geqq 4}(\mathcal{G}'') \geqq 3$ and $V_{\leqq 2}(\mathcal{G}'') = 0$. Therefore by I $\mathcal{G}'' \supset \bigcirc^2$, whereas $\mathcal{G}_0 \not\supset \bigcirc^2$. This contradiction proves II in the case considered.

If $\mathcal{G}_0$ contains a vertex, say d, having valency 4 which is joined to two vertices of valency 1, say d_1 and d_2, then let d_3 and d_4 denote the remaining to vertices of $\mathcal{G}_0$ joined to d. Let $\mathcal{G}'' = (\mathcal{G}_0 - d) \cup (d_1, d_2) \cup (d_1, d_3) \cup (d_1, d_4)$. $\mathcal{G}''$ satisfies the conditions of II (by (6)) and therefore contains two independent circuits (because of the minimal property of G_0), it follows as in the proof of Theorem 1. I that $G_0 \supset \bigcirc^2$, whereas $G_0 \not\supset \bigcirc^2$. This contradiction proves II in the remaining case.

Proof of III. For $k = 2$ III follows from II in the same way as Theorem 1. V follows from Theorem 1. IV. For $k \geqq 3$ III will be proved by *reductio ad absurdum*. Assume that III is untrue. Let $\varkappa$ denote the least value of k for which the assertion of III is false, and among the planar graphs to which the assertion of III with $k = \varkappa$ does not apply let $\mathcal{G}_0$ be one with the least number of vertices and barely satisfying the conditions of III with $k = \varkappa$.

(12) *At least one of the end-vertices of every edge of $\mathcal{G}_0$ has valency $2\varkappa$ or $2\varkappa - 1$.*

For $\mathcal{G}_0$ barely satisfies the conditions of III with $k = \varkappa$.

(13) $V_{\geqq 2\varkappa - 1}(\mathcal{G}_0) \geqq 1$.

Because $\varkappa \geqq 3$ and $\mathcal{G}_0$ is planar.

(14) $V_{\leqq 1}(\mathcal{G}_0) = 0$.

For if x is a vertex of valency $\leqq 1$ in $\mathcal{G}_0$ then $V_{\geqq 2k}(\mathcal{G}_0 - x) - V_{\leqq 2k-2}(\mathcal{G}_0 - x) \geqq 5\varkappa - 7$ and $\mathcal{G}_0 - x \not\supset \bigcirc^{\varkappa}$, which contradicts the minimal property of $\mathcal{G}_0$.

(15) *$\mathcal{G}_0$ contains at least one triangle.*

Replace $\varkappa^2 + 2\varkappa - 4$ by $5\varkappa - 7$ in the proof of (5) in the proof of Theorem 3.

Let x_1, x_2, x_3 be the vertices of a triangle contained in $\mathcal{G}_0$ and let $\mathcal{G}'' = \mathcal{G} - x_1 - x_2 - x_3$.

(16) $\mathcal{G}'' \not\supset \bigcirc^{\varkappa - 1}$.

Because $\mathcal{G}_0 \not\supset \bigcirc^{\varkappa}$.

Let s denote the set of those vertices of $\mathcal{G}''$ which are in $\mathcal{G}_0$ joined to all three vertices x_1, x_2, x_3.

(17) $|s| \leqq 2$.

Because $\mathcal{G}_0$ is planar.

From (17) and the conditions of III it follows as in the proof of Theorem 3 that

(18) $V_{\geqq 2\varkappa-2}(\mathcal{G}'') - V_{\leqq 2\varkappa-4}(\mathcal{G}'') \geqq V_{\geqq 2\varkappa}(\mathcal{G}_0) - V_{\leqq 2\varkappa-2}(\mathcal{G}_0) - 5 \geqq 5(\varkappa - 1) - 7$.

It follows from (18) that $\varkappa = 3$, because if $\varkappa \geqq 4$ then, by (18) and the minimal property of $\varkappa$, $\mathcal{G}'' \supset \bigcirc^{\varkappa-1}$ contrary to (16). If $\varkappa = 3$ then $V_{\geqq 6}(\mathcal{G}_0) - V_{\leqq 4}(\mathcal{G}_0) \geqq 8$. In this case obviously $V(\mathcal{G}_0) \geqq 11$ because $\mathcal{G}_0$ is planar. Hence $V(\mathcal{G}'') \geqq 8$. Therefore, from the first part of III, $\mathcal{G}'' \supset \bigcirc^2$, wich contradicts (16). The assumption that III is false therefore leads to a contradiction. Theorem 5 is now proved.

REMARKS. 1. *The condition of* I *cannot be relaxed*, $V(\mathcal{G}) \geqq 6$, $V_{\leqq 2}(\mathcal{G}) = 0$ and $V_{\geqq 4}(\mathcal{G}) = 1$ does not necessarily imply that the planar graph $\mathcal{G}$ contains two independent circuits. This is illustrated by the planar graphs consisting of a circuit together with a vertex not belonging to the circuit joined to every vertex of the circuit.

2. If $\mathcal{G}$ is planar and $V_{\geqq 4}(\mathcal{G}) \geqq \frac{1}{2} V(\mathcal{G}) + 1\frac{1}{2}$ then $V(\mathcal{G}) \geqq 7$, as the reader can easily verify. If $V_{\geqq 4}(\mathcal{G}) \geqq \frac{1}{2} V(\mathcal{G}) + 1\frac{1}{2}$ and $V(\mathcal{G}) = 7$ then the planar graph $\mathcal{G}$ does not necessarily contain two independent circuits. For example if $\mathcal{V}(X) = \{x_1, \ldots, x_7\}$ and $\mathcal{E}(X) = \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_2, x_3), (x_2, x_4), (x_2, x_5), (x_3, x_4), (x_3, x_5), (x_4, x_6), (x_5, x_7)\}$ then X is planar and contains no loops or multiple edges, and $V_4(X) = 5$, $V_1(X) = 2$, but $X \not\supset \bigcirc^2$.

If $\mathcal{G}$ is planar and $V(\mathcal{G}) = 8$ and $V_{\geqq 4}(\mathcal{G}) = \frac{1}{2} V(\mathcal{G}) + 1$, then $\mathcal{G}$ does not necessarily contain two independent circuits, as is shown by any graph Y obtained from X by subdividing an arbitrary edge through inserting a new vertex.

3. X shows that if $\mathcal{G}$ is planar, $V_{\geqq 4}(\mathcal{G}) - V_{\leqq 2}(\mathcal{G}) = 3$ and $V(\mathcal{G}) = 7$ then $\mathcal{G}$ does not necessarily contain two independent circuits. $X - x_7$ shows that if $\mathcal{G}$ is planar, $V_{\geqq 4}(\mathcal{G}) - V_{\leqq 2}(\mathcal{G}) = 3$ and $V(\mathcal{G}) = 6$ then $\mathcal{G}$ does not necessarily contain two independent circuits. Y shows that if $\mathcal{G}$ is planar, $V_{\geqq 4}(\mathcal{G}) - V_{\leqq 2}(\mathcal{G}) = 2$ and $V(\mathcal{G}) = 8$ then $\mathcal{G}$ does not necessarily contain two independent circuits. III is surely not best possible for $k \geqq 3$.

4. The remarks after Theorem 2 apply equally to planar graphs, because the counterexamples described there are planar.

5. Concerning the existence of two circuits without a common edge in finite planar graphs

THEOREM 6. *If* $\mathcal{G}$ *is a finite planar graph which may contain multiple edges and loops, and if* $E(\mathcal{G}) \geqq V(\mathcal{G})+3$, *then* $\mathcal{G} \supseteqq 2\bigcirc$.

NOTE. If $\mathcal{G}$ is finite and contains a loop and if $E(\mathcal{G}) \geqq V(\mathcal{G})+2$, a loop being counted as two edges, then $\mathcal{G} \supseteqq 2\bigcirc$, whether $\mathcal{G}$ is planar or not. For if l is a loop then $E(\mathcal{G}-l) \geqq V(\mathcal{G})$, hence $\mathcal{G}-l \supseteqq \bigcirc$, therefore $\mathcal{G} \supseteqq 2\bigcirc$. If $E(\mathcal{G})=V(\mathcal{G})+1$ then $\mathcal{G}$ need not contain two circuits without a common edge, a simple example is a path with a loop incident with one of its edges.

PROOF OF THEOREM 6 by *reductio ad absurdum*. Assume that Theorem 6 is untrue. Then there exists a finite planar graph $\mathcal{G}_0$ such that $E(\mathcal{G}_0)=V(\mathcal{G}_0)+3$, and $\mathcal{G}_0 \not\supseteq 2\bigcirc$, and Theorem 6 is true for all graphs with fewer vertices than $\mathcal{G}_0$. Clearly $V(\mathcal{G}_0) \geqq 3$.

(1) $\mathcal{G}_0$ *contains no circuit having fewer than four edges.*

For if C is a circuit with fewer than four edges contained in $\mathcal{G}_0$ then $E(\mathcal{G}_0-E(C)) \geqq V(\mathcal{G}_0)$, so $\mathcal{G}_0-\mathcal{E}(C) \supseteqq \bigcirc$ and consequently $\mathcal{G}_0 \supseteqq 2\bigcirc$ contrary to hypothesis.

(2) $V_{\leqq 2}(\mathcal{G}_0)=0$.

$V_0(\mathcal{G}_0)=0$ because if a is an isolated vertex of $\mathcal{G}_0$ then $E(\mathcal{G}_0-a)= = V(\mathcal{G}_0-a)+4$, so $\mathcal{G}_0-a \supseteqq 2\bigcirc$ from the minimal property of $\mathcal{G}_0$, whereas $\mathcal{G}_0 \supseteqq 2\bigcirc$. $V_1(\mathcal{G}_0)=V_2(\mathcal{G}_0)=0$ can be proved by the same argument as is used in the proof of Theorem 3 in [2].

It follows from (2) that $2E(\mathcal{G}_0) \geqq 3V(\mathcal{G}_0)$. Consequently, since $E(\mathcal{G}_0)=V(\mathcal{G}_0)+3$, $V(\mathcal{G}_0) \leqq 6$. On the other hand it follows easily from (1) and (2) that $V(\mathcal{G}_0) \geqq 6$. Hence $V(\mathcal{G}_0)=6$. From this and (1) and (2) it follows easily that $\mathcal{G}_0$ contains a circuit having exactly four edges and vertices.

Let the vertices of $\mathcal{G}_0$ be denoted by $g_1, g_2, \ldots, g_6$ and suppose that $\mathcal{G}_0$ contains a circuit whose vertices are g_1, g_2, g_3, g_4 in this order. By (1) g_5 and g_6 are each joined to at most two of $g_1, \ldots, g_4$, therefore by (2) $(g_5, g_6) \in \mathcal{G}_0$ and by (1) it may be supposed without loss of generality that (g_1, g_6), (g_3, g_6), (g_2, g_5), $(g_4, g_5) \in \mathcal{G}_0$. So $\mathcal{G}_0$ contains the circuit $[g_1, g_2, g_3, g_4, g_5, g_6]$ and the edges (g_1, g_4), (g_2, g_5), (g_3, g_6). Therefore $\mathcal{G}_0$ is not planar. This contradiction proves Theorem 6.

REMARKS. 1. If $V(\mathcal{G}) \leqq 3$ and $E(\mathcal{G}) \geqq V(\mathcal{G})+2$ then $\mathcal{G} \supseteqq 2\bigcirc$, as the reader may very easily verify.

2. If $V(\mathcal{G}) \geqq 4$ and $E(\mathcal{G}) \leqq V(\mathcal{G})+2$ then the planar graph $\mathcal{G}$ need not contain two circuits without a common edge. This is illustrated by the graph containing

four vertices, each pair of distinct vertices joined by one edge (this graph is 3-fold connected) and by all graphs obtained from this graph through subdividing edges by the insertion of new vertices (these graphs are all 2-fold connected) — in other words by the complete 4-graph and the topological complete 4-graphs.

MATHEMATISCHES SEMINAR
DER UNIVERSITÄT HAMBURG

MATHEMATICAL INSTITUTE,
EÖTVÖS LORÁND UNIVERSITY,
BUDAPEST

(Received 27 March 1962)

References

[1] K. CORRÁDI and A. HAJNAL, On the maximal number of independent circuits of a graph (to appear).

[2] P. ERDŐS and L. PÓSA, On the maximal number of disjoint circuits of a graph. *Publ. Math. Debrecen*, **9** (1962), pp. 3–12.