REPRESENTATIONS OF REAL NUMBERS AS SUMS AND PRODUCTS OF LIOUVILLE NUMBERS

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A real number x is a *Liouville number* if to each natural number m there corresponds a rational number h_m/k_m, with $k_m > 1$, such that

$$0 < |x - h_m/k_m| < (1/k_m)^m .$$

Some years ago I showed (possibly jointly with Mahler), that every real number is the sum of two Liouville numbers. A proof of the proposition may now be in the literature, but I do not know of any reference. In any case, the following slightly stronger theorem is now needed (see [1]), and therefore I publish a proof.

THEOREM. *To each real number* t $(t \neq 0)$ *there correspond Liouville numbers* x, y, u, v *such that*

$$t = x + y = uv .$$

The reciprocal of a Liouville number is again a Liouville number, and therefore we obtain immediately the following proposition.

COROLLARY. *Each real number other than* 0 *is the solution of a linear equation whose coefficients are Liouville numbers.*

Proof of the theorem. Since the theorem is trivial for rational t, we assume that t is irrational. We also assume, without loss of generality, that $0 < t < 1$. Let

$$t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} \qquad (\varepsilon_k = 0, 1),$$

and write

$$x = \sum_{k=1}^{\infty} \xi_k 2^{-k}, \qquad y = \sum_{k=1}^{\infty} \eta_k 2^{-k},$$

where, for $n! \leq k < (n+1)!$,

$$\xi_k = \varepsilon_k \quad \text{and} \quad \eta_k = 0 \qquad (n = 1, 3, 5, \cdots),$$

$$\xi_k = 0 \quad \text{and} \quad \eta_k = \varepsilon_k \qquad (n = 2, 4, 6, \cdots).$$

Then $t = x + y$, and since x and y are Liouville numbers, half of the theorem is proved.

To prove the other half, we assume, again without loss of generality, that $t > 1$, and we choose a representation of t of the form

Received October 23, 1961.

$$t = \prod_{k=1}^{\infty} (1 + \varepsilon_k/k) \qquad (\varepsilon_k = 0, 1).$$

(Clearly, infinitely many nonterminating representations of this form are possible.) Let $m_0 = 0$, and let $\{m_i\}_1^\infty$ denote an increasing sequence of positive integers which are to be chosen presently. We write

$$s_i = \prod_{m_{i-1} < k \leq m_i} (1 + \varepsilon_k/k),$$

$$u_r = \prod_{i=1}^{r} s_{2i-1}, \qquad v_r = \prod_{i=1}^{r} s_{2i},$$

$$u = \lim_{r \to \infty} u_r, \qquad v = \lim_{r \to \infty} v_r.$$

Let m_1 be arbitrary. Once $m_1, m_2, \cdots, m_{2r-1}$ have been chosen, we can make the differences $u - u_r$ and $v - v_r$ as small as we like by choosing first m_{2r}, and thereafter m_{2r+1}, sufficiently large. Since u_r and v_r are rational and have denominators that are independent of m_{2r} and m_{2r+1}, respectively, we can choose the sequence $\{m_r\}$ in such a way that u and v are Liouville numbers. This completes the proof.

The following proof is not constructive, but it may be of interest because of its generality. The set L of Liouville numbers, being a dense set of type G_δ, is residual (in other words, it is the complement of a set of first category). Let A and B be any two residual sets of real numbers. For each real number t, the set B_t of numbers $t - b$ $(b \in B)$ is also residual, and therefore it contains a point x of A. Let $y = t - x$. Then $y \in B$, and since $t = x + y$, we have shown that each real number is the sum of a number in A and a number in B. We now obtain the first part of our theorem by choosing $A = B = L$. The second part can be proved similarly, under the hypothesis that $t \neq 0$.

REFERENCE

1. Z. A. Melzak, *On the algebraic closure of a plane set*, Michigan Math. J. 9 (1962), 61-64.

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