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ON A THEOREM OF RADEMACHER-TURÁN

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY
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A set of points some of which are connected by an edge will be called a graph G. Two vertices are connected by at most one edge, and loops (i.e., edges whose endpoints coincide) will be excluded. Vertices will be denoted by $\alpha, \beta, \cdots$, edges will be denoted by $e_1, e_2, \cdots$ or by (α, β) where the edge (α, β) connects the vertices α and β.

$G - e_1 - \cdots - e_k$ will denote the graph from which the edges $e_1, \cdots, e_k$ have been omitted, and $G - \alpha_1 - \cdots - \alpha_k$ denotes the graph from which the vertices $\alpha_1, \cdots, \alpha_k$ and all the edges emanating from them have been omitted; similarly $G + e_1 + \cdots + e_k$ will denote the graph to which the edges $e_1, \cdots, e_k$ have been added (without generating a new vertex).

The valency $v(\alpha)$ of a vertex will denote the number of edges emanating from it. $G_u^{(v)}$ will denote a graph having v vertices and u edges. The graph $G_{\binom{k}{2}}^{(k)}$ (i.e., the graph of k vertices any two of which are connected by an edge) will be called the complete k-gon.

A graph is called *even* if every circuit of it has an even number of edges.

Turán[1] proved that every

$$G_{V+1}^{(n)}, \qquad V = \frac{k-2}{2(k-1)}(n^2 - r^2) + \binom{r}{2}$$

for $n = (k-1)t + r$, $0 \leqq r < k - 1$, contains a complete k-gon, and he determined the structure of the $G_V^{(n)}$'s which do not contain a complete k-gon. Thus if we put $f(2m) = m^2$, $f(2m+1) = m(m+1)$, a special case of Turán's theorem states that every $G_{f(n)+1}^{(n)}$ contains a triangle.

In 1941 Rademacher proved that for even n every $G_{f(n)+1}^{(n)}$ contains at least $[n/2]$ triangles and that $[n/2]$ is best possible. Rademacher's proof was not published. Later on[2] I simplified Rademacher's proof and proved more generally that for $t \leqq 3$, $n > 2t$, every $G_{f(n)+t}^{(n)}$ contains at least $t[n/2]$ triangles. Further I conjectured that for $t < [n/2]$ every $G_{f(n)+t}^{(n)}$ contains at least $t[n/2]$ triangles. It is easy to see that for $n = 2m$, $2m > 4$, the conjecture is false for $t = n/2$. To see this, consider a graph $G_{m^2+m}^{(2m)}$ whose vertices are

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[1] P. TURÁN, Matematikai és Fizikai Lapok, vol. 48 (1941), pp. 436–452 (in Hungarian); see also *On the theory of graphs*, Colloq. Math., vol. 3 (1954), pp. 19–30.

[2] P. ERDÖS, *Some theorems on graphs*, Riveon Lematematika, vol. 9 (1955), pp. 13–17 (in Hebrew with English summary).

$\alpha_1, \cdots, \alpha_{2m}$ and whose edges are

$$(\alpha_i, \alpha_j), \qquad 1 \leqq i \leqq m+1 < j \leqq 2m,$$

and the $m + 1$ further edges

$$(\alpha_i, \alpha_{i+1}), \quad 1 \leqq i \leqq m, \quad \text{and} \quad (\alpha_1, \alpha_{m+1}).$$

It is easy to see that this graph contains $m^2 - 1$ triangles (for $2m = 4$ an unwanted triangle $(\alpha_1, \alpha_2, \alpha_3)$ enters and ruins the counting, and in fact it is easy to see that for $2m = 4$ the conjecture holds for $t = m = 2$). For odd $n = 2m + 1$ perhaps every $G^{(2m+1)}_{f(2m+1)+t}$, $t \leqq 2m - 2$, contains at least tm triangles. But here is a $G^{(2m+1)}_{f(2m+1)+2m-1}$, $2m + 1 \geqq 9$, which contains fewer than $m(2m - 1)$ triangles. The vertices of our graph are $\alpha_1, \cdots, \alpha_{2m+1}$, the edges are

$$(\alpha_i, \alpha_j), \qquad 1 \leqq i \leqq m+2 < j \leqq 2m+1,$$

and the following $2m + 1$ edges:

$$(\alpha_1, \alpha_k), \quad (\alpha_2, \alpha_k), \quad (\alpha_3, \alpha_4), \quad (\alpha_3, \alpha_5), \quad (\alpha_3, \alpha_6),$$
$$3 \leqq k \leqq m + 2.$$

It is easy to see that this graph contains $2m^2 - m - 1 < m(2m - 1)$ triangles. For $2m + 1 = 5$ we must have $t \leqq 4$, and it is easy to see that the conjecture holds for all these t. For $2m + 1 = 7, t \leqq 9$, and by a little longer argument one can easily convince oneself that the conjecture holds for all these t.

In the present paper we are going to prove the following

THEOREM. *There exists a constant $c_1 > 0$ so that for $t < c_1 n/2$ every $G^{(n)}_{f(n)+t}$ contains at least $t[n/2]$ triangles.*

First we need three lemmas.

LEMMA 1. *Every $G^{(n)}_{f(n-1)+2}$ which is not even contains a triangle.*

Lemma 1 was found jointly by Gallai and myself. (The lemma was also found by Mr. Andrásfai independently.)

Let G be a graph with n vertices which is not even and contains no triangle. Let $\alpha_1, \cdots, \alpha_{2k+1}$ be the vertices of the odd circuit of our graph having the least number of vertices. We can assume $3 < 2k + 1 \leqq n$. The subgraph of G spanned by $\alpha_1, \cdots, \alpha_{2k+1}$ can have no other edges; otherwise our graph would contain an odd circuit having fewer than $2k + 1$ edges. Let $\beta_1, \cdots, \beta_{n-2k-1}$ be the other vertices of G. Any of the β's can be connected with at most two of the α's, for otherwise G contains an odd circuit of fewer than $2k + 1$ edges. Finally by Turán's theorem the subgraph of G spanned by $\beta_1, \cdots, \beta_{n-2k-1}$ can have at most $f(n - 2k - 1)$ edges. Thus the number of edges of G is at most

$$2k + 1 + 2(n - 2k - 1) + f(n - 2k - 1) \leqq f(n - 1) + 1$$

by a simple calculation (equality only for $2k + 1 = 5$). This completes the proof of our lemma.

Our proof in fact gives that a graph G of n vertices whose smallest odd circuit has $2k + 1$ vertices, $k > 1$, has at most $2n - 2k - 1 + f(n - 2k - 1)$ edges, and the following simple example shows that this result is best possible. Let the vertices of G be

$$\alpha_1, \cdots, \alpha_v, \quad \beta_1, \cdots, \beta_u, \quad \gamma_1, \cdots, \gamma_{2k+1},$$
$$v = \left[\frac{n-2k-1}{2}\right], \quad u = n - \left[\frac{n-2k-1}{2}\right].$$

The edges of G are (α_i, β_j), (γ_1, α_i), (γ_3, α_i), $1 \leqq i \leqq v$, (γ_2, β_i), (γ_4, β_i), $1 \leqq i \leqq u$, further the edges (γ_i, γ_{i+1}), $1 \leqq i \leqq 2k$, $(\gamma_1, \gamma_{2k+1})$.

LEMMA 2. *There exists a constant $c_2 > 0$ so that every $G^{(n)}_{f(n)+1}$ contains at least $[c_2 n]$ triangles having a common edge (α_1, α_2) (i.e., all the edges (α_1, β_i), (α_2, β_i), (α_1, α_2), $1 \leqq i \leqq [c_2 n]$, are in $G^{(n)}_{f(n)+1}$).*

Let $(\alpha_i, \beta_i, \gamma_i)$, $1 \leqq i \leqq r$, be a maximal system of disjoint triangles of our graph $G^{(n)}_{f(n)+1}$. In other words if we omit the vertices $\alpha_i, \beta_i, \gamma_i$, $1 \leqq i \leqq r$, the subgraph of $G^{(n)}_{f(n)+1}$ spanned by the remaining $n - 3r$ vertices contains no triangle and has therefore at most $f(n - 3r)$ edges (by Turán's theorem).

Denote by $G(n, i)$ the graph $G^{(n)}_{f(n)+1} - \sum_{j=1}^{i-1} (\alpha_j + \beta_j + \gamma_j)$, and let $v^{(i)}(a_i)$, $v^{(i)}(\beta_i)$, $v^{(i)}(\gamma_i)$ be the valencies of $\alpha_i, \beta_i, \gamma_i$ in $G(n, i)$. Now we show that for some i, $1 \leqq i \leqq r$, we must have

$$v^{(i)}(\alpha_i) + v^{(i)}(\beta_i) + v^{(i)}(\gamma_i) > n(1 + 9c_2) - 3i. \tag{1}$$

For if (1) would not hold for any i, $1 \leqq i \leqq r$, then the number of edges of $G^{(n)}_{f(n)+1}$ would be not greater than

$$\textstyle\sum_{i=1}^{r} (n(1 + 9c_2) - 3i) + f(n - 3r) < f(n) \tag{2}$$

by a simple calculation for sufficiently small c_2. But (2) is an evident contradiction since $G^{(n)}_{f(n)+1}$ has by definition $f(n) + 1$ edges.

Thus (1) holds for say $i = i_0$. Then a simple computation shows that there are at least $3[c_2 n]$ vertices of $G(n, i_0)$ which are connected with more than one of the vertices $\alpha_{i_0}, \beta_{i_0}, \gamma_{i_0}$. Therefore there are at least $[c_2 n]$ of them which are connected with the same pair, i.e., $G(n, i_0)$, and therefore $G^{(n)}_{f(n)+1}$, contains the configuration required by Lemma 2, which completes the proof of the lemma.

By more careful considerations we can prove that every $G^{(n)}_{f(n)+1}$ contains $n/6 + O(1)$ triangles $(\alpha_1, \alpha_2, \beta_i)$, $1 \leqq i \leqq n/6 + O(1)$, and that this result is best possible.

LEMMA 3. *Let $\delta > 0$ be a fixed number. Consider any graph*

$$G^{(n)}_u, \qquad u > f(n) - (n/2)(1 - \delta), \quad n > n_0(\delta),$$

which contains a triangle. Then $G_u^{(n)}$ contains an edge (α_1, α_2) and $[c_3 n] + 1, c_3 = c_3(\delta)$, vertices $\beta_1, \cdots, \beta_r$, $r = [c_3 n] + 1$, so that all the triangles $(\alpha_1, \alpha_2, \beta_i)$, $1 \leqq i \leqq r$, are in $G_u^{(n)}$.

By assumption $G_u^{(n)}$ contains a triangle $(\alpha_1, \alpha_2, \alpha_3)$. Assume first that

$$v(\alpha_1) + v(\alpha_2) + v(\alpha_3) > n(1 + 9c_3) + 9. \tag{3}$$

Then as in the proof of Lemma 2 we can show that $G_u^{(n)}$ contains the required configuration.

If (3) is not satisfied, then $G_u^{(n)} - \alpha_1 - \alpha_2 - \alpha_3$ has $n - 3$ vertices and at least $u - n(1 + 9c_3) - 9$ edges. But if $c_3 < \delta/18$, then for $n > n_0$

$$\begin{aligned} u &- n(1 + 9c_3) - 9 \\ &> f(n) - (n/2)(1 - \alpha) - n(1 + 9c_3) - 9 > f(n - 3). \end{aligned}$$

Thus by Lemma 2, $G_u^{(n)} - \alpha_1 - \alpha_2 - \alpha_3$, and therefore $G_u^{(n)}$, contains the configuration required by Lemma 3, which completes the proof of Lemma 3.

Now we can prove our Theorem. Let there be given a $G_{f(n)+t}^{(n)}$, $t < c_1 n/2$. Assume first that after the omission of any $r = [c_1 n/2c_3]$, $c_3 = c_3(\frac{1}{4})$ ($\delta = \frac{1}{4}$ in Lemma 3), edges the graph will still contain a triangle. For sufficiently small c_1, $c_1/2c_3 < \frac{1}{4}$; thus it will be permissible to apply Lemma 3 during the omission of these edges.

By Lemma 3 (or Lemma 2) there exists an edge e_1 which is contained in at least $[c_3 n] + 1$ triangles of $G_{f(n)+t}^{(n)}$; again by Lemma 3 in $G_{f(n)+t}^{(n)} - e_1$ there exists an edge e_2 which is contained in $[c_3 n] + 1$ triangles of $G_{f(n)+t}^{(n)} - e_1$. Suppose we have already chosen the edges $e_1, \cdots, e_r$ each of which is contained in at least $[c_3 n] + 1$ triangles. By our assumption $G_{f(n)+t}^{(n)} - e_1 - \cdots - e_r$ contains at least one triangle; thus by Lemma 3 there is an edge e_{r+1} in $G_{f(n)+t}^{(n)} - e_1 - \cdots - e_r$ which is contained in at least $[c_3 n] + 1$ triangles in this graph. These triangles incident on the edges $e_1, \cdots, e_{r+1}$ are evidently distinct; thus $G_{f(n)+t}^{(n)}$ contains at least $(r + 1)([c_3 n] + 1) > c_1 n^2/2 > tn/2$ triangles, which proves our Theorem in this case.

Therefore we can assume that there are $l \leqq r < n/4$ edges $e_1, \cdots, e_l$ so that $G = G_{f(n)+t}^{(n)} - e_1 - \cdots - e_l$ contains no triangle, and we can assume that l is the smallest integer with this property. By $l \leqq r < n/4$, G has

$$f(n) + t - l > f(n) - n/4 > f(n - 1) + 1$$

edges. Thus by Lemma 1, G is even.

By Turán's theorem, $l \geqq t$. Assume first $l = t$ (it is not necessary to treat the cases $l = t$ and $l > t$ separately, but perhaps it will be easier for the reader to do so). Then G has $f(n)$ edges, and by Turán's theorem G is of the following form: The vertices of G are $\alpha_1, \cdots, \alpha_{[n/2]}, \beta_1, \cdots, \beta_{n-[n/2]}$, and the edges are (α_i, β_j), $1 \leqq i \leqq [n/2]$, $1 \leqq j \leqq n - [n/2]$. A simple argument shows that the addition of every further edge introduces at least

$[n/2]$ triangles and that these triangles are distinct. Thus $G^{(n)}_{f(n)+t}$ contains at least $t[n/2]$ triangles, and our Theorem is proved in this case too.

Assume finally $l = t + w$, $0 < w < n/4$ (since $l < n/4$). It will be more convenient to assume first that n is even. Put $n = 2m$. Since G is even, it is contained in a graph $G(E, u)$ whose vertices are $\alpha_1, \cdots, \alpha_{m-u}, \beta_1, \cdots, \beta_{m+u}$ and whose edges are (α_i, β_j), $1 \leqq i \leqq m - u$, $1 \leqq j \leqq m + u$ (since G has more than $f(2m) - m/2 = m^2 - m/2$ edges, we have $0 \leqq u < (m/2)^{1/2}$).

Clearly every one of the edges $e_1, \cdots, e_l$ connect two α's or two β's. For if say e_i would connect an α with a β, then

$$G^{(n)}_{f(n)+t} - e_1 - \cdots - e_{1-1} - e_{i+1} - \cdots - e_l$$

would be even, and hence would contain no triangle, which contradicts the minimum property of l.

By our assumption G is a subgraph of $G(E, u)$. Assume that G is obtained from $G(E, u)$ by the omission of x edges. Then we evidently have

$$l = x + u^2 + t \quad (\text{or } w = x + u^2),$$

and $G^{(n)}_{f(n)+t}$ is obtained from G by adding l edges $e_1, \cdots, e_l$ which are all of the form $(\alpha_{i_1}, \alpha_{i_2})$ or $(\beta_{i_1}, \beta_{i_2})$. Put $e_i = (\beta_{i_1}, \beta_{i_2})$, and let us estimate the number of triangles $(\beta_{i_1}, \beta_{i_2}, \alpha_j)$ in $G(E, u) + e_i$. Clearly at most x of the edges (β_{i_1}, α_j), (β_{i_2}, α_j) are not in $G(E, u)$; thus $G(E, u) + e_i$ contains at least

$$m - u - x$$

triangles (if e_i connects two α's, then $G(E, u) + e_i$ contains at least $m + u - x$ triangles). For different e_i's these triangles are clearly different; thus

$$G^{(n)}_{f(n)+t} = G + e_1 + \cdots + e_l$$

contains at least

$$(4) \qquad (m - u - x)l = (m - u - x)(x + u^2 + t) \geqq tm = t(n/2)$$

triangles. (4) follows by simple computation from $l = u^2 + x + t < m/2$. (4) completes the proof of our Theorem for $n = 2m$. For $n = 2m + 1$ the proof is almost identical and can be omitted. Thus the proof of our Theorem is complete.

It seems possible that a slight improvement of this proof will give the conjecture that every $G^{(n)}_{f(n)+t}$, $t < [n/2]$ contains at least $t[n/2]$ triangles, but I have not been successful in doing this.

I have not succeeded in formulating a reasonable conjecture about the minimum number of triangles a $G^{(n)}_{f(n)+t}$ must contain if $[n/2] \leqq t \leqq \binom{n}{2} - f(n)$. It is easy to see that if t is close to $\binom{n}{2} - f(n)$, then $G^{(n)}_{f(n)+t}$ contains more than $t[n/2]$ triangles, and it would be easy to obtain a best possible result in this case. But I have not investigated the range of t for which this is possible. I just remark that $G^{(n)}_{\binom{n}{2}-l}$, $l \leqq 2$, contains at least $\binom{n}{3} - l(n - 2)$ triangles and that $G^{(n)}_{\binom{n}{2}-3}$ contains at least $\binom{n}{3} - 3(n - 2) + 1$ triangles, and that these results are best possible. The simple proofs are left to the reader.

Turán's theorem implies that every $G^{(3n)}_{3n^2+1}$ contains a complete 4-gon. As an analogue of the theorem of Rademacher I can prove by very much more complicated arguments that it contains at least n^2 complete 4-gons; this result is easily seen to be best possible.

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