PROBLEMS AND RESULTS ON THE THEORY OF INTERPOLATION. II

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Let $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$ be n arbitrary points in the interval $(-1, +1)$. $\omega_n(x) = \prod_{i=1}^{n} (x - x_i)$, $l_k(x) = \omega_n(x)/\omega_n'(x_k)(x - x_k)$. It is well known that the sum $\sum_{k=1}^{n} |l_k(x)|$ plays a decisive role in the convergence and divergence properties of the Lagrange interpolation polynomials. FABER [1] proved that $\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)|$ tends to infinity with n, in fact he proved that

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > \frac{1}{12} \log n. \tag{1}$$

Later FEJÉR [2] obtained a very simple proof for (1). The problem of determining the n points for which $\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)|$ is minimal is unsolved up to the present. BERNSTEIN [3] asserts that for every $\varepsilon > 0$, if $n > n_0$,

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > (1-\varepsilon) \frac{2}{\pi} \log n. \tag{2}$$

BERNSTEIN in his important paper proved (2) in full detail for trigonometric interpolation. He states that (2) for interpolation in $(-1, +1)$ is a simple consequence of this result. I was not able to reconstruct the proof. However, we proved with TURÁN [4] that (2) is true, even if the right side is replaced by $\frac{2}{\pi} \log n - c \log \log n$; here and throughout this paper $c, c_1, c_2, \ldots$ will denote positive absolute constants.

The main task of the present paper is the proof of the following

THEOREM 1. *Let* $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$. *Then*

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > \frac{2}{\pi} \log n - c_1.$$

This result can not be improved very much, since it is known that for the roots of the n^{th} Chebyshev polynomial $T_n(x)$

$$\max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}|l_k(x)|<\frac{2}{\pi}\log n+c_2.$$

In fact, it is known and can be shown by a simple calculation that if $y_1<y_2<\cdots<y_n$ are the roots of $T_n(x)$, then

$$\frac{2}{\pi}\log n-c_2<\max_{y_i<x<y_{i+1}}\sum_{k=1}^{n}|l_k(x)|<\frac{2}{\pi}\log n+c_2.$$

Let $\begin{matrix} x_1^{(1)} \\ x_1^{(2)}\ x_2^{(2)} \\ \cdots\cdots \end{matrix}$ be a triangular matrix called point group in the theory of interpolation, $-1\leqq x_1^{(n)}<x_2^{(n)}<\cdots<x_n^{(n)}\leqq 1$. BERNSTEIN [3] proved that there exists an x_0 $(-1<x_0<1)$ so that

$$\overline{\lim}\sum_{k=1}^{n}|l_k(x_0)|=\infty.$$

More precisely, he proved that for every fixed $-1\leqq a<b\leqq 1$

$$\max_{a<x<b}\sum_{k=1}^{n}|l_k(x)|>\left(\frac{1}{4}-\varepsilon\right)\log n \tag{3}$$

for $n>n_0(\varepsilon,a,b)$. I think that in (3) $\frac{1}{4}$ can be replaced by $\frac{2}{\pi}$, but I have not been able to prove this.

In my paper [5] I stated that I can prove that there exists an x_0 so that for infinitely many n

$$\sum_{k=1}^{n}|l_k(x_0)|>\frac{2}{\pi}\log n-c. \tag{4}$$

(4) is quite possibly true, but unfortunately I am very far from being able to prove it.

To prove our Theorem we first need some lemmas.

LEMMA 1. *Let* $\cos\theta_i=y_i$ $(1\leqq i\leqq n)$ *be the roots of the* n^{th} *Chebyshev polynomial* $T_n(x)$. *Then for every* $-1\leqq x\leqq 1$ *and* $t>c_3$

$$\frac{1}{n}\sum\nolimits_t\left|\frac{(1-y_i^2)^{\frac{1}{2}}}{x-y_i}\right|>\frac{2}{\pi}\log n-c_4\log t,$$

where Σ_t *denotes that the summation is extended only over those* y_i*'s for which* $|\theta-\theta_i|>t\pi/n$, $\cos\theta=x$.

The proof of Lemma 1 is by simple computation and is left to the reader.

$\cos\vartheta_0 = x_0$ will denote the point in $(-1, +1)$ where $|\omega_n(x)|$ assumes its absolute maximum. $\bar{I}_t$ will denote the intersection with $(0, \pi)$ of an interval of length $t\pi/n$, one endpoint of which is ϑ_0, I_t will be the interval in $(-1, +1)$ obtained from $\bar{I}_t$ by the mapping $\cos\vartheta = x$. There are two intervals I_t, one to the right, the other to the left of x_0.

LEMMA 2. *Assume that there exists a $t > c_3$ so that for every $t' \geqq t$ every interval $I_{t'}$ contains more than $t'\left(1 - \frac{1}{(\log t')^2}\right)$ x_i's. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i}\right| > \frac{2}{\pi}\log n - c_5 \log t.$$

The term $|(1-x_i^2)^{\frac{1}{2}}|$ is really understood to mean $\max\left(|(1-x_i^2)^{\frac{1}{2}}|, \frac{1}{n}\right)$, to save space I will always replace this by $|(1-x_i^2)^{\frac{1}{2}}|$.

Let y_i be such that there are k y's in the interval (x_0, y_i), and let $x_{i'}$ be such that there are k x's in $(x_0, x_{i'})$. Clearly $\theta_i - \theta_0 = \frac{k\pi + O(1)}{n}$ and by our condition on the x's

$$\vartheta_{i'} - \vartheta_0 < \frac{k\pi}{n} + \frac{c_6 k\pi}{n(\log k)^2} + \frac{t\pi}{n} < \frac{k\pi}{n} + \frac{c_7 k\pi}{n(\log k)^2} \tag{5}$$

for $k > t^2$. From (5) we obtain by a simple trigonometrical calculation for $k > t^2$

$$\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i}\right| - \left|\frac{(1-y_i^2)^{\frac{1}{2}}}{y_0 - y_i}\right| > -\frac{c_8}{k(\log k)^2}. \tag{6}$$

Lemma 2 immediately follows from (6) and Lemma 1.

LEMMA 3. *Assume that the x_i's and x_0 have the same properties as in Lemma 2 and the further property that for some $t' > t$ there is an $I_{t'}$ which contains more than t'^3 x_i's. Then if $t > c_3$,*

$$\sum = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i}\right| > \frac{2}{\pi}\log n.$$

Let t^* be the greatest t' for which an interval I_{t^*} contains t^{*3} x's. Write

$$\sum = \sum{}' + \sum_{t^*}$$

where in Σ' $|\vartheta_0 - \vartheta_i| \leqq \frac{t^*\pi}{n}$ and in Σ_{t^*} $|\vartheta_i - \vartheta_0| > \frac{t^*\pi}{n}$.

As in the proof of Lemma 2 we can show that

(7) $$\sum_{t^*} > \frac{2}{\pi} \log n - c_9 \log t^*.$$

A simple trigonometrical computation shows that for the x_i's in Σ' $\left(\text{here } |\vartheta_i - \vartheta_0| \leqq \frac{t^* \pi}{n} \text{ and by our remark } |(1-x_i^2)^{\frac{1}{2}}| \geqq \frac{1}{n}\right)$

$$\frac{1}{n} \left| \frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i} \right| > \frac{c_{10}}{t^{*2}}.$$

Thus, since there are at least t^{*3} summands in Σ', we have

(8) $$\Sigma' > ct'.$$

(7) and (8) imply Lemma 3 for sufficiently large $t > c_3$.

LEMMA 4. *Let* $\cos \lambda_0 = x_0$ *be any point in* $(-1, +1)$. *There exists a polynomial* $F_r(x)$ *of degree* r *for which* $F_r(z_0) = 1$ *and*

$$\left| F_r \left[\cos \left(\lambda_0 + s \frac{\pi}{n} \right) \right] \right| < \frac{c_{11}}{|s|}$$

if $\lambda_0 + \frac{s\pi}{n}$ *is in* $(0, \pi)$.

Lemma 4 is well known [6].

LEMMA 5. *Let* $g_m(x)$ *be any polynomial of degree* m, *assume that it assumes its absolute maximum in* $(-1, +1)$ *at* $\cos \lambda_0 = z_0$. *Then if* $\cos \lambda_i = z_i$ *is any root of* $g_m(x)$, *we have*

$$|\lambda_0 - \lambda_i| \geqq \frac{\pi}{2m},$$

equality only holds if $g_m(x) = T_m(x)$.

This is a theorem of M. RIESZ [7].

LEMMA 6. *Assume that the* x_i'*s are such that there is a* $t > c_{12}$ *so that at least one of the intervals* I_t *contains fewer than* $t\left(1 - \frac{1}{(\log t)^2}\right)$ x_i'*s, and that for* $t' \geqq t$ *the intervals* $I_{t'}$ *contain not more than* t'^3 x_i'*s. Then*

$$\max_{x_k \subset\!\!\!= I_t} \max_{x \text{ in } J_t} |l_k(x)| > t$$

where by J_t $(J_t \subset I_t)$ *we denote the interval*

$$J_t = \left\} \cos \left(\vartheta_0 + \frac{t\pi}{n(\log t)^3} \right), \cos \left(\vartheta_0 + \frac{t\pi}{n} - \frac{t\pi}{n(\log t)^3} \right) \right\{.$$

Lemma 6 is very far from being best-possible, the conditions could be weakened and the conclusions strengthened, but it will suffice for our purpose in its present form. The proof of Lemma 6 is the most difficult part of the paper [8].

Let $g(x)$ be a polynomial whose roots in I_t coincide with those of $\omega_n(x)=\prod_{i=1}^{n}(x-x_i)$ and outside of J_t they coincide with the roots of the m^{th} Chebyshev polynomial $T_m(x)$, $m=\left[n\left(1-\frac{1}{(\log t)^3}\right)\right]$. By our assumptions the degree of $g(x)$ is less than

$$(9) \qquad t-\frac{t}{(\log t)^2}+m-t\left(1-\frac{2}{(\log t)^3}\right)<m$$

for $t>c_{12}$ (i. e. the degree of $g_m(x)$ equals the number of x_i in I_t plus m minus the number of roots of $T_m(x)$ in J_t).

From Lemma 5 and (9) it follows that $g(x)$ must assume its absolute maximum for $(-1,+1)$ in J_t at the point $\cos\lambda_0=z_0$, say.

Denote by $I_t^{(l)}$ $(l=1,2,\ldots)$ the intersection with $(-1,+1)$ of the intervals

$$(10) \qquad \left\}\cos\left(\vartheta_0+\frac{2^{l-1}t\pi}{n}\right),\ \cos\left(\vartheta_0+\frac{2^{l}t\pi}{n}\right)\right\{$$

and

$$\left\}\cos\left(\vartheta_0-\frac{(2^{l}-1)t\pi}{n}\right),\ \cos\left(\vartheta_0-\frac{(2^{l-1}-1)t\pi}{n}\right)\right\{.$$

We now apply Lemma 4 with $r=\left[\frac{n(\log t)^4}{t}\right]$. Since $\cos\lambda_0=z_0$ is in J_t and the distance of the endpoints of $\bar{J}_t$ from the endpoints of $\bar{I}_t$ (in ϑ) is $\frac{t\pi}{n(\log t)^3}$, we obtain from Lemma 4 by a simple computation that for the x's in $I_t^{(l)}$

$$(11) \qquad |F_r(x)|<\frac{1}{2^l}$$

for sufficiently large t (i. e. the s in Lemma 4 is for $l=1$ not less than $\log t$ [z_0 is in J_t] and for $l>1$ it is not less than $2^{l-1}\log t$).

Consider now

$$(12) \qquad G(x)=Ag(x)\,(F_r(x))^{[t/(\log t)^8]}$$

where A is chosen so that $G(z_0)=1$. The degree of $G(x)$ is not greater than $\left(m=\left[n\left(1-\frac{t}{(\log t)^3}\right)\right]\right)$

$$n-\frac{n}{(\log t)^3}+\frac{t}{(\log t)^8}\frac{n(\log t)^4}{t}<n.$$

Thus by the Lagrange interpolation formula (taken on $x_1, x_2, \dots, x_n$) we have by (12)

$$1=G(z_0)=\sum_{i=1}^{n} G(x_i)\, l_i(z_0). \tag{13}$$

For the x_i's in I_t $G(x_i)=0$. Thus we can write (13) as

$$1=\sum_{l=1}^{\infty}\sum\nolimits^{(l)} G(x_i)\, l_i(z_0) \tag{14}$$

where in $\Sigma^{(l)}$ the summation is extended over the x_i's in $I_t^{(l)}$. The summation in (14) clearly has to be extended only over a finite number of l's.

Since $|g(z_0)|\geqq|g(x)|$ for $-1\leqq x\leqq 1$ and $F_r(z_0)=1$, we obtain from (11) and (12) that

$$|G(x_i)|<\left(\frac{1}{2^l}\right)^{[t/(\log t)^8]} \quad \text{for the } x_i\text{'s in } I_t^{(l)}. \tag{15}$$

Assume now that our Lemma is false. Then for all $i\not\subset I_t$

$$|l_i(z_0)|\leqq t. \tag{16}$$

Further by the assumptions of our Lemma the number of the x_i's in $I_t^{(l)}$ is not greater than $2^{3l+1}t^3$ (since $I_t^{(l)}$ is contained in the union of the two intervals $I_{2^l t}$). Thus, finally, we obtain from (14), (15) and (16) that

$$1<t^4\sum_{l=1}^{\infty} 2^{3l+1}\left(\frac{1}{2^l}\right)^{[t/(\log t)^8]}. \tag{17}$$

The terms of the series (17) drop faster than a geometric series of quotient $\frac{1}{2}$, thus (17) implies

$$1<32t^4\left(\frac{1}{2}\right)^{[t/(\log t)^8]}$$

which is clearly false for $t>c_{12}$. This contradiction proves the Lemma.

Now we are ready to prove our Theorem. In fact, we shall show that if x_0 is the place in $(-1,+1)$ where $\omega_n(x)$ assumes its absolute maximum, then

$$\sum_{k=1}^{n}|l_k(x_0)|>\frac{2}{\pi}\log n-c_1 \tag{18}$$

for sufficiently large c_1. We can clearly assume $\omega_n(x_0)=1$ (replacing $\omega_n(x)$ by $c\omega_n(x)$), and thus by the classical theorem of Bernstein

$$(19)\qquad |\omega_n'(x_k)| \leqq \min\left(n^2, \frac{n}{|1-x_k^2|^{\frac{1}{2}}}\right).$$

Thus from (19)

$$(20)\qquad \sum_{k=1}^{n} |l_k(x_0)| \geqq \frac{1}{n}\sum_{k=1}^{n}\left|\frac{(1-x_k^2)^{\frac{1}{2}}}{x_0-x_k}\right|.$$

Let the constant c_{12} be sufficiently large. If for every $t>c_{12}$ every I_t contains more than $t\left(1-\frac{1}{(\log t)^2}\right)$ x's, then our Theorem follows from (20) and Lemma 2. Assume next that there exists a $t>c_{12}$ for which I_t contains not more than $t\left(1-\frac{1}{(\log t)^2}\right)$ x's, and let t_0 be the largest such t. Assume first that there exists a $t'\geqq t_0$ for which $I_{t'}$ contains more than t'^3 x's, then our Theorem follows from (20) and Lemma 3. If no such t' exists, consider the largest interval I_{t_0} which contains not more than $t_0\left(1-\frac{1}{(\log t_0)^2}\right)$ x_k's. By Lemma 6 there is an x_i not in I_{t_0} so that for a certain z_0 in J_{t_0}

$$(21)\qquad |l_i(z_0)|>t_0.$$

Now since z_0 is in J_{t_0} ($\cos\lambda_0=z_0$, $\cos\vartheta_0=x_0$, $\cos\vartheta_i=x_i$, $x_i\not\subset I_{t_0}$),

$$(22)\qquad |\vartheta_i-\vartheta_0| \leqq (\log t_0)^3\,|\vartheta_i-\lambda_0|.$$

Thus from (22) by a simple computation

$$(23)\qquad |x_i-x_0|<(\log t_0)^6\,|x_i-z_0|.$$

From (23), (21) and $|\omega_n(x_0)|\geqq|\omega_n(z_0)|$ we have

$$(24)\qquad |l_i(x_0)|>\frac{t_0}{(\log t_0)^6}.$$

From Lemma 2 we have

$$(25)\qquad \frac{1}{n}\sum_{k=1}^{n}{}'\left|\frac{(1-x_k^2)^{\frac{1}{2}}}{x_0-x_k}\right| > \frac{2}{\pi}\log n - c_{13}\log t_0$$

where the dash indicates that $k=i$ is omitted. (25) holds, since a simple computation shows from Lemma 5 that

$$\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0-x_i}\right| < c_{14}n.$$

Thus, finally, from (20), (24) and (25) we have

$$\sum_{k=1}^{n} |l_k(x_0)| \geqq \frac{1}{n} \sum_{k=1}^{n}{}' \left| \frac{(1-x_k^2)^{\frac{1}{2}}}{x_0-x_k} \right| + \tag{26}$$

$$+ |l_i(x_0)| > \frac{2}{\pi} \log n - c_{13} \log t_0 + \frac{t_0}{(\log t_0)^6} > \frac{2}{\pi} \log n$$

if t is sufficiently large ($t > c_{13}$, say). Thus the proof of Theorem 1 is complete.

It would have been possible to organize the proof differently, since it can be shown that I_t can never contain more than t^3 x_i's. In fact, we have the following

THEOREM 2. *Let* $\omega_n(x) = \prod_{i=1}^{n}(x-x_i)$ *(we do not assume that the* x_i*'s are in* $(-1, +1)$*). Assume that* $\omega_n(x)$ *assumes its absolute maximum in* $(-1, +1)$ *at* $\cos \vartheta_0 = x_0$. *Then every interval* I_t *contains at most* $c_{14}t$ *of the* x_i*'s.*

We do not give the proof of Theorem 2. The best value of c_{14} is not known. Perhaps $c_{14} = 2$.

The problem of determining the points $-1 \leqq x_1 < \cdots < x_n \leqq 1$ for which

$$\int_{-1}^{+1} \sum_{k=1}^{n} |l_k(x)|\, dx$$

is a minimum is unsolved, and so far as I know has not yet been considered. I believe that to every $\varepsilon > 0$ there exists an n_0 so that for $n > n_0$

$$\int_{-1}^{+1} \sum_{k=1}^{n} |l_k(x)|\, dx > (1-\varepsilon) \int_{-1}^{+1} \sum_{k=1}^{n} |L_k(x)|\, dx \tag{27}$$

where $L_k(x) = \dfrac{T_n(x)}{T_n'(y_k)\,(x-y_k)}$ are the fundamental functions of the Lagrange interpolation taken at the roots $y_1, y_2, \ldots, y_n$ of the n^{th} Chebyshev polynomial. I have not been able to prove (27), but I can prove the following weaker

THEOREM 3. *There exists a constant* c_{15} *so that for every* $-1 \leqq x_1 < < x_2 < \cdots < x_n \leqq 1$ *we have*

$$\int_{-1}^{+1} \sum_{k=1}^{n} |l_k(x)|\, dx > c_{15} \log n. \tag{28}$$

In fact, to every ε there exists a δ so that the number of indices $1 \leqq k \leqq n$, for which

$$\int_{-1}^{+1} |l_k(x)|\, dx < \frac{\delta \log n}{n}, \tag{29}$$

is less than εn, and the number of k's, for which $\int_{-1}^{+1} |l_k(x)|\, dx > \frac{c_{16}}{n}$ is less than $c_{17} \frac{n}{\log n}$.

We do not give the proof of Theorem 3, it can be obtained by using the methods of my paper [5].

As far as I know the problem of determining the sequence $-1 \leqq x_1 < < x_2 < \cdots < x_n \leqq 1$ for which

$$\int_{-1}^{+1} \sum_{k=1}^{n} l_k^2(x)\, dx \tag{30}$$

is minimal has not been considered. It is possible that the integral (30) is minimal if the x_i's are the roots of the integral of the Legendre polynomial. FEJÉR [9] proved that these are the only points for which

$$\sum_{k=1}^{n} l_k^2(x) \leqq 1 \quad \text{for} \quad -1 \leqq x \leqq 1.$$

THEOREM 4. *To every ε there exists an n_0 so that for every $n > n_0$ the integral* (30) *is greater than* $2 - \varepsilon$.

We only outline the idea of the proof. If the projections of the points $x_1, x_2, \ldots, x_n$ on the unit circle are not asymptotically uniformly distributed, then there exists a k so that [10]

$$\max_{-1 \leqq x \leqq 1} |l_k(x)| > (1+\delta)^n, \tag{31}$$

and from (31) by Markov's theorem

$$\int_{-1}^{+1} l_k^2(x)\, dx > \frac{(1+\delta)^{2n}}{8n^2} > 2$$

for $n > n_0$. Thus we can assume that the projections of the x_k's on the unit circle are asymptotically uniformly distributed. In this case we obtain our Theorem by showing that

$$\int_{-1}^{+1} \sum_{k=1}^{n} l_k^2(x)\, dx > (1-\varepsilon) \int_{-1}^{+1} \sum_{k=1}^{n} L_k^2(x)\, dx \tag{32}$$

where $L_k(x) = \frac{P_n(x)}{P'_n(z_k)(x-z_k)}$ $\left(P_n(x) = \prod_{k=1}^{n}(x-z_k)\right)$ is the n^{th} Legendre polynomial. The proof of (32) follows easily from the fact that

$$\int_{-1}^{+1} L_k^2(x)\,dx \leqq \int_{-1}^{+1} f_{n-1}^2(x)\,dx$$

where $f_{n-1}(x)$ is any polynomial of degree $\leqq n-1$ for which $f_{n-1}(z_k) = 1$, and by a simple computation. We suppress the details.

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[8] The proof of Lemma 6 is similar to the one used in P. ERDŐS, *Annals of Math.,* **43** (1942), pp. 59—64; see also P. ERDŐS and P. TURÁN, *ibid.,* **41** (1940), pp. 510—553.

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