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APPLICATIONS OF PROBABILITY TO COMBINATORIAL PROBLEMS

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In this lecture elementary probabilistic methods will be employed to obtain partial solutions to various combinatorial problems which, as far as I know, have not **yet** been obtained by different methods.

I. Denote by $f(n)$ the smallest integer so that every graph of $f(n)$ vertices either contains a complete subgraph of n vertices or contains a set of n independent points. We have

$$(1) \qquad 2^{\frac{1}{2}n} < f(n) \leq \binom{2n-2}{n-1}$$

The upper bound (due to G. Szeheres) is obtained by non-probabilistic arguments. Very likely $\lim f(n)^{1/n}$ exists.

More generally, denote by $f(k,n)$ the smallest integer so that every graph of $f(k,n)$ vertices which does not contain a set of k independent points contains a complete n-gon Szeheres's proof gives

$$(2) \qquad f(k,n) < \binom{k+n-2}{k-1}$$

In the most interesting case $k=3$ I can prove by elementary but complicated probabilistic arguments that

$$(3) \qquad f(3,n) > cn^2 (\log n)^{-2}$$

Perhaps $f(3,n) > cn^2$, but it seems that my methods are not sharp enough to decide this question.

Another problem, where graph theory could be successfully applied, is the following one: Tutte proved that for every k there exists a graph which contains no triangle, and whose chromatic number is $\geq k$ (the chromatic number is the least integer k so that we can color the vertices of our graph with k colors so that no two vertices which have the same color are connected by an edge). The Kellys constructed such a graph whose smallest circuit has at least six edges. I proved that to every l there exists an ε_l so that for every $n > n_0(l, \varepsilon_l)$ there exists a graph of n vertices, the smallest circuit of which has at least l edges and whose chromatic number is $> n^{\varepsilon_l}$ [1,2] .

II. Miller defines that a family of sets F has property B if there exists a set S which intersects all sets of the family F and does not contain any of them. In a recent paper Hajnal and I investigate this property extensively. Several unsolved problems remain for infinite sets, but also some for finite ones and to the problems on finite sets probability methods can succesfully be applied.

Denote by $f(n)$ the smallest integer so that there exists a family of $f(n)$ sets $S_1, \ldots, S_{f(n)}$ each having n elements which do not have property B. Hajnal and I remarked that $f(n) \leq \binom{2n-1}{n}$ and $f(3)=7$. Now I can show $f(n) > (1-\varepsilon)2^n \log 2$. I can not even show that $\lim f(n)^{1/n}$ exists.

More generally, assume that S_k has a_k elements. Then, using an old theorem of Chung, I can show that if

$$\prod (1-2^{-a_k}) < \tfrac{1}{2}$$

the family $\{S_k\}$ has property B. My paper on this subject will appear in Nordisk Mat. Tidsskr. [3,4] .

Gallai asked the following question:

Does there exist a finite family of sets $\{S_k\}$ each S_k has $\leq n$ elements, the intersection of any two S_k's has at most one element so that the family $\{S_k\}$ does not have property B ? For $n=2$ this is trivial, for $n=3$ easy , but for larger values of n I can prove it only by probabilistic methods.

III. A few days ago Schütte asked me the following question:

Denote by $f(k)$ the smallest integer so that there should exist a directed complete graph of $f(k)$ vertices $X_1,\ldots,X_{f(k)}$ so that to every k of its vertices $X_{l_1},\ldots,X_{l_k}$, $1\leq l_1<\ldots<l_k\leq f(k)$ there exists a vertex X_u, $1\leq u\leq f(k)$ so that all the edges (X_u,X_{i_r}), $1\leq r\leq k$ are directed away from X_u. $f(2)=7$, but for $k>2$ it seems hard to determine $f(k)$. I can show by induction that $f(k)>2^k$ and by probabilistic methods that $f(k)<ck^2 2^k$. By using deep results on quadratic residues I could obtain by non-probabilistic methods an estimate for $f(k)$ from above but it would be worse than $ck^2 2^k$.

References.

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3. P. Erdös and A. Hajnal, On a property of families of sets, Acta Math. Hung., 12 (1961), 87-123.
4. K.L. Chung, On mutually favorable events, Ann. Math. Statist. 13 (1942), 338-349.