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On a classification of denumerable order types and an application to the partition calculus

by

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1. Introduction. In this paper we are going to give a classification of denumerable order types, namely we are going to prove that every order type of a denumerable set which does not contain a dense subset can be built up from the order types $0, 1$ by a transfinite induction process taking at every step the so-called *ω-sum* and *ω^*-sum* of order types previously defined. Thus every order type Θ of such kind can have an ordinal number $\varrho(\Theta)$ less than ω_1, called the *rank of* Θ, associated with it — and several properties of denumerable order types can be verified by carrying out a transfinite induction on $\varrho(\Theta)$ (1).

As an application of the above-mentioned result a problem in the partition calculus for sets will be solved. Finally we are going to state some unsolved problems concerning non denumerable types (2).

2. Notations. Definitions. We are going to use the usual notations of set theory and we list only those where there is a danger of misunderstanding.

Capital Roman letters denote sets, $x, y, \ldots, a, b, \ldots$ denote elements of sets, $\alpha, \beta, \gamma, \varrho, \nu, \ldots$ denote ordinal numbers, Θ, φ, Φ denote order types, n, k, l denote non-negative integers. No distinction will be made between finite cardinal numbers and ordinal numbers.

η will denote the type of rational numbers ordered according to magnitude.

$\overline{\overline{X}}, \overline{\varphi}$ denote the cardinal number of X and φ respectively.

If S is a set ordered by a relation R, then for an arbitrary pair $x, y \in S$ "x is less than y" will be denoted by $x < y(R)$ and the order type of X will be denoted by $\overline{X}(R)$. If there is no danger of misunderstanding (R) will be omitted.

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(1) This classification seems to be so simple and natural that probably it is already described somewhere in the literature; however, the authors have been unable to find it. Therefore it seems worthwhile to give the proofs in detail.

(2) For another application of the classification see [1].

If an ordered set of type Θ_1 contains a subset of type Θ_2 we briefly write $\Theta_2 \leqslant \Theta_1$.

If S is a set ordered by the relation R and $A, B \subseteq S$ then $A \prec B(R)$ denotes that

$$a < b(R) \quad \text{for every pair} \quad a \in A,\ b \in B.$$

Definition 2.1. Let Z be a set $\bar{Z} = \varphi(T)$ and let Θ_x be defined for every $x \in Z$. We define $\Theta = \sum_{x \in Z} \Theta_x$ as usual in the following way. Let S_x be a system of disjoint sets ordered by the relations R_x such that $\bar{S}_x = \Theta_x(R_x)$ for every $x \in Z$. Then Θ is the type of the set $S = \bigcup_{x \in Z} S_x$ ordered by the following relation R.

Let $a, b \in S$, $a \in S_x$, $b \in S_y$, $a < b(R)$ if and only if either $x < y(T)$ or $x = y$ and $a < b(R_x)$.

It is well known that Θ depends only on the ordered set Z and on the function Θ_x.

Θ will be briefly termed a *sum* of type φ of the Θ_x's.

If $\varphi = \omega$ or $\varphi = \omega^*$ we may denote the Θ_x's by Θ_n and we can speak of the ω-*sum* or ω^*-*sum* of the sequence $(\Theta_n)_{n<\omega}$, which will be denoted by

$$\Theta_0 + \ldots + \Theta_n + \ldots, \quad \ldots + \Theta_n + \ldots + \Theta_0, \quad \text{respectively}.$$

Remarks. 1. If $\Theta_x = \psi$ for every $x \in \omega$, then Θ depends only on φ and ψ and will be denoted by $\psi \cdot \varphi$, as usual.

2. Note that some of the Θ_x's may be equal to 0, and thus, e.g. $\omega \leqslant \Theta_0 + \ldots + \Theta_n + \ldots$ does not follow from Definition 2.1.

Now we are going to redefine the partition symbol defined in [2] in the special cases needed for our purpose.

Let $[X]^m$ denote the set $\{Y \colon Y \subseteq X \text{ and } \bar{\bar{Y}} = m\}$

Definition 2.2. $\Theta_1 \to (\Theta_2, \Theta_3)^2$ indicates that the following statement is true.

Whenever S is an ordered set, $\bar{S} = \Theta_1$ and $[S]^2 = I_1 \cup I_2$ is a partition of the set $[S]^2$, then either there exists a set $S' \subset S$, $\bar{S}' = \Theta_2$ such that $[S']^2 \subseteq I_1$ or there exists an $S'' \subset S$, $\bar{S}'' = \Theta_3$ such that $[S'']^2 \subseteq I_2$.

$\Theta_1 \nrightarrow (\Theta_2, \Theta_3)^2$ denotes the negation of the above statement.

If m_1, m_2, m_3 are cardinal numbers, then the symbol $m_1 \to (m_2, m_3)^2$ has a similar self-explanatory meaning.

However, in this paper we are going to deal with the case when types and cardinals may appear in the same symbol.

Definition 2.3. Let Θ, Θ_1 be types and let m be a cardinal number.

$\Theta \to (\Theta_1, m)^2$ indicates the following statement. Whenever S is an ordered set, $\bar{S} = \Theta$ and $[S]^2 = I_1 \cup I_2$ is an arbitrary partition of $[S]^2$, then either there exists an $S' \subseteq S$, $\bar{S}' = \Theta_1$ such that $[S']^2 \subseteq I_1$ or there exists an $S'' \subseteq S$, $\bar{\bar{S}}'' = m$ such that $[S'']^2 \subseteq I_2$.

$\Theta \nrightarrow (\Theta_1, m)^2$ indicates the negation of this statement.

The symbol just defined has the following obvious monotonicity properties

$$\Theta \to (\Theta_1, m)^2 \quad \text{implies} \quad \Theta' \to (\Theta_1, m)^2 \quad \text{for every} \quad \Theta \leqslant \Theta',$$
$$\Theta \to (\Theta_1, m)^2 \quad \text{implies} \quad \Theta \to (\Theta_1', m')^2 \quad \text{for every} \quad \Theta_1' \leqslant \Theta_1,\ m' \leqslant m.$$

3. Classification of the denumerable order types. Let S be an ordered set ordered by the relation R, and let $a \leqslant b(R)$ be two arbitrary elements of S. $(a, b)(R)$ denotes, as usual, the interval $\{x\colon x \in S \text{ and } a < x < b(R)\}$. The *ordered set S is said to be dense* if $(a, b) \neq 0$ for every pair $a < b \in S$. The *order type Θ is said to be a discrete type* if $\overline{S} = \Theta$ and S does not contain a dense subset.

Let Δ denote the set of all denumerable order types and let Δ_D be the set of all discrete denumerable order types and put $\Delta_S = \Delta - \Delta_D$.

Considering that every denumerable dense set is of type η, $1+\eta$, $\eta+1$, or $1+\eta+1$, the following statements are immediate consequences of the above definitions.

3.1. *If $\Theta \in \Delta$ then $\Theta \in \Delta_D$ if and only if $\eta \nleqslant \Theta$ and $\Theta \in \Delta_S$ if and only if $\eta \leqslant \Theta$.*

Now we are going to define a class O of denumerable order types.

DEFINITION 3.2. We define the classes O_ϱ for every $\varrho < \omega_1$ by transfinite induction on ϱ as follows. O_0 consists of 0 and 1. Suppose that $O_{\varrho'}$ is defined for every $\varrho' < \varrho$ for a $\varrho < \omega_1$. Put $G_\varrho = \bigcup_{\varrho' < \varrho} O_{\varrho'}$. Let O_ϱ consist of the ω-sums and of the ω^*-sums of the sequences $\Theta_0, \ldots, \Theta_n, \ldots$ satisfying the condition $\Theta_n \in G_\varrho$ for every $n < \omega$.

It is obvious that $O_0 \subseteq \ldots \subseteq O_\varrho \subseteq \ldots$ for $\varrho < \omega_1$. Put $O = \bigcup_{\varrho < \omega_1} O_\varrho$. Then there exists a least $\varrho < \omega_1$, corresponding to every $\Theta \in O$, such that $\Theta \in O_\varrho$. Put $\varrho = \varrho(\Theta)$ for this ϱ. $\varrho(\Theta)$ will be called the *rank* of Θ.

The main aim of this section is to prove the following

THEOREM 1. *The discrete denumerable order types coincide with the elements of O and the non-discrete ones are sums of type η, $1+\eta$, $\eta+1$, or $1+\eta+1$ of non-vanishing discrete ones.*

To prove Theorem 1 we have to verify the following statements.

3.3. $O = \Delta_D$.

3.4. *If $\Theta \in \Delta_S$ then there exists a function Θ_x defined on a set Z of type η (or $1+\eta$, or $\eta+1$, or $1+\eta+1$) satisfying the conditions $\Theta_x \neq 0$, $\Theta_x \in O$ for every $x \in Z$ and $\Theta = \sum_{x \in Z} \Theta_x$.*

Before proving these we need some further preliminaries.

3.5. *Every order type $\Theta \in O$ of rank $\varrho(\Theta) > 0$ is either the ω-sum or the ω^*-sum of order types $\Theta_n \in O$ of rank less than $\varrho(\Theta)$.*

In fact, $\Theta \epsilon O_{\varrho'}$ implies $\varrho(\Theta) \leqslant \varrho'$, hence the statement follows from 3.2.

Considering that 0 and 1 are discrete types and that the ω-sum as well as the ω^*-sum of discrete types is again discrete, it follows from 3.5 by transfinite induction on $\varrho(\Theta)$ that the elements of O are discrete types, i.e. we have

3.6. $O \subset \Delta_D$.

To prove the inverse inclusion we need another classification of the elements of Δ_D.

Let S be a set ordered by a relation R.

DEFINITION 3.7. The collection S^* of subsets of S is briefly said to be a *splitting* of S if it satisfies the following conditions:

$$\bigcup_{X \epsilon S^*} X = S\,,$$

and either $X < Y(R)$ or $Y < X(R)$ for every pair $X \neq Y \epsilon S^*$.

Then S^* may be considered as a set ordered by the relation R^* defined by the stipulation

$$X < Y(R^*) \quad \text{if and only if} \quad X < Y(R)\,.$$

Let $S^*(x)$ denote for every $x \epsilon S$ the uniquely determined element of S^* for which $x \epsilon S^*(x)$.

Let S_1^*, S_2^* be two splittings of S. S_1^* is said to be a *refinement* of S_2^* if $S_1^*(x) \subseteq S_2^*(x)$ for every $x \epsilon S$. S_1^* is a proper refinement of S_2^* if there is an $x \epsilon S$ such that $S_1^*(x) \subset S_2^*(x)$[2].

DEFINITION 3.8. Let $(S_\beta^*)_{\beta<\alpha}$ be a sequence of splittings of S such that S_β^* is a refinement $S_{\beta'}^*$, for every $\beta < \beta' < \alpha$. Put $S_\alpha^*(x) = \bigcup_{\beta<\alpha} S_\beta^*(x)$. Then the set S_α^*, which consists of all different $S_\alpha^*(x)$'s, is a splitting of S called the *sum* of S_β^*'s and every S_β^* is a refinement of it.

Proof. If $S_\beta^*(x) = S_\beta^*(y)$ for a $\beta < \alpha$ then $S_{\beta'}^*(x) = S_{\beta'}^*(y)$ for every $\beta \leqslant \beta' < \alpha$, hence $S_\alpha^*(x) = S_\alpha^*(y)$. If $S_\beta^*(x) \neq S_\beta^*(y)$ for every $\beta < \alpha$ and, for instance, $x < y(R)$, then by 3.7,

$$S_\beta^*(x) < S_\beta^*(y) \quad \text{for every} \quad \beta < \alpha$$

and thus

$$S_\alpha^*(x) < S_\alpha^*(y)(R)$$

DEFINITION 3.9. Let S be an ordered set.

Put $N(x) = \{y\colon (y \leqslant x \text{ and } |(yx)| < \aleph_0) \text{ or } (x \leqslant y \text{ and } |(xy)| < \aleph_0)\}$.

It is easy to verify from 3.7 and 3.9 that the set S', which consists of all different $N(x)$'s, is a splitting of S which satisfies $S'(x) = N(x)$ for every $x \epsilon S$, and it is easy to see that

3.10. $\overline{S'(x)}(R) = \overline{N(x)}(R)$ *is* ω, ω^*, $\omega^* + \omega$, *or finite for every* $x \epsilon S$.

DEFINITION 3.11. Suppose now that S is a set ordered by R and that S^* is a splitting of it. Consider the set S^* ordered by R^*. Apply to it the operation defined in 3.9. Then we get a splitting $S^{*\prime}$ of it. Define the splitting S'' of S induced by S^* by the stipulation

$$S''(x) = \bigcup_{S^*(y) \in N(S^*(x))} S^*(y) .$$

It is obvious that S'' is a splitting of S and S^* is a refinement of it.

On the other hand, it follows immediately from the definitions 2.1, 3.10 and 3.11 that

3.12. *Under the notations of* 3.11, $\overline{S''(x)}(R)$ *is an* ω*-sum, an* ω^**-sum, an* $\omega^*+\omega$*-sum or a finite sum of the order types* $\overline{S^*(y)}(R)$ *for* $S^*(y) \in N\big(S^*(x)\big)$

DEFINITION 3.13. Let S be a denumerable set ordered by the relation R, $\bar{S} = \Theta\,(R)$.

We are going to define a sequence S^*_a of splittings of S for every $a < \omega_1$ by transfinite induction on a as follows.

Define S^*_0 by the stipulation $S^*_0(x) = \{x\}$ for every $x \in S$. Suppose that $0 < a < \omega_1$ and that S^*_β is defined for every $\beta < a$ in such a way that $S^*_{\beta'}$ is a refinement of S^*_β for every $\beta' < \beta < a$. Distinguish two cases

(i) $a = \gamma + 1$ for a $\gamma < a$,

(ii) a is of the second kind.

In case (i) let S^*_a be the splitting S'' of S induced by S^*_γ (defined in 3.11).

In case (ii) let S^*_a be the sum of the splittings S^*_β $(\beta < a)$ (defined in 3.8).

It follows from 3.8 and 3.11 that S^*_β is a refinement of S_a for every $\beta < a$ in both cases, and thus S^*_a is defined for every $a < \omega_1$.

Put $\varphi(\Theta, a) = \bar{S}^*_a\,(R^*_a)$ for every $a < \omega_1$.

In the rest of this section S denotes a fixed, non-empty denumerable ordered set, $\bar{S} = \Theta\,(R)$. We need the following lemmas.

3.14. *If* $S^*_\gamma = S^*_{\gamma+1}$ *for a* $\gamma < \omega_1$ *then either* $\varphi(\Theta, \gamma) = 1$ *or* $\varphi(\Theta, \gamma) = \eta$ (*or* $1+\eta$, *or* $\eta+1$, *or* $1+\eta+1$).

Proof. By 3.13, $S^*_{\gamma+1}$ is the splitting S'' of S induced by S^*_γ defined in 3.11. But then by 3.11

$$S^*_{\gamma+1} = \bigcup_{S^*_\gamma(y) \in N(S^*_\gamma(x))} S^*_\gamma(y) \qquad \text{for every} \qquad x \in S .$$

This means by 3.9 that in the ordered set $S^*_\gamma(R^*_\gamma)$, $N(X) = \{X\}$ for every $X \in S^*_\gamma$. But then again by 3.9 either S^*_γ contains exactly one element or, for every pair $X < Y \in S^*_\gamma$, $|(X, Y)\,(R^*_\gamma)| \geqslant \aleph_0$. But this means that S^*_γ is either of type 1 or dense, and—being denumerable—it is of type η (or $1+\eta$, or $\eta+1$, or $1+\eta+1$).

Now we prove that

3.15. *There exists an ordinal number* $\gamma_0 < \omega_1$ *such that* $S^*_{\gamma_0} = S^*_{\gamma_0+1}$.

Proof. By the definitions 3.7 and 3.11 corresponding to every element x of S, $S^*_\gamma(x)$ is a non decreasing sequence of subsets of S, and thus—S being denumerable—there exists a $\gamma(x) < \omega_1$ such that $S^*_{\gamma(x)} = S^*_\gamma$ for every $\gamma \geqslant \gamma(x)$. Using again the fact that $\overline{S} \leqslant \aleph_0$ we infer that there exists a $\gamma_0 < \omega_1$ such that $\gamma_0 \geqslant \gamma(x)$ for every $x \in S$ and consequently $S^*_{\gamma_0}(x) = S^*_{\gamma_0+1}(x)$ for every $x \in S$, whence $S^*_{\gamma_0} = S^*_{\gamma_0+1}$.

Definition 3.16. By 3.15 we can make the following definition. Let $\gamma(\Theta) = \gamma$ be the least ordinal number $\gamma < \omega_1$ for which $S^*_\gamma = S^*_{\gamma+1}$. $\gamma(\Theta)$ will be called the *order* of Θ.

Remark. It is obvious from the above considerations that S^*_β is a proper refinement of S^*_a for every $\beta < a \leqslant \gamma(\Theta)$ and that $S^*_{\gamma(\Theta)} = S^*_\gamma$ for every $\gamma \geqslant \gamma(\Theta)$. It follows that the sequence $\varphi(\Theta, \gamma)$ is non-increasing ($\varphi(\Theta, \gamma) \leqslant \varphi(\Theta, \gamma')$ for $\gamma \geqslant \gamma'$) but it is not strictly decreasing even if Θ is an ordinal number. For example, put $\Theta = \omega^\omega$, $S = W(\omega^\omega)$. Then $\gamma(\omega^\omega) = \omega$, $\varphi(\omega^\omega, \omega) = 1$ but $\varphi(\omega^\omega, n) = \omega^\omega$ for every integer n.

By 3.14 we have $\varphi(\Theta, \gamma(\Theta)) = 1$, η, $1+\eta$, $\eta+1$ or $1+\eta+1$. Considering that $S^*_\gamma(x) \neq 0$ for every $x \in S$, $\gamma < \omega_1$, $\varphi(\Theta, \gamma(\Theta)) \neq 1$ implies $\eta \leqslant \Theta$. It follows from 3.1 that

3.17. *If* $\Theta \in \Delta_D$ *then* $\varphi(\Theta, \gamma(\Theta)) = 1$.

Now we need preliminaries concerning the class O.

3.18. *Suppose that* $\overline{Z} = \varphi$, $\varphi \in O$, *and* $\Theta_x \in O$ *for every* $x \in Z$. *Then* $\Theta = \sum_{x \in Z} \Theta_x \in O$.

Proof: By induction on $\varrho(\varphi)$. The statement is obvious for $\varrho(\varphi) = 0$. Suppose that it is true for every type φ' with $\varrho(\varphi)' < \varrho$ for a $0 < \varrho < \omega_1$.

Then, by 3.5, Z is either the ω-sum or the ω^*-sum of the sets Z_n of type φ_n of rank less than ϱ.

The types $\Theta_n = \sum_{x \in Z_n} \Theta_x$ then belong to O by the induction hypothesis and Θ is either the ω-sum or the ω^*-sum of them, whence $\Theta \in O$.

3.19. $a, a^* \in O$ *for every* $a < \omega_1$.

Proof. By symmetry it is enough to prove this for a. We use induction on a. $0 \in O$ and if $a > 0$ then either $a = \beta + 1$ or a is of the second kind and consequently is cofinal with ω. Hence in both cases it is the ω-sum of ordinals less than a which belong to O by the induction hypothesis.

Now we are going to prove that

3.20. $\Delta_D \subseteq O$, $\varphi(\Theta, \gamma(\Theta)) = 1$ *implies* $\Theta \in O$ *for every* $\Theta \in \Delta$.

Proof. If $\Theta \in \Delta_D$ then, by 3.17, $\varphi(\Theta, \gamma(\Theta)) = 1$. We are going to prove by induction on $\gamma(\Theta)$ that $\varphi(\Theta, \gamma(\Theta)) = 1$ implies $\Theta \in O$. If $\gamma(\Theta) = 0$ then, by 3.13, $\overline{S}(R) = \overline{S}_0^*(R_0^*) = 1$, whence $\Theta = 1$, $\Theta \in O$.

Suppose that $\gamma(\Theta) = \gamma > 0$, $\gamma < \omega_1$ and that $\Theta' \in O$ for every Θ' provided $\gamma(\Theta') < \gamma$ and $\varphi(\Theta', \gamma(\Theta')) = 1$.

We distinguish two cases: (i) $\gamma = \beta + 1$, (ii) γ is of the second kind.

Ad (i). $\overline{S}_\gamma^* = 1\ (R_\gamma^*)$. Hence $S_\gamma^*(x) = S$ for every $x \in S$. By 3.13, S_γ^* is the splitting S'' of S induced by the splitting S_β^* (defined in 3.11) and thus

$$S = S_\gamma^*(x) = \bigcup_{S_\beta^*(y) \in N(S_\beta^*(x))} S_\beta^*(y)\ .$$

It is obvious that the order of the sets $S_\beta^*(y)$ ordered by R is $\leqslant \beta < \gamma$, and thus $\overline{S_\beta^*(y)}\ (R)$ belongs to O by the induction hypothesis. Considering that by 3.12 Θ is the ω-sum, the ω^*-sum, the $\omega^* + \omega$-sum, or a finite sum of them, Θ belongs to O.

Ad (ii). $\overline{S}_\gamma^* = 1\ (R_\gamma^*)$, whence by 3.8 and 3.13 $S = S_\gamma^*(x_0) = \bigcup_{\beta < \gamma} S_\beta^*(x_0)$ for an arbitrary fixed $x_0 \in S$. Considering that the order of every $S_\beta^*(x)$ is $\leqslant \beta < a$, we infer from the induction hypothesis that $\overline{S_\beta^*(x_0)}\ (R)$ belongs to O for every $\beta < a$. Put

$$A_\beta = \{x\colon\ x \in S \text{ and } x < x_0\ (R) \text{ and } x \in S_\beta^*(x_0) - \bigcup_{\beta' < \beta} S_{\beta'}^*(x_0)\}\ ,$$

$$B_\beta = \{x\colon\ x \in S \text{ and } x \geqslant x_0 \text{ and } x \in S_\beta^*(x_0) - \bigcup_{\beta' < \beta} S_\beta^*(x_0)\}\ .$$

Considering that every section of an element of O belongs to O, we get $\overline{A_\beta}(R)$, $\overline{B_\beta}(R) \in O$. $\gamma, \gamma^* \in O$ by 3.19, hence the sum of type γ or γ^* of the sets B_β, A_β as well as their sum Θ belongs to O by 3.18.

3.6 and 3.20 prove $O = \Delta_D$, hence 3.3, which is the first part of Theorem 1, is proved. If we replace in Definition 3.2 the ω-sums and ω^*-sums by $\omega^* + \omega$-sums, then it is easy to verify that $\varrho(\Theta) = \gamma(\Theta)$ for every $\Theta \in O = \Delta_D$, but we do not need this and so we omit the proof.

Now we are going to prove 3.4. Suppose that $\Theta \in \Delta_S$, $\gamma(\Theta) = \gamma$. Then $\eta \leqslant \Theta$ by 3.1. By 3.20, $\varphi(\Theta, \gamma(\Theta)) = 1$ implies $\Theta \in O = \Delta_D$, whence we have $\varphi(\Theta, \gamma) = \eta$, $1 + \eta$, $\eta + 1$ or $1 + \eta + 1$. Thus $\overline{S}_\gamma^*(R_\gamma^*) = \eta$, $1 + \eta$, $\eta + 1$ or $1 + \eta + 1$. By the definition 3.7, S_γ^* consists of the different $S_\gamma^*(x)$'s for $x \in S$ and thus by the definitions 2.1, 3.7, Θ the type of $S(R)$ is an η, $1 + \eta$, $\eta + 1$ or $1 + \eta + 1$ sum of their types. Thus to prove 3.4 it is sufficient to see that $\overline{S_\gamma^*(x)\ (R) \in O}$ for every $x \in S$. Put $\Theta_x = \overline{S_\gamma^*(x)}\ (R)$. It is obvious that $\gamma(\Theta_x) \leqslant \gamma$ and $\varphi(\Theta_x, \gamma) = 1$, whence $\varphi(\Theta_x, \gamma(\Theta_x)) = 1$ and consequently $\Theta_x \in O$ by 3.20.

Thus the proof of Theorem 1 is finished.

It is obvious that the above constructions can be generalized to non-denumerable ordered sets. If in the definition 3.2 we replace the ω- and ω^*-sums by ω_α- and ω_α-sums we get a class $O(\aleph_\alpha)$ of order types of power at most $\aleph_\alpha$ $(O = O(\aleph_0))$. One can associate with every $\Theta \in O(\aleph_\alpha)$ a rank $\varrho(\Theta) < \omega_{\alpha+1}$ and one can prove in the same way as in case $\alpha = 0$ that $O(\aleph_\alpha)$ consists of all discrete types of power at most $\aleph_\alpha$ and further that every type $\overline{\Theta} \leqslant \aleph_\alpha$ is a sum $\sum_{x \in Z} \Theta_x$ of discrete types where the ordered set Z is dense.

However, here the dense sets cannot be characterized so simply as in the case of denumerable sets and therefore we do not give the detailed proof of this result.

4. Results concerning the partition symbol. As a consequence of the well-known theorem of Ramsay we have $\aleph_0 \to (\aleph_0, \aleph_0)^2$ and this clearly implies $\omega \to (\omega, \aleph_0)^2$ and $\omega^* \to (\omega^*, \aleph_0)^2$. On the other hand, it is proved in [2] that $\eta \to (\eta, \aleph_0)^2$ holds. Considering that $\Theta \leqslant \eta$ for every denumerable type, it follows that

$$\Theta \to (\Theta, \aleph_0)^2 \text{ holds for every } \Theta \in \Delta_S .$$

The following problem arises now: are there any other denumerable order types Θ satisfying $\Theta \to (\Theta, \aleph_0)^2$? We are going to prove that the answer is negative.

THEOREM 2. *If $\Theta \in \Delta$ then $\Theta \to (\Theta, \aleph_0)^2$ holds if and only if $\Theta = \omega$ or $\Theta = \omega^*$ or $\eta \leqslant \Theta$.*

We have to prove that if $\Theta \in \Delta_D$ and $\Theta \neq \omega$ or $\Theta \neq \omega^*$, then $\Theta \nrightarrow (\Theta, \aleph_0)^2$ holds.

Instead of this we are going to prove the following

THEOREM 3. *Suppose $\Theta \in \Delta_D$. Then there exists a partition $[S]^2 = I_1 \cup \cup I_2$ of $[S]^2$ satisfying the following conditions:*

(α) *Whenever $S', S'' \subseteq S$, $S' < S''$, $\overline{\overline{S}}' = \overline{\overline{S}}'' = \aleph_0$, then*

$$[S', S'']^2 \nsubseteq I_1 \quad \textit{where} \quad [S', S'']^2 = \left\{\{xy\}\colon x \in S' \text{ and } y' \in S''\right\} .$$

($\alpha\alpha$) *Whenever $S' \subseteq S$, $\overline{\overline{S}}' = \aleph_0$, then*

$$[S']^2 \nsubseteq I_2 .$$

First we prove that Theorem 3 implies Theorem 2. The implication is obvious if $\Theta \in \Delta_D$ is such that $\omega \cdot 2 \leqslant \Theta$, $\omega^* \cdot 2 \leqslant \Theta$, $\omega^* + \omega \leqslant \Theta$ or $\omega + \omega^* \leqslant \Theta$. But it is easy to see that if none of these conditions hold, then either $\Theta = \omega + n$ or $\Theta = n + \omega^*$, and a trivial construction shows that $\omega + n \nrightarrow (\omega + n, \aleph_0)^2$, $n + \omega^* \nrightarrow (n + \omega^*, \aleph_0)^2$.

Proof of Theorem 3: By Theorem 1. $O = \Delta_D$, hence we may prove our theorem by induction on $\varrho(\Theta) = \varrho$. For $\varrho = 0$, $\Theta = 0$ or

$\Theta = 1$ and the statement is trivial. Suppose that $\varrho = \varrho(\Theta) > 0$ and that the statement is true for every order type $\Theta' \in \Delta_D$ of rank less than ϱ.

By 3.5 there exists a sequence $\Theta_n \in \Delta$ of types of rank less than ϱ such that Θ is either the ω-sum or the ω^*-sum of the sequence $(\Theta_n)_{n<\omega}$. By symmetry, we may suppose that

$$\Theta = \Theta_0 + \ldots + \Theta_n + \ldots \tag{1}$$

Then there exists a sequence $(S_n)_{n<\omega}$ of subsets of S satisfying the following conditions:

$$\bigcup_{n<\omega} S_n = S\,, \quad S_n < S_{n'} \quad \text{and} \quad \overline{S}_n = \Theta_n\,, \quad \text{provided } n < n' < \omega\,. \tag{2}$$

By the induction hypothesis, for every $n < \omega$ there exists a partition $[S_n]^2 = I_1^n + I_2^n$ of the set $[S_n]^2$ satisfying the following conditions:

(3) *Whenever* $S', S'' \subseteq S_n$, $S' < S''$, $\overline{\overline{S}}' = \overline{\overline{S}}'' = \aleph_0$, *then* $[S', S'']^2 \nsubseteq I_1^n$.

(4) *Whenever* $S' \subseteq S_n$, $\overline{\overline{S}}' = \aleph_0$, *then* $[S']^2 \nsubseteq I_2^n$.

The sets S_n are denumerable, whence there exists a $\delta_n \leqslant \omega$ such that

$$\begin{aligned} S_n &= \{x_{n,k}\}_{k<\delta_n} \quad (\text{if } S_n \text{ is empty } \delta_n = 0), \\ x_{n,k} &\neq x_{n,k'} \quad \text{for} \quad k \neq k' < \delta_n\,. \end{aligned} \tag{5}$$

Define the partition $[S]^2 = I_1 \cup I_2$ of $[S]^2$ as follows.

(6) *Let* $\{x, y\} \in [S]^2$ *be arbitrary. Then* $x = x_{n,k}$, $y = x_{n',k'}$ *for some* $k < \delta_n$, $k' < \delta_{n'}$.

Distinguish two cases: (i) $n = n'$, (ii) $n \neq n'$.
In case (i) put $\{x_{n,k}, x_{n',k'}\} \in I_j$ if and only if

$$\{x_{n,k}, x_{n',k'}\} \in I_j^n \quad \text{for} \quad j = 1, 2\,.$$

In case (ii) we may suppose $n < n'$ and put

$$\begin{aligned} \{x_{n,k}, x_{n',k'}\} \in I_1 \quad &\text{if and only if} \quad k \leqslant k'\,, \\ \{x_{n,k}, x_{n',k'}\} \in I_2 \quad &\text{if and only if} \quad k > k'\,. \end{aligned}$$

Suppose now that $S', S'' \subseteq S$, $S' < S''$, $\overline{\overline{S}}' = \overline{\overline{S}}'' = \aleph_0$. Then by (2) there exist $n_0 \leqslant n_0'$ such that $\overline{\overline{S' \cdot S_{n_0}}} = \aleph_0$ and $\overline{\overline{S'' \cdot S_{n_0'}}} = \aleph_0$.

If $n_0 = n_0'$ then $[S', S'']^2 \nsubseteq I_1$ by (3) and (6). If $n_0 < n_0'$, then there is a k_0' such that $x_{n_0' k_0'} \in S''$, and considering that $\overline{\overline{S' \cdot S_{n_0}}} = \aleph_0$ there is a $k_0 > k_0'$ such that $x_{n_0, k_0} \in S'$, whence $\{x_{n_0,k_0}, x_{n_0',k_0'}\} \in I_1$ by (6), and consequently $[S', S'']^2 \nsubseteq I_1$ also in this case. This proves that (α) holds.

Suppose now that $S' \subseteq S$, $\overline{\overline{S'}} = \aleph_0$ and $[S']^2 \subseteq I_2$. Then $\overline{\overline{S' \cdot S_n}} < \aleph_0$ for every $n < \omega$ by (4) and (6). Hence there exists an increasing sequence $\{n_j\}_{j<\omega}$ of integers such that $\{x_{n_j} k_{n_j}\}_{j<\omega} \subseteq S'$. But then $[S']^2 \subseteq I_2$ would imply by (6) that $k_{n_j} > k_{n_{j'}}$ for every $j < j' < \omega$, but this is a contradiction, whence ($\alpha\alpha$) holds. Q.e.d.

We obtain from Theorem 3, the following

COROLLARY 1. *$\Theta \nrightarrow (\Theta', \aleph_0)^2$ for every $\Theta \in \Delta_D$ provided $\Theta' \neq \omega + n$ or $\Theta' \neq n + \omega^*$.*

Thus to complete our results it would be necessary to decide under what conditions for Θ we have $\Theta \rightarrow (\omega + n, \aleph_0)^2$ or $\Theta \rightarrow (n + \omega^*, \aleph_0)^2$ for $1 \leqslant n < \omega$. Here we have the following

THEOREM 4. (a) *$\Theta \rightarrow (\omega + n, \aleph_0)^2$ if and only if $\omega \cdot \omega^* \leqslant \Theta$.*

(b) *$\Theta \rightarrow (n + \omega^*, \aleph_0)^2$ if and only if $\omega^* \cdot \omega \leqslant \Theta$ for every $1 \leqslant n < \omega$ and for every denumerable type Θ.*

Proof (in outline). By symmetry it is sufficient to prove part (a) of our theorem. First we prove the negative part of it.

$$(1) \qquad \Theta \nrightarrow (\omega + 1, \aleph_0)^2 \quad \text{provided} \quad \omega \cdot \omega^* \nleqslant \Theta.$$

By $\omega \cdot \omega^* \nleqslant \Theta$, Θ is discrete and by Theorem 1 it has a rank $\varrho(\Theta)$. It is easy to verify, for example by induction on $\varrho(\Theta)$, that Θ is of the form $\sum_{\nu < \alpha} \beta_\nu^*$, where α and β_ν $(\nu < \alpha)$ are ordinal numbers.

Suppose $\overline{S} = \Theta(R)$. Then there exists a sequence $\{S_\nu\}_{\nu < \alpha}$ of subsets of S satisfying the following conditions.

$$(2) \quad S = \bigcup_{\nu < \alpha} S_\nu, \quad \overline{S}_\nu = \beta_\nu^*(R), \quad S_\nu < S_{\nu'}(R) \quad \text{for every} \quad \nu < \nu' < \alpha.$$

Let $W(\alpha) = \{\nu_n\}_{n < \omega}$ be a well-ordering of type ω of the denumerable set $W(\alpha)$.

Define the partition $I_1 \cup I_2$ of $[S]^2$ as follows. Let $\{x, y\} \in [S]^2$ be arbitrary. Suppose $x \in S_{\nu_n}$, $y \in S_{\nu_{n'}}$,

(3) Put $\{x, y\} \in I_1$ if $\nu_n = \nu_{n'}$. If $\nu_n \neq \nu_{n'}$ and, for instance, $n < n'$, put $\{x, y\} \in I_1$ if $\nu_n < \nu_{n'}$, and put $\{x, y\} \in I_2$ if $\nu_n > \nu_{n'}$.

Suppose $S' \subseteq S$, $[S']^2 \subseteq I_1$, $\overline{S}' = \omega + 1\,(R)$. Then considering that $\overline{S_{\nu_n}} = \beta_{\nu_n}^*(R)$ for every $n < \omega$ we have $\overline{S' \cdot S_{\nu_n}} < \omega$ for every $n < \omega$. Thus we may suppose $\overline{S' \cdot S_{\nu_n}} = 1$ for every $n < \omega$ and then, by (3), $\overline{S}' \leqslant \omega\,(R)$, which contradicts our assumptions. Hence we have

$$(4) \qquad S' \subseteq S, \quad [S']^2 \subset I_1 \quad \text{implies that} \quad \overline{S}' \neq \omega + 1.$$

On the other hand, suppose that $S' \subseteq S$, $[S']^2 \subset I_2$. Then $\overline{\overline{S' \cdot S_{\nu_n}}} = 1$ for every $n < \omega$ by (3).

Then $\overline{\overline{S'}} = \aleph_0$ would imply by (3) the existence of a decreasing infinite sequence of ordinal numbers, which would be a contradiction; thus we find that

(5) $$S' \subseteq S\,, \quad [S']^2 \subseteq I_2 \quad \text{implies} \quad \overline{\overline{S'}} < \aleph_0\,.$$

(4) and (5) prove (1).

To prove the positive part of part (a) of Theorem 4 it is sufficient to prove

(6) $$\omega\cdot\omega^* \to (\omega + n, \aleph_0)^2 \quad \text{for every} \quad n < \omega\,.$$

For $n = 0$ this follows from Theorem 2. We prove it by induction on n for every $n < \omega$. Suppose that the theorem is true for an $n < \omega$ and let S be an ordered set $\overline{S} = \omega\cdot\omega^*\ (R)$.

Put

$$T_1(x) = \{y\colon\ y \in S\,,\ y < x\ (R) \text{ and } \{xy\} \in I_1\}\,,$$
$$T_2(x) = \{y\colon\ y \in S\,,\ y < x\ (R) \text{ and } \{xy\} \in I_2\}$$

for an arbitrary $x \in S$. It is obvious that either $\overline{T_1(x)} = \omega\cdot\omega^*\ (R)$ or $\overline{T_2(x)} = \omega\cdot\omega^*\ (R)$ for an arbitrary $x \in S$. Suppose that $T_1(x) = \omega\cdot\omega^*$ for an $x \in S$. Then by the induction hypothesis there exists a subset $S' \subseteq T_1(x)$, $\overline{S'} = \omega + n\ (R)$ such that $[S']^2 \subseteq I_1$, and then $S'' = S' + \{x\}$ satisfies the condition

(7) $$S'' \subseteq S\,, \quad [S'']^2 \subset I_1\,, \quad \overline{S''} = \omega + n + 1\ (R)\,.$$

Thus we may suppose that $\overline{T_1(x)}\ (R) < \omega\cdot\omega^*$ for every $x \in S$.

We define a sequence $\{x_k\}_{k<\omega}$ by induction on k. x_0 is an arbitrary element of S. Suppose that $x_0, \ldots, x_k$ are already defined; then $\overline{T_1(x_0) \cup \ldots \cup T_1(x_k)}\ (R) < \omega\cdot\omega^*$, whence there exists an $x_{k+1} \in S$ such that $x_{k+1} \in T_2(x_i)$ for every $i < k+1$. The set $S' = \{x_k\}_{k<\omega}$ then satisfies the condition

(8) $$S' \subseteq S\,, \quad \overline{\overline{S'}} = \aleph_0\,, \quad [S']^2 \subseteq I_2\,.$$

(7) and (8) prove (6) and thus Theorem 4 is proved.

As to the case of non-denumerable types, the problems are more difficult. Generally one can ask the following question: which are the order types Θ, $\overline{\Theta} = p$ satisfying the condition $\Theta \to (\Theta, m)^2$? It is obvious that if we have $p \nrightarrow (p, m)^2$, then there are no such order types. Thus the genuine cases are when the corresponding partition symbol for cardinals is true.

For the results concerning this symbol see [2] (a complete discussion of it will be given in a forthcoming paper by P. Erdös, A. Hajnal and R. Rado).

If $m > \aleph_0$, then we have $m \nrightarrow (m, m)^2$, at least if m is not strongly inaccessible, and it is not known whether $m \to (m, m)^2$ holds for any $m > \aleph_0$.

Thus a direct generalization of the question treated in Theorem 2 cannot be asked.

However, using the generalized continuum hypothesis, one can prove that

$$\aleph_{a+1}\to(\aleph_{a+1},\aleph_a)^2 \textit{ is true provided } \aleph_a \textit{ is regular } (^3),$$

and this implies that

$$\Theta\to(\Theta,\aleph_a)^2 \textit{ holds provided } \Theta=\omega_{a+1} \textit{ or } \Theta=\omega^*_{a+1}.$$

P. Erdös and R. Rado have proved (4) that the same is true for $\Theta=\eta_{a+1}$ provided $\aleph_a$ is regular and the generalized continuum hypothesis holds where η_{a+1} is the normal type of power $2^{\aleph_a}$ given by Hausdorff (5).

It is not known whether there are other types $\eta_{a+1}\nleq\Theta$ of power $\aleph_{a+1}\,(=2^{\aleph_a})$ for which $\Theta\to(\Theta,\aleph_a)^2$ holds.

Thus the simplest unsolved problem is

PROBLEM 1. Suppose that $(2^{\aleph_0}=\aleph_1)$, $\overline{\Theta}=\aleph_1$. It is true that $\Theta\to(\Theta,\aleph_0)^2$ holds if and only if $\Theta=\omega_1$ or $\Theta=\omega_1^*$ or $\eta_1\leqslant\Theta$?

Remark. Using the methods of this paper it is easy to prove that under the condition of Problem 1 $\Theta\nrightarrow(\Theta,\aleph_0)^2$ holds for every discrete type Θ.

We would like to mention a few further results without proof. $\omega_1\omega^*\to(\omega_1+a,\aleph_0)^2$ for every $a<\omega_1$, but $\omega_1\omega^*\nrightarrow(\omega_1\cdot 2,\aleph_0)^2$; in fact the same holds if $\omega_1\omega^*$ is replaced by any discrete type.

We further have $\omega_2\omega_1^*\to(\omega_2+n,\aleph_1)^2$ for every $n<\omega$ provided the generalized continuum hypothesis holds. We can not decide whether $\omega_2\omega_1^*\to(\omega_2+\omega,\aleph_1)^2$ is true or not. Clearly many more problems could be stated, but we do not discuss them here.

The investigation of the statement $\Theta\to(\Theta',n)^2$ for $n<\omega$ leads to more ramified problems, even in cases where Θ is a denumerable ordinal number or order type. For a recapitulation of problems and results of this kind see a forthcoming paper of E. C. Milner and R. Rado and [4].

Here we mention only one problem of this kind. Let λ denote the order type of the continuum. It is easy to see that for every $\Phi\leqslant\lambda$, $\overline{\Phi}>\aleph_0$ we have $\Phi\nrightarrow(\Phi,3)^2$ provided $2^{\aleph_0}=\aleph_1$. It would be interesting to characterize those non-denumerable order types for which $\lambda\to(\Phi,3)^2$ holds. Although we have $\aleph_1\to(\aleph_1,3)^2$, we do not even know whether such Φ's exist.

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(3) For singular $\aleph_a$'s this is false.

(4) See a forthcoming paper of P. Erdös and R. Rado.

(5) See [3], § 8, Normaltypen.

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