ON SOME PROBLEMS INVOLVING INACCESSIBLE CARDINALS

PAUL ERDÖS AND ALFRED TARSKI
Haifa, Israel; *Berkeley, California*

Introduction[1]

Many problems in set theory and related domains are known which have the following features in common. Each of them can be formulated as the problem of determining all the infinite cardinals which have a given property P. It is known that the property P applies to all infinite cardinals which are not (strongly) inaccessible while it does not apply to the smallest inaccessible cardinal, $\aleph_0$. On the other hand, the problem remains open whether P applies to all inaccessible cardinals greater than $\aleph_0$, and in some cases it is not even known whether P applies to any such cardinal. (The meaning of most terms used in this introduction will be explained below.)

In the present paper we shall be concerned with four problems of the kind described; the corresponding properties will be denoted by P_1, P_2, P_3, and P_4. A cardinal λ is said to have the property P_1 if there is a set A with power λ which is simply ordered by a relation $\leqq$ in such a way that no subset of A with power λ is well ordered by the same relation $\leqq$ or by the converse relation $\geqq$. The property P_2 applies by definition to a cardinal λ if there is a complete graph on a set of power λ that can be divided in two subgraphs neither of which includes a complete subgraph on a set of power λ. P_3 applies to λ if in the set algebra of all subsets of a set of power λ every λ-complete prime ideal is principal. Finally, P_4 applies to λ if there is a λ-complete and λ-distributive Boolean algebra $\mathfrak{B}$ which is not isomorphic to any λ-complete set algebra. We do not attempt here to solve fully any of these problems, but we establish some implications among them; in fact, our main result is that, for every infinite cardinal λ, each property P_m with $m = 1, 2, 3$ implies P_{m+1}. The problem remains open whether any of these implications holds in the opposite direction as well. Various equivalent formulations of the properties P_3 and P_4 are known; some relevant equivalences will be explicitly formulated and established.

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In the last section of the paper we consider a further property of cardinals, the property Q. This property applies to a cardinal λ if there is a ramification system $\mathfrak{R}$ formed by a set A and a partial ordering relation $\leqq$ which satisfies the following conditions: (i) $\mathfrak{R}$ is of order λ; (ii) for every $\xi < \lambda$, the set of all elements of A of order ξ has power $< \lambda$; (iii) every subset of A which is well ordered by $\leqq$ has power $< \lambda$. It seems likely that Q will eventually prove to be an intermediate property between P_2 and P_3, i.e., to be implied by P_2 and to imply P_3 for every infinite λ. Actually we establish the second of these implications in its whole generality, while the first implication is shown to hold for all inaccessible λ's. In opposition to what is known about $P_1 - P_4$, the problem whether Q holds for all accessible cardinals has not yet been completely solved. In the domain of inaccessible cardinals Q turns out to be closely related to P_4, and in fact to be equivalent to a property R which is a specialized form of P_4; R is obtained from P_4 by imposing on the algebra $\mathfrak{B}$ involved the additional condition that $\mathfrak{B}$ is λ-generated by λ elements. We show that R implies Q for all inaccessible cardinals. The implication in the opposite direction is one of the results of the note [14], which also contains the discussion of another, quite different, property of cardinals that proves to be equivalent to Q. (The numbers in square brackets refer to the References at the end of the paper.)(2)

The present paper is to a large extent self-contained. For the convenience of the reader we outline the proofs of several results which can be found in the literature; we have here in mind the results which show that the properties $P_1 - P_4$ hold for all accessible cardinals. On the other hand, we only mention the well known results to the effect that none of these properties applies to the cardinal $\aleph_0$, as well as the recent results of one of the authors by which the properties P_3, P_4, Q, and R apply to a very comprehensive class of inaccessible cardinals $> \aleph_0$. At the end of the paper the reader will find bibliographical references to various problems, in several domains of mathematics, which are closely related to the problems discussed in this paper.

The material presented in this paper was reported on in the seminar in the foundations of set theory conducted by Andrzej Mostowski and Alfred Tarski at the University of California, Berkeley, during the academic year 1958–59. Certain proofs offered below appear in the

(2) The main results of this paper were stated without proof in Erdös-Tarski [3], pp. 326 ff. (see in particular p. 328, footnote 4). All the results which are not provided here with references to earlier publications should be regarded as joint results of both authors.

form in which they were reconstructed by some members of the seminar, Andrzej Ehrenfeucht, James D. Halpern, and Donald Monk, and may differ from those originally found by the authors. The authors wish to express their most sincere appreciation to Donald Monk for his great help in preparing this paper for publication. Some related results obtained by Monk during his work on the paper are included in [14].

Terminology and notation

We employ the usual set-theoretical terminology and symbolism. Thus, for example, $\subseteq, \subset, 0, \cup$, and $\bigcup$ denote respectively the relations of inclusion and proper inclusion, the empty set, and the operations of forming unions of two sets and of arbitrarily many sets. $A \sim B$ is the set-theoretic difference of A and B (which, in case $B \subseteq A$, is also referred to as the complement of B with respect to A). $S(A)$ is the set of all subsets of A. A symbolic expression of the form $\{x \mid \phi\}$ where ϕ is to be replaced by any formula containing x (as a free variable) denotes the set of all those x which satisfy this formula. $\{x\}$ is the set whose only element is x, $\{x, y\}$ is the unordered pair with the elements x and y, and $\langle x, y\rangle$ is the ordered pair with x as the first term and y as the second term. If f is a function and x is an element of the domain D of f, then the value of f at x is denoted by $f(x)$ or sometimes by f_x; thus f is the set of all ordered couples $\langle x, f(x)\rangle$ where $x \in D$. By A^B we denote the set of all functions on B to A, i.e., of all functions with domain B and with range included in A. Given a function F on a set I assuming sets F_i as values, we denote by $\mathsf{P}_{i \in I} F_i$ the Cartesian product of all sets F_i with $i \in I$, i.e., the set of all functions f on I such that $f_i \in F_i$ for every $i \in I$.

By a *graph* we understand any set of unordered pairs $\{x, y\}$ with $x \neq y$. Given any set A, the set of all unordered pairs $\{x, y\}$ such that $x \neq y$ and $x, y \in A$ is referred to as the *complete graph on A* and will be denoted by $G(A)$.

We assume that *ordinals* have been introduced in such a way that every ordinal coincides with the set of all smaller ordinals. In consequence, the intersection $\bigcap_{\xi \in A} \xi$ of all members of a non-empty set A of ordinals is again an ordinal, and in fact the smallest ordinal belonging to A. We put

$$\mu(A) = \bigcap_{\xi \in A} \xi$$

for every non-empty set of ordinals A, and in addition

$$\mu(0) = 0.$$

A function on an ordinal α is sometimes referred to as a *sequence of type* α or an *α-termed sequence.*

Cardinals are identified with those ordinals which are not of the same power as some smaller ordinal; in particular, the cardinal $\aleph_0$ is identified with the ordinal ω. We shall use small Greek letters $\alpha, \beta, \ldots, \xi, \zeta, \ldots$ to represent arbitrary ordinals, and specifically the letters κ, λ, ν, $\pi, \ldots$ to represent arbitrary cardinals. The power, or the cardinal, of a set A is denoted by $\kappa(A)$. The order relation between ordinals and the operations of addition and multiplication on cardinals are denoted in the usual manner. For each cardinal λ the symbol λ^+ denotes the least cardinal greater than λ. κ and λ being any cardinals, $\lambda^{\cdot\kappa}$ is the cardinal power with base λ and exponent κ; thus the cardinal $\lambda^{\cdot\kappa}$ is the power of the set λ^{κ}. An infinite cardinal λ is called *regular* if it cannot be represented as a sum of less than λ cardinals each of which is less than λ or, what amounts to the same, if $\sum_{\xi<\kappa}\nu_\xi < \lambda$ whenever $\kappa < \lambda$, and $\nu_\xi < \lambda$ for every $\xi < \kappa$; otherwise it is called *singular*. An infinite cardinal λ is said to be *strongly inaccessible* or simply *inaccessible* if it is regular and if $\kappa < \lambda$ always implies $2^{\cdot\kappa} < \lambda$; otherwise λ is said to be *accessible*. Several equivalent formulations of the definition of inaccessibility are known. For instance, each of the following conditions (*i*)–(*iii*) is necessary and sufficient for an infinite cardinal λ to be inaccessible: (*i*) $\prod_{\xi<\kappa}\nu_\xi < \lambda$ whenever $\kappa < \lambda$, and $\nu_\xi < \lambda$ for every $\xi < \kappa$; (*ii*) $2^{\cdot\kappa} < \lambda = \lambda^{\cdot\kappa}$ for every cardinal κ such that $0 < \kappa < \lambda$; (*iii*) $\lambda = \lambda^{\cdot 2^{\cdot\kappa}}$ for every $\kappa < \lambda$.(3) The smallest inaccessible cardinal is obviously ω. It is well known that the existence of inaccessible cardinals $> \omega$ cannot be established on the basis of familiar axiomatic systems of set theory such as the systems of Zermelo-Fraenkel or Bernays.

The notion of a *Boolean algebra* and those of an *atom*, an *ideal*, a *principal ideal*, a *prime ideal*, and a *subalgebra* of a Boolean algebra are assumed to be familiar to the reader; the same applies to the notion of a quotient algebra $\mathfrak{A}/I$ (where $\mathfrak{A}$ is a Boolean algebra and I one of its ideals). We use the symbols $+$, $\cdot$, and $-$ to denote the fundamental Boolean-algebraic operations of addition (join), multiplication (meet) and complementation, respectively; $\leqq$ denotes the Boolean-algebraic inclusion, while $\sum$ and $\prod$ denote the usual infinite generalizations

(3) For various equivalent definitions of inaccessible cardinals see [30] where, in particular, conditions (*i*) and (*ii*) are discussed. The fact that (*iii*) is a necessary and sufficient condition for λ to be inaccessible was stated in [22], p. 231.

of $+$ and $\cdot$. Whenever a Boolean algebra is represented by a capital German letter, say, $\mathfrak{A}$, we stipulate that the set of all its elements is denoted by the corresponding capital italic letter, A; thus $\mathfrak{A}$ is a system (ordered quadruple) formed by the set A and the fundamental operations $+$, $\cdot$, and $-$.

An ideal I in a Boolean algebra $\mathfrak{A}$ is called *λ-complete* if $\sum_{x \in X} x$ exists and is in I whenever $X \subseteq I$ and $\kappa(X) < \lambda$; hence every ideal is ω-complete. In case $I = A$, the Boolean algebra itself is called λ-complete. The algebra $\mathfrak{A}$, or an ideal I in $\mathfrak{A}$, is called *absolutely complete* or (simply) *complete* if it is λ-complete for every cardinal λ. A λ-complete Boolean algebra is said to be *λ-generated* by a set $X \subseteq A$ if A does not include properly any set B which is closed under the fundamental operations of $\mathfrak{A}$ and under the addition $\sum$ restricted to sets $Y \subseteq B$ with power $< \lambda$.

A Boolean algebra $\mathfrak{A}$ is said to be *λ-distributive* if it is λ-complete and satisfies the following condition: Let I be any set with $\kappa(I) < \lambda$, J be any function which assigns a set J_i with $\kappa(J_i) < \lambda$ to every element $i \in I$, and K be the Cartesian product $\mathsf{P}_{i \in I} J_i$; let x be a function which assigns an element $x_{i,j}$ to every ordered couple $\langle i,j \rangle$ with $i \in I$ and $j \in J_i$; under these assumptions $\sum_{f \in K} \prod_{i \in I} x_{i,f(i)}$ exists and equals $\prod_{i \in I} \sum_{j \in J_i} x_{i,j}$. (In the present paper we do not extend the notion of λ-distributivity to Boolean algebras which are not λ-complete—although such an extension often proves useful.) A Boolean algebra is called *completely distributive* if it is λ-distributive for every cardinal λ.

A *set algebra* is a Boolean algebra $\mathfrak{A}$ in which A is a family of subsets of certain set U (called the unit set of the algebra), and the fundamental operations $+$, $\cdot$, and $-$ coincide with the set-theoretic operations $\cup$, $\cap$, and complementation with respect to U. A set algebra $\mathfrak{A}$ is referred to as a *λ-complete set algebra* if $\mathfrak{A}$ is λ-complete as a Boolean algebra and if the Boolean-algebraic sum $\sum_{X \in F} X$ coincides with the set-theoretic union $\bigcup_{X \in F} X$ for every family $F \subseteq A$ with $\kappa(F) < \lambda$. Notice that a set algebra may be λ-complete as a Boolean algebra without being a λ-complete set algebra in our sense. $\mathfrak{A}$ is an (*absolutely*) *complete set algebra* if it is a λ-complete set algebra for every cardinal λ; thus, e.g., the set algebra $\mathfrak{A}$ in which A consists of all subsets of the unit set U is a complete set algebra.

By a *measure* on a set A we understand a function m on $S(A)$ to the set of non-negative real numbers such that $m(X \cup Y) = m(X) + m(Y)$ for any two disjoint subsets X, Y of A and in addition $m(A) = 1$. The measure m is said to be *trivial* if $m(\{x\}) \neq 0$ for some $x \in A$, and otherwise it is called *non-trivial*. It is called *λ-additive* if for every family $G \subseteq S(A)$ of mutually exclusive sets with $\kappa(G) < \lambda$, the family $H = \{X \mid X \in G \text{ and } m(X) \neq 0\}$ is at most denumerable, and we have

$$m\left(\bigcup_{X \in G} X\right) = \sum_{Y \in H} m(Y).$$

The measure m is said to be *two-valued* if its range contains only the numbers 0 and 1.

A function F on $S(A)$ to any family of sets is called respectively a *λ-additive*, or *λ-multiplicative*, *set function on A* if

$$F\left(\bigcup_{X \in G} X\right) = \bigcup_{X \in G} F(X), \text{ or } F\left(\bigcap_{X \in G} X\right) = \bigcap_{X \in G} F(X),$$

for every family $G \subseteq S(A)$ such that $0 < \kappa(G) < \lambda$. F is called *completely additive*, or *completely multiplicative*, if it is λ-additive, or λ-multiplicative, for every cardinal λ.

We assume to be known under what conditions a set A is said to be partially ordered, simply ordered, or well ordered by a binary relation (i.e., a set of ordered couples) $\preccurlyeq$. A *ramification system* is an ordered pair $\langle A, \preccurlyeq \rangle$ such that the set A is partially ordered by $\preccurlyeq$ and, for every $x \in A$, the set $P(x) = \{y \mid y \in A, y \preccurlyeq x, \text{ and } y \neq x\}$ is well ordered by $\preccurlyeq$; the type of the well-ordering of $P(x)$ is called the *order of* x, and the least ordinal greater than the orders of all elements of A is called the *order of the whole ramification system*.

1. Properties P_1 and P_2

A cardinal λ is said to have the *property* P_1 if there is a relation $\preccurlyeq$ which simply orders λ in such a way that every subset of λ which is well ordered by $\preccurlyeq$ or by the converse relation $\succcurlyeq$ has power $< \lambda$. We obviously obtain an equivalent formulation of this property if, instead of simple orderings on λ, we consider simple orderings on any given set of power λ. An analogous remark applies to several other formulations in this paper.

It is easy to see that *ω does not have the property P_1*. On the other hand we have

THEOREM 1.1. *Every accessible cardinal λ has the property P_1.*(4)

PROOF. We distinguish two cases.

(4) Theorem 1.1 is essentially due to Hausdorff; he does not state this theorem explicitly, but the theorem can easily be deduced from his results in [6], pp. 477 f. In the proof of Theorem 1.1 given below we follow, in part (i), the argument of Sierpiński in [19] (the proof of Lemma II).

Case (i): λ is regular. Then, by the definition of accessible cardinals, there is a cardinal ν such that

$$\nu < \lambda \leqq 2^{\cdot\nu}.$$

Consequently there is a set $A \subseteq 2^{\nu}$ with power λ and, instead of simple orderings on λ, we can consider simple orderings on A.

We obtain a relation $\preccurlyeq$ which simply orders the set 2^{ν}, and hence also the set A, by stipultating that, for any $f, g \in 2^{\nu}$, $f \preccurlyeq g$ if and only if either $f_{\xi} = g_{\xi}$ for every ordinal $\xi < \nu$, or else there is an ordinal $\alpha < \nu$ such that $f_{\alpha} < g_{\alpha}$ while $f_{\xi} = g_{\xi}$ for every $\xi < \alpha$. (The resulting simple ordering of 2^{ν} is the so-called *lexicographic ordering.*) Consider any subset B of 2^{ν} which is well ordered either by $\preccurlyeq$ or by $\succcurlyeq$; since the argument in both cases is essentially the same, we assume that B is well ordered by $\preccurlyeq$. We wish to show that $\kappa(B) \leqq \nu$ and hence *a fortiori* $\kappa(B) < \lambda$.

Suppose, on the contrary, that $\kappa(B) > \nu$. Then B contains a subset C with the order type ν^{+}. For each $f \in C$ let $\phi(f)$ be the smallest ordinal $\alpha < \nu$ such that, for some $g \in C$, we have $f_{\alpha} < g_{\alpha}$, and $f_{\xi} = g_{\xi}$ for every $\xi < \alpha$; for each $\alpha < \nu$ let

(1) $$D_{\alpha} = \{f \mid f \in C \text{ and } \phi(f) = \alpha\}.$$

Then

$$C = \bigcup\nolimits_{\alpha < \nu} D_{\alpha};$$

since, in addition, any two sets D_{α}, D_{β} with $\alpha \neq \beta$ are disjoint, we conclude that

(2) $$\kappa(C) = \nu^{+} = \sum\nolimits_{\alpha < \nu} \kappa(D_{\alpha}).$$

We shall show that

(3) $$\kappa(D_{\alpha}) \leqq \nu \text{ for every } \alpha < \nu.$$

This is obvious in case $D_{\alpha} = 0$. Otherwise, let $f \in D_{\alpha}$. By (1) and the definition of ϕ we see that there exists an element $g \in C$ such that

(4) $$f_{\alpha} < g_{\alpha}, \text{ and } f_{\xi} = g_{\xi} \text{ for every } \xi < \alpha.$$

Consider now an arbitrary element $h \in D_\alpha$. Since

$$\phi(f) = \phi(h) = \alpha,$$

we must have $f_\xi = h_\xi$ for every $\xi < \alpha$ whence, by (4),

$$h_\xi = g_\xi \text{ for every } \xi < \alpha. \tag{5}$$

Moreover, we see (as before, in the case of f) that

$$h_\alpha < k_\alpha \tag{6}$$

for some $k \in C$. Remembering that $f, g, h, k, \in 2^\nu$, we conclude from (4) and (6):

$$f_\alpha = h_\alpha = 0, \quad g_\alpha = k_\alpha = 1,$$

and hence

$$h_\alpha < g_\alpha. \tag{7}$$

By (5) and (7) we have $h \preccurlyeq g$ (in the lexicographic order established by $\preccurlyeq$ on 2^ν). Thus we have proved that there exists an element $g \in C$ such that $h \preccurlyeq g$ for every $h \in D_\alpha$; since the order type of C is ν^+, this immediately implies (3).

From (2) and (3) we obtain at once a contradiction:

$$\nu^+ \leqq \nu \cdot \nu = \nu.$$

Consequently, every subset of A which is well ordered by $\preccurlyeq$ has power $< \lambda$.

Case (ii): λ is singular. This means that λ can be represented as a sum of less than λ cardinals each of which is $< \lambda$. Consequently, there is an infinite cardinal ν and a sequence E satisfying the conditions:

$$\nu < \lambda, \ E \in [S(\lambda)]^\nu, \ \kappa(E_\xi) < \lambda \text{ for every } \xi < \nu; \tag{8}$$

$$\lambda = \bigcup_{\xi < \nu} E_\xi, \text{ and } E_\xi \cap E_\eta = 0 \text{ for any } \xi, \eta < \nu \text{ with } \xi \neq \eta. \tag{9}$$

We define a binary relation $\preccurlyeq$ between ordinals $< \lambda$ by stipulating as follows:

(10) $\alpha \preccurlyeq \beta$ if and only if either $\alpha, \beta \in E_\xi$ for some $\xi < \nu$ and $\alpha \leqq \beta$, or else $\alpha \in E_\xi$ and $\beta \in E_\eta$ for some ξ and η such that $\eta < \xi < \nu$.

From (9) and (10) we easily see that λ is simply ordered by $\preccurlyeq$. In view of (9) we have for every subset B of λ

$$B = \bigcup_{\xi < \nu} (B \cap E_\xi). \tag{11}$$

Consider a set $B \subseteq \lambda$ which is well ordered by $\preccurlyeq$. If the set

$$C = \{\xi \mid \xi < \nu \text{ and } B \cap E_\xi \neq 0\} \tag{12}$$

were infinite, we could choose an infinite sequence $\gamma \in C^\omega$ such that $\gamma_\eta < \gamma_{\eta+1}$ for every $\eta < \omega$; by (9), (10), and (12) we would then have

$$\mu(B \cap E\gamma_{\eta+1}) \prec \mu(B \cap E\gamma_\eta) \text{ [i.e., } \mu(B \cap E\gamma_{\eta+1}) \preccurlyeq \mu(B \cap E\gamma_\eta)$$
$$\text{and } \mu(B \cap E\gamma_{\eta+1}) \neq \mu(B \cap E\gamma_\eta)]$$

as well as

$$\mu(B \cap E\gamma_\eta) \in B$$

for every $\eta < \omega$. This clearly contradicts the assumption that B is well ordered by $\preccurlyeq$. Therefore the set C is finite. Since, by (11) and (12),

$$B = \bigcup_{\xi \in C} (B \cap E_\xi)$$

and since the sum of finitely many cardinals $< \lambda$ is always $< \lambda$, we conclude by (8) that $\kappa(B) < \lambda$.

Now consider a set $B \subseteq \lambda$ which is well ordered by $\succcurlyeq$. If, for some $\xi < \nu$, the set $B \cap E_\xi$ were infinite, we could choose a sequence $\beta \in (B \cap E_\xi)^\omega$ such that $\beta_\eta < \beta_{\eta+1}$ and hence, by (10), $\beta_\eta \prec \beta_{\eta+1}$ for every $\eta < \omega$. This is again a contradiction. Hence $\kappa(B \cap E_\xi) < \omega$ for every $\xi < \nu$. Consequently, by (8) and (11), $\kappa(B) \leqq \nu \cdot \omega = \nu < \lambda$, and the proof is complete.

A cardinal λ has the *property* P_2 if and only if the complete graph $G(\lambda)$ can be divided into two graphs G_1 and G_2 neither of which includes the complete graph on a set of power λ. We have

THEOREM 1.2. *If an infinite cardinal λ has the property P_1, then it also has the property P_2.*

PROOF. By hypothesis, there is a relation $\preccurlyeq$ which simply orders λ in such a way that every subset of λ which is well ordered by $\preccurlyeq$ or by $\succcurlyeq$ has power $< \lambda$. Let

(1) $G_1 = \{x \mid$ for some ξ and ζ, $x = \{\xi, \zeta\}$ and either $\xi < \zeta < \lambda$ and $\xi \preccurlyeq \zeta$, or else $\zeta < \xi < \lambda$ and $\zeta \preccurlyeq \xi\}$,

(2) $G_2 = \{x \mid$ for some ξ and ζ, $x = \{\xi, \zeta\}$ and either $\xi < \zeta < \lambda$ and $\zeta \preccurlyeq \xi$, or else $\zeta < \xi < \lambda$ and $\xi \preccurlyeq \zeta$l$\}$.

Obviously,

$$G_1 \cup G_2 = G(\lambda).$$

Let A be any subset of λ. If $G(A) \subseteq G_1$, then, by (1), A is well ordered by $\preccurlyeq$; if $G(A) \subseteq G_2$, then, by (2), A is well ordered by $\succcurlyeq$. In either case $\kappa(A) < \lambda$. Thus neither G_1 nor G_2 includes the complete graph on a set of power λ, and this completes the proof.

A result known as Ramsey's theorem (see [16], Theorem A) shows that *ω fails to have the property P_2*. On the other hand, from Theorems 1.1 and 1.2 we derive at once

THEOREM 1.3. *Every accessible cardinal λ has the property P_2.*(5)

The problem is open whether all, or at least some, inaccessible cardinals greater than ω have the property P_1 or P_2.

2. Property P_3

In this and the following sections we shall make use of some elementary facts concerning principal and prime ideals in Boolean algebras. It is obvious, e.g., that in a λ-complete Boolean algebra (where λ is an arbitrary cardinal) every principal ideal is λ-complete; in a complete Boolean algebra principal ideals coincide with complete ideals. A prime ideal I in an arbitrary Boolean algebra is principal if and only if there is an atom of the algebra which does not belong to I. If I is an ideal in a λ-complete algebra $\mathfrak{A}$ $(\lambda \geqq \omega)$ and $I \neq A$, then I is both prime and λ-complete if and only if it satisfies the following condition: $X \cap I \neq 0$ whenever $X \subseteq A$, $0 < \kappa(X) < \lambda$, and $\prod_{y \in X} y \in I$.

(5) This result is due to Erdös; its original proof was independent of Theorem 1.2. See [1].

A cardinal λ is said to have the *property* P_3 if every λ-complete prime ideal in the set algebra formed by all the subsets of λ is principal. The main result of this section is:

THEOREM 2.1. *Every infinite cardinal λ which has the property P_2 also has the property P_3.*

PROOF. Since λ has the property P_2, the complete graph on λ can be divided into two sets, G_1 and G_2, such that neither G_1 nor G_2 includes a complete graph on a set of power λ. Let I be any λ-complete prime ideal in the set algebra of all subsets of λ. We wish to show that I is principal.

We define two functions $F_1, F_2 \in [S(\lambda)]^\lambda$ by letting

(1) $F_i(\alpha) = \{\beta \mid \beta < \lambda$ and $\{\alpha,\beta\} \in G_i\}$ for $i = 1, 2$ and for every $\alpha < \lambda$.

Suppose there is a set X with the following property:

(2) $X \in S(\lambda) \sim I$, and for every $Y \in S(X) \sim I$ there is an $\alpha \in Y$ such that $Y \cap F_1(\alpha) \notin I$.

By means of double recursion we define two sequences $H \in [S(\lambda)]^\lambda$ and $\phi \in \lambda^\lambda$ satisfying the conditions:

(3) $H_0 = X$, $H_{\alpha+1} = H_\alpha \cap F_1(\phi_\alpha)$ for every $\alpha < \lambda$, $H_\alpha = \bigcap_{\xi<\alpha} H_\xi$ for every limit ordinal $\alpha < \lambda$;

$$\phi_\alpha = \mu(\{\xi \mid \xi \in H_\alpha \text{ and } H_\alpha \cap F_1(\xi) \notin I). \tag{4}$$

We show that for every $\alpha < \lambda$ the following formulas hold:

$$H_\alpha \subseteq \bigcap{}_{\xi<\alpha} H_\xi,\ H_\alpha \in S(X) \sim I, \tag{5}$$

$$\phi_\alpha \in H_\alpha,\ H_\alpha \cap F_1(\phi_\alpha) \notin I. \tag{6}$$

These formulas are obtained by transfinite induction based upon (3) and (4): we make essential use of (2) and of the fact that I is a λ-complete prime ideal.

Consider any two different ordinals $\alpha,\beta < \lambda$. If $\alpha < \beta$, we have, by (3), (5), and (6),

$$\phi_\beta \in H_\beta \subseteq H_{\alpha+1} \subseteq F_1(\phi_\alpha)$$

whence by (1)

$$\{\phi_\alpha, \phi_\beta\} \in G_1.$$

By symmetry, we obtain the same conclusion if $\beta < \alpha$. Thus $\phi_\alpha \neq \phi_\beta$ whenever $\alpha \neq \beta$; therefore the range of ϕ, i.e., the set

$$A = \{\phi_\xi \mid \xi < \lambda\},$$

is of power λ, and G_1 includes $G(A)$. This contradicts, however, our original assumption. Hence no set X satisfies (2); in other words,

(7) for every $X \in S(\lambda) \sim I$ there is a $Y \in S(X) \sim I$ such that $Y \cap F_1(\alpha) \in I$ for all $\alpha \in Y$.

By an entirely symmetric argument we obtain:

(8) for every $Y \in S(\lambda) \sim I$ there is a $Z \in S(Y) \sim I$ such that $Z \cap F_2(\alpha) \in I$ for all $\alpha \in Z$.

In accordance with (7), we can find a set $Y \in S(\lambda) \sim I$ such that $Y \cap F_1(\alpha) \in I$ for all $\alpha \in Y$. Hence, by (8), we can choose a set $Z \in S(Y) \sim I$ such that $Z \cap F_2(\alpha) \in I$ for all $\alpha \in Z$. Since $Z \notin I$, we have $Z \neq 0$. Let α be any element of Z. Then $Z \cap F_2(\alpha) \in I$ and also $Y \cap F_1(\alpha) \in I$, whence $Z \cap F_1(\alpha) \in I$. Since, by (1),

$$Z = [Z \cap F_1(\alpha)] \cup [Z \cap F_2(\alpha)] \cup \{\alpha\} \notin I,$$

we infer that the set $\{\alpha\}$ (which is an atom of our set algebra) does not belong to I and that consequently the prime ideal I is principal. Thus λ has the property P_3, and the proof is complete.

It is known that *ω does not have the property P_3*; this is a particular case of a general result established in [31], p. 49. On the other hand, as an immediate consequence of Theorems 1.3 and 2.1 we obtain

THEOREM 2.2. *Every accessible cardinal λ has the property P_3.*(6)

(6) This is essentially Theorem 3.9 in [26], part 1, p. 58; the proof given there is independent of Theorem 2.1.

Theorem 2.2 has thus been established in a round-about way, using the property P_2 and hence also, implicitly, the property P_1. It may be interesting to notice that direct proofs of this result are also known. Such a proof can be based, for instance, upon the following

LEMMA 2.3. *In the set algebra of all subsets of any given set A every λ^+-complete prime ideal I (where λ is any infinite cardinal) is also $(2^{\cdot\lambda})^+$-complete.*(7)

PROOF. Let G be any family $\subseteq I$ with power $2^{\cdot\lambda}$. By a familiar method, applying the well-ordering principle, we construct a family H with the following properties:

(1) H is a family of pairwise disjoint sets;

(2) every set of H is included in some set of G, and hence $H \subseteq I$;

(3) $$\bigcup_{X \in H} X = \bigcup_{X \in G} X;$$

(4) $$\kappa(H) \leqq 2^{\cdot\lambda}.$$

In view of (4) there is a family $K \subseteq S(\lambda)$ and a function F mapping K onto H in a one-one way. We extend this function to the whole family $S(\lambda)$ by putting $F(X) = 0$ for $X \in S(\lambda) \sim K$. By (1) and (2) we have:

(5) if $X, Y \in S(\lambda)$ and $X \neq Y$, then $F(X) \cap F(Y) = 0$;

(6) $$F(X) \in I \text{ for every } X \in S(\lambda).$$

Let

(7) $$L_{\alpha,0} = \bigcup_{\alpha \in X \subseteq \lambda} F(X) \quad \text{and} \quad L_{\alpha,1} = \bigcup_{\alpha \notin X \subseteq \lambda} F(X),$$

whence

(8) $$L_{\alpha,0} \cup L_{\alpha,1} = \bigcup_{X \in H} X$$

for every $\alpha < \lambda$. Suppose

(9) $$\bigcup_{X \in H} X \notin I.$$

(7) This result, in a more general form, is stated in [29], p. 162, Theorem 3.19.

From (8) and (9) we conclude that $L_{\alpha,0} \notin I$ or $L_{\alpha,1} \notin I$. We define a function $\phi \in 2^\lambda$ by putting $\phi(\alpha) = 0$ if $L_{\alpha,1} \in I$ and $\phi(\alpha) = 1$ otherwise; clearly,

$$(10) \qquad L_{\alpha,\phi(\alpha)} \notin I \text{ for every } \alpha < \lambda.$$

Since I is *a* λ^+-complete prime ideal, (10) implies

$$(11) \qquad \bigcap_{\alpha<\lambda} L_{\alpha,\phi(\alpha)} \notin I.$$

On the other hand, letting

$$(12) \quad M_\alpha = \{X \mid \text{either } \alpha \in X \subseteq \lambda \text{ and } \phi(\alpha)=0, \text{ or } \alpha \notin X \subseteq \lambda \text{ and } \phi(\alpha)=1\}$$

we obtain from (7)

$$\bigcap_{\alpha<\lambda} L_{\alpha,\phi(\alpha)} = \bigcap_{\alpha<\lambda} \bigcup_{X \in M_\alpha} F(X).$$

Hence, in view of (5),

$$(13) \qquad \bigcap_{\alpha<\lambda} L_{\alpha,\phi(\alpha)} = \bigcup_{X \in N} F(X), \text{ where } N = \bigcap_{\alpha<\lambda} M_\alpha.$$

From (12) we see that $\bigcap_{\alpha<\lambda} M_\alpha$ consists of just one set, namely, the set

$$X_0 = \{\alpha \mid \alpha < \lambda \text{ and } \phi(\alpha) = 0\}.$$

Consequently, by (13) and (6),

$$(14) \qquad \bigcap_{\alpha<\lambda} L_{\alpha,\phi(\alpha)} = F(X_0) \in I.$$

Formulas (11) and (14) contradict each other. We must therefore reject supposition (9). Thus, by (3), we have

$$\bigcup_{X \in G} X \in I.$$

Since this conclusion applies to every family $G \subseteq I$, the ideal I is $(2^{\cdot\lambda})^+$-complete, and this is what was to be proved.

To derive Theorem 2.2 from our lemma, consider an accessible cardinal λ and a λ-complete prime ideal I in the algebra of all subsets of λ. If λ is singular, then I is clearly λ^+-complete (since in this case every λ-complete ideal is λ^+-complete); hence it is $(2^{\cdot\lambda})^+$-complete by

Lemma 2.3, therefore (absolutely) complete and principal. (The application of Lemma 2.3 at this point is not essential.) If λ is accessible and regular, then there is a cardinal ν such that $\nu^+ \leqq \lambda \leqq 2^{\cdot\nu}$. Since I is λ-complete, it is *a fortiori* ν^+-complete, hence $(2^{\cdot\nu})^+$-complete by Lemma 2.3, therefore λ^+-complete, and, as before, it is principal.

Lemma 2.3 will find an important application ir the next section (in the proof of Theorem 3.1).

Many interesting and important equivalent formulations of the property P_3 are known. We list a few of them, $P_3^{(1)}-P_3^{(3)}$. A cardinal λ has the *property* $P_3^{(1)}$ if every family F of sets which covers λ (i.e., for which $\lambda \subseteq \bigcup_{X \in G} X$) and is such that $S(\lambda) \sim F$ contains no two disjoint sets includes a subfamily G with power $< \lambda$ which also covers λ; λ has the *property* $P_3^{(2)}$ if every λ-additive two-valued measure on λ is trivial; it has the *property* $P_3^{(3)}$ if every λ-additive and λ-multiplicative set function on λ is completely additive and completely multiplicative.

THEOREM 2.4. *For every infinite cardinal λ the property P_3 is equivalent to each of the properties $P_3^{(1)}$, $P_3^{(2)}$ and $P_3^{(3)}$.*[8]

PROOF. The arguments leading to these equivalences are not difficult. As a sample we outline the proof that the properties P_3 and $P_3^{(3)}$ are equivalent.

Let λ be an infinite cardinal with the property P_3, and F be a λ-additive and λ-multiplicative set function on λ. Thus

(1) $F(\bigcup_{X \in G} X) = \bigcup_{X \in G} F(X)$ and $F(\bigcap_{X \in G} X) = \bigcap_{X \in G} F(X)$ for every $G \subseteq S(\lambda)$ such that $0 < \kappa(G) < \lambda$.

Hence we easily conclude that F is increasing and 3-multiplicative, i.e.

$$F(X) \subseteq F(Y) \text{ whenever } X \subseteq Y \subseteq \lambda; \tag{2}$$

$$F(X \cap Y) = F(X) \cap F(Y) \text{ for all } X, Y \in S(\lambda). \tag{3}$$

Suppose that F is not completely additive. Then, for some non-empty family of sets $H \subseteq S(\lambda)$, we must have

$$F(\bigcup_{X \in H} X) \neq \bigcup_{X \in H} F(X).$$

(8) Mutual implications between the properties P_3, $P_3^{(1)}$, $P_3^{(2)}$, and $P_3^{(3)}$ were discussed on various occasions by Tarski. See, e.g., [25], pp. 152 f.; [26], part 2, pp. 55 ff.; [29], pp. 161 f.

Since, by (2),

$$\bigcup_{X \in H} F(X) \subseteq F\left(\bigcup_{X \in H} X\right),$$

we conclude that there is an element y such that

$$y \in F\left(\bigcup_{X \in H} X\right) \text{ and } y \notin \bigcup_{X \in H} F(X). \tag{4}$$

Let

$$I = \{X \mid X \subseteq \lambda \text{ and } y \notin F(X)\}. \tag{5}$$

If $y \in F(0)$, then, by (2), $y \in F(X)$ for every $X \in H$, in contradiction to (4). Hence

$$y \notin F\,(0) \tag{6}$$

and therefore, by (5),

$$0 \in I. \tag{7}$$

From (1), (5), and (7) we obtain:

$$\text{if } G \subseteq I \text{ and } \kappa(G) < \lambda, \text{ then } \bigcup_{X \in G} X \in I; \tag{8}$$

(2) and (5) imply:

$$\text{if } X \in I \text{ and } Y \subseteq X, \text{ then } Y \in I. \tag{9}$$

By (7)–(9), I is a λ-complete ideal in the set algebra formed by $\boldsymbol{S}(\lambda)$. By (3) we have for every $X \subseteq \lambda$

$$F(0) = F(X) \cap F(\lambda \sim X)$$

whence, with the help of (5) and (6), we infer that either $X \in I$ or $\lambda \sim X \in I$. Thus the ideal I is prime. We also notice that, by (4) and (5), $\bigcup_{X \in H} X \notin I$ while $H \subseteq I$. Therefore the λ-complete prime ideal I is not complete and hence not principal. Since this conclusion contradicts the assumption that λ has the property P_3, we must assume that the function F is completely additive.

In an entirely analogous way we show that F is completely multiplicative. The proof that P_3 implies $P_3^{(3)}$ has thus been completed.

Assume now, conversely, that λ has the property $P_3^{(3)}$. Consider any λ-complete prime ideal I in the set algebra formed by $S(\lambda)$. Define a function F on $S(\lambda)$ to 2 by letting

$$F(X) = 0 \text{ if } X \in I, \text{ and } F(X) = 1 \text{ if } X \in S(\lambda) \sim I.$$

We easily show that F is a λ-additive and λ-multiplicative set function on λ. Hence, by $P_3^{(3)}$, F is completely additive; consequently, the ideal I is complete and principal. Thus $P_3^{(3)}$ implies P_3.

3. Property P_4

We agree to say that the cardinal λ has the *property* P_4 if there is a λ-distributive Boolean algebra which is not isomorphic to any λ-complete set algebra.

In discussing this property we shall apply various results concerning λ-complete and λ-distributive Boolean algebras, their ideals and quotient algebras, and their relation to λ-complete set algebras; they are partly quite elementary, and most of them can be found in the literature. We wish to state here these results explicitly.

Obviously, every λ-complete set algebra is λ-distributive. If a λ-distributive Boolean algebra has less than λ elements, then it is completely distributive; hence it is atomistic and isomorphic to a complete, and *a fortiori* to a λ-complete, set algebra (cf. [27], paper XI, pp. 337 ff.). In general, however, as we shall see from Theorem 3.1, a λ-distributive Boolean algebra is not nescessarily isomorphic to a λ-complete set algebra. It is always isomorphic to a quotient algebra $\mathfrak{A}/I$ where $\mathfrak{A}$ is a λ-complete set algebra and I is a λ-complete ideal in $\mathfrak{A}$; for a proof see [21]. For a λ-complete Boolean algebra $\mathfrak{A}$ to be isomorphic to a λ-complete set algebra it is necessary and sufficient that every proper principal ideal in $\mathfrak{A}$ can be extended to a λ-complete prime ideal. (The necessity of this condition is almost obvious; the proof of the sufficiency is entirely analogous to the proof of the representation theorem for Boolean algebras given by Stone in [24], pp. 106–107.) Hence, in particular, $\mathfrak{A}$ is not isomorphic to any λ-complete set algebra if it has at least two elements and no λ-complete prime ideals.

If $\mathfrak{A}$ is a λ-complete Boolean algebra and I is a λ-complete ideal in $\mathfrak{A}$, then the quotient $\mathfrak{A}/I$ is clearly λ-complete. If A is λ-distributive, then, under each of the following three hypotheses, $\mathfrak{A}/I$ proves to be λ-distributive: (*i*) I is principal; (*ii*) λ is of the form $\lambda = \nu^+$ and I is

$(2^{\cdot\nu})^+$-complete; (*iii*) λ is inaccessible and I is λ-complete. (In cases (*i*) and (*ii*) the proof is direct; in case (*iii*) we first show, with the help of (*ii*), that $\mathfrak{A}/I$ is ν^+-distributive for every $\nu < \lambda$, and hence we conclude that $\mathfrak{A}/I$ is λ-distributive.)

Given any Boolean algebra $\mathfrak{A}$ and any ideal I in $\mathfrak{A}$, we can establish a one-one correspondence between all the ideals in $\mathfrak{A}/I$ and all those ideals in $\mathfrak{A}$ which include I. To this end we correlate, with any given ideal J in $\mathfrak{A}/I$, the ideal $H \supseteq I$ in $\mathfrak{A}$ defined by the formula

$$H = \{x \mid x \in A \text{ and } x/I \in J\}.$$

As is well known, the algebras $\mathfrak{A}/H$ and $(\mathfrak{A}/I)/J$ are isomorphic; if one of the ideals J and H is prime, then the other is also prime; if $\mathfrak{A}$, I, and one of the ideals J and H are λ-complete, then the other of these two ideals is also λ-complete.

Returning now to the property P_4, we first notice that, by the well known representation theorem for Boolean algebras established in [24], *the property P_4 fails to apply* to ω. On the other hand, we have

THEOREM 3.1. *Every accessible cardinal λ has the property P_4.*(9)

PROOF. We distinguish two cases.

Case (i): λ is regular. Hence, by the definition of accessible cardinals, there is a cardinal ν such that $\nu < \lambda \leqq 2^{\cdot\nu}$. Let $\mathfrak{A}$ be the set algebra formed by all the subsets of $S(S(\lambda))$ and let I be the family of all subsets of $S(S(\lambda))$ with power $\leqq 2^{\cdot\lambda}$. Obviously, $\mathfrak{A}$ is completely distributive and I is a $(2^{\cdot\lambda})^+$-complete ideal in $\mathfrak{A}$. Hence $\mathfrak{A}/I$ is a λ^+-distributive and *a fortiori* λ-distributive Boolean algebra.

Suppose $\mathfrak{A}/I$ is isomorphic to a λ-complete set algebra. Consequently, $\mathfrak{A}/I$ has some λ-complete prime ideal. Hence $\mathfrak{A}$ has a λ-complete prime ideal J which includes I and therefore contains all one-element sets, i.e., all atoms of $\mathfrak{A}$. From this we conclude that J is not principal. On the other hand, however, since $\nu < \lambda \leqq 2^{\cdot\nu}$, the prime ideal J is ν^+-complete, hence $(2^{\cdot\nu})^+$-complete by Lemma 2.3, and therefore λ^+-complete. By applying Lemma 2.3 again (three times in succession), we infer that J is $(2^{\cdot 2^{\cdot 2^{\cdot\lambda}}})^+$-complete; since the algebra $\mathfrak{A}$ has $2^{\cdot 2^{\cdot 2^{\cdot\lambda}}}$ elements, the ideal J is (absolutely) complete and hence is principal. Thus we have arrived at a contradiction. Consequently, $\mathfrak{A}/I$ is an instance of a λ-distributive algebra which is not isomorphic to any λ-complete set algebra.

(9) This result is due to Tarski and was first stated in [3], p. 328.

Case (ii): λ is singular. Since λ^+ is regular and accessible, we can apply the result obtained in part (i) of our proof. Thus there is a λ^+-distributive Boolean algebra $\mathfrak{A}$ which is not isomorphic to any λ^+-complete set algebra. Obviously, $\mathfrak{A}$ is λ-distributive. Moreover, since λ is singular, every λ-complete set algebra is also λ^+-complete, and hence $\mathfrak{A}$ is not isomorphic to any λ-complete set algebra. The proof has thus been completed.

THEOREM 3.2. *Every infinite cardinal λ which has the property P_3 also has the property P_4.*

PROOF. In view of Theorem 3.1 we can restrict ourselves to the case when λ is inaccessible. Since λ is assumed to have the property P_3, every λ-complete prime ideal in the set algebra $\mathfrak{A}$ formed by all the subsets of λ is principal. Let I be the family of all subsets of λ with power $< \lambda$. I is obviously a λ-complete ideal in the λ-complete set algebra $\mathfrak{A}$ and it contains all atoms of $\mathfrak{A}$. Hence I cannot be extended to any λ-complete prime ideal in $\mathfrak{A}$, since every such prime ideal would be non-principal. Therefore the quotient algebra $\mathfrak{A}/I$ has no λ-complete prime ideals and, in consequence, is not isomorphic to any λ-complete set algebra. On the other hand, from the facts that λ is inaccessible, $\mathfrak{A}$ is a λ-distributive Boolean algebra, and I is a λ-complete ideal in $\mathfrak{A}$, we conclude that $\mathfrak{A}/I$ is λ-distributive. Thus λ has the property P_4.

To conclude this section, we list several properties which will be shown (in the next theorem) to be equivalent to the property P_4. The cardinal λ has the *property* $P_4^{(1)}$ if (and only if) there is a λ-distributive Boolean algebra, with more than one element, which has no λ-complete prime ideals; it has the *property* $P_4^{(2)}$ if there is a λ-distributive Boolean algebra, with at least λ elements, in which every λ-complete prime ideal is principal; it has the *property* $P_4^{(3)}$ if there is a λ-distributive Boolean algebra $\mathfrak{A}$ and a proper λ-complete ideal I in $\mathfrak{A}$ such that I cannot be extended to any λ-complete prime ideal in $\mathfrak{A}$; finally, λ has the *property* $P_4^{(4)}$ if there is a λ-complete set algebra $\mathfrak{A}$ and a λ-complete ideal I in $\mathfrak{A}$ such that the quotient algebra $\mathfrak{A}/I$ is not isomorphic to any λ-complete set algebra.

THEOREM 3.3 *For every infinite cardinal λ the property P_4 is equivalent to each of the properties $P_4^{(1)}$–$P_4^{(4)}$.*

PROOF. (i) P_4 implies $P_4^{(1)}$. In fact, by P_4, there is a λ-distributive Boolean algebra $\mathfrak{A}$ which is not isomorphic to any λ-complete set algebra. As we know, this implies that some proper principal ideal I in

$\mathfrak{A}$ cannot be extended to any λ-complete prime ideal. Consequently, $\mathfrak{A}/I$ is an example of a λ-distributive Boolean algebra with more than one element and with no λ-complete prime ideals.

(ii) $P_4^{(1)}$ implies $P_4^{(2)}$. By $P_4^{(1)}$ there is a λ-distributive Boolean algebra $\mathfrak{A}$, with more than one element, which has no λ-complete prime ideals; *a fortiori* we can say that every λ-complete prime ideal in $\mathfrak{A}$ is principal. If $\mathfrak{A}$ had less than λ elements, it would be completely distributive and therefore atomistic; hence it would have at least one λ-complete (and actually principal) prime ideal. Thus $\mathfrak{A}$ has at least λ elements, and λ has the property $P_4^{(2)}$.

(iii) $P_4^{(2)}$ implies $P_4^{(3)}$. It is obvious that $P_4^{(1)}$ implies $P_4^{(3)}$. Hence, by part (i) of our proof, P_4 implies $P_4^{(3)}$. Therefore, by virtue of Theorem 3.1, $P_4^{(3)}$ certainly holds for every accessible λ, and we can restrict ourselves to the case when λ is inaccessible. In accordance with $P_4^{(2)}$, let $\mathfrak{A}$ be a λ-distributive Boolean algebra, with at least λ elements, in which every λ-complete prime ideal is principal. Let I be the set of all elements $x \in A$ which can be represented as sums of less than λ atoms of $\mathfrak{A}$. Clearly, I is a λ-complete ideal in $\mathfrak{A}$. Moreover, I is a proper ideal, i.e., $I \neq A$. This is obvious in case $\mathfrak{A}$ has at least λ atoms, for 1 cannot then belong to I. However, this is also clear in case $\mathfrak{A}$ has less than λ, say ν, atoms; for, λ being inaccessible, we have then

$$\kappa(I) = 2^{\cdot\nu} < \lambda \leqq \kappa(A).$$

Finally, I cannot be extended to any λ-complete prime ideal J in $\mathfrak{A}$, since J would contain all atoms of $\mathfrak{A}$ and hence could not be principal. This shows that λ has the property $P_4^{(3)}$.

(iv) $P_4^{(3)}$ implies $P_4^{(4)}$. In fact, consider a λ-distributive Boolean algebra $\mathfrak{A}$ and a proper λ-complete ideal I in $\mathfrak{A}$ which cannot be extended to any λ-complete prime ideal. By a known result mentioned at the beginning of this section, $\mathfrak{A}$ is isomorphic to a quotient algebra $\mathfrak{B}/H$ where $\mathfrak{B}$ is a λ-complete set algebra and H is a λ-complete ideal in $\mathfrak{B}$. Hence we conclude that there is a proper λ-complete ideal J in $\mathfrak{B}/H$ which cannot be extended to any λ-complete prime ideal. Consequently, $(\mathfrak{B}/H)/J$ is an algebra with more than one element and with no λ-complete prime ideals. As is well known, we can construct a λ-complete ideal K in $\mathfrak{B}$ such that $\mathfrak{B}/K$ is isomorphic to $(\mathfrak{B}/H)/J$. Thus $\mathfrak{B}/K$ is a Boolean algebra which has more than one element and no λ-complete prime ideals, and which therefore is not isomorphic to any λ-complete set algebra. Consequently, λ has the property $P_4^{(4)}$.

(v) $P_4^{(4)}$ implies P_4. In view of Theorem 3.1 we can again restrict ourselves to the case when λ is inaccessible. By $P_4^{(4)}$ there is a λ-complete set algebra $\mathfrak{A}$ and a λ-complete ideal I in $\mathfrak{A}$ such that $\mathfrak{A}/I$ is not isomorphic to any λ-complete set algebra. $\mathfrak{A}$ is of course λ-distributive and therefore, λ being inaccessible, $\mathfrak{A}/I$ is λ-distributive as well. Hence λ has the property P_4.

By (i)–(v) the proof is complete.

4. Properties Q and R

The *property* Q applies by definition to a cardinal λ if there is a ramification system $\langle A, \preccurlyeq \rangle$ of order λ such that (*i*) the set of all elements $x \in A$ of order ξ has power $< \lambda$ for every $\xi < \lambda$, and (*ii*) every subset of A well ordered by $\preccurlyeq$ has power $< \lambda$. Under these assumptions A is clearly a union of λ mutually exclusive sets each of which has power $< \lambda$. Hence, in case λ is infinite, A is of power λ, and we obtain an equivalent formulation of the property Q if we replace A by λ. The discussion of the property Q is the main topic of the present section. It will be seen that the results of this discussion are less complete than those obtained in the preceding sections for the properties P_1–P_4.

In the first two theorems we shall establish certain implications between the property Q and the properties previously discussed.

THEOREM 4.1. *If λ is an inaccessible cardinal which has the property P_2, then λ also has the property Q.*

PROOF. By the hypothesis and the definition of the property P_2, there are two sets G_1 and G_2 satisfying the conditions:

$$(1) \qquad G_1 \cup G_2 = \boldsymbol{G}(\lambda) \text{ and } G_1 \cap G_2 = 0;$$

$$(2) \qquad \text{if } A \subseteq \lambda \text{ and } \boldsymbol{G}(A) \subseteq G_1 \text{ or } \boldsymbol{G}(A) \subseteq G_2, \text{ then } \kappa(A) < \lambda.$$

We define two functions $F_1, F_2 \in [S(\lambda)]^{S(\lambda)}$ by putting for every set $A \subseteq \lambda$

$$(3) \qquad F_i(A) = \{\xi \mid \xi \in A \text{ and } \{\mu(A), \xi\} \in G_i\},\ i = 1, 2.$$

(1) and (3) imply at once:

(4) if $A \subseteq \lambda$ and $A \neq 0$, then $A = \{\mu(A)\} \cup F_1(A) \cup F_2(A)$, where $\mu(A) \notin F_1(A) \cup F_2(A)$ and $F_1(A) \cap F_2(A) = 0$;

(5) if $A \subseteq \lambda$, $B \subseteq F_i(A)$ ($i = 1$ or $i = 2$), and $B \neq 0$, then $\mu(A) < \mu(B)$ and $\{\mu(A), \mu(B)\} \in G_i$.

By transfinite recursion, we correlate with every ordinal ξ a family of sets K_ξ determined by the following formulas:

(6) $$K_0 = \{\lambda\};$$

(7) $K_{\xi+1} = \{Y \mid \text{for some } X,\ X \in K_\xi \text{ and } Y = F_1(X) \neq 0 \text{ or } Y = F_2(X) \neq 0\}$;

(8) $K_\xi = \{Y \mid \text{for some } H,\ H \in \mathsf{P}_{\zeta<\xi} K_\zeta \text{ and } Y = \bigcap_{\zeta<\xi} H_\zeta \neq 0\}$ in case ξ is a limit ordinal.

From (4) and (6)–(8) we conclude by transfinite induction that, for arbitrary ordinals ξ and ζ,

(9) K_ξ is a family of mutually exclusive non-empty subsets of λ;

(10) if $\xi < \zeta$ and $Z \in K_\zeta$, then there is just one set $X \in K_\xi$ such that $Z \subset X$ and there is no set $X' \in K_\xi$ such that $X' \subseteq Z$ (so that, in particular, $Z \notin K_\xi$).

Hence, with the help of (5) and (7),

(11) if $X \in K_\xi$, $Z \in K_\zeta$, and $Z \subset X$, then $Z \subseteq F_1(X)$ and $\{\mu(X), \mu(Z)\} \in G_1$ or else $X \subseteq F_2(Z)$ and $\{\mu(X), \mu(Z)\} \in G_2$, and in either case $\mu(X) < \mu(Z)$.

Furthermore, (10) implies

(12) if $X \in K_\xi$, $Y \in K_\eta$, $Z \in K_\zeta$, $Z \subseteq X$, and $Z \subseteq Y$, then either $X \subseteq Y$ and $\eta \leqq \xi \leqq \zeta$ or else $Y \subseteq X$ and $\xi \leqq \eta \leqq \zeta$.

From (10) and (11) we see that

(13) $\langle \bigcup_{\xi<\lambda} K_\xi, \supseteq \rangle$ is ramification system and, for every $\xi < \lambda$, K_ξ is the set of all elements of this system with order ξ.

We wish to show that the system $\langle \bigcup_{\xi<\lambda} K_\xi, \supseteq \rangle$ satisfies all the conditions listed in the formulation of the property Q.

By (6)–(8) we have:

$$\kappa(K_0) = 1 < \lambda; \quad \kappa(K_{\xi+1}) \leqq \kappa(K_\xi) \cdot 2 \text{ for every ordinal } \xi;$$
$$\kappa(K_\xi) \leqq \prod_{\zeta<\xi} \kappa(K_\zeta) \text{ for every limit ordinal } \xi.$$

Hence, making essential use of the fact that λ is inaccessible, we derive by induction:

(14) $$\kappa(K_\xi) < \lambda \text{ for every } \xi < \lambda.$$

From (10) we obtain

$$\bigcup_{X\in K_\xi} X \supseteq \bigcup_{X\in K_\zeta} X, \text{ whenever } \xi \leqq \zeta;$$

moreover, (6) and (8) imply:

$$\bigcup_{X\in K_0} X = \lambda,$$

$$\bigcup_{X<K_\xi} X = \bigcap_{\zeta<\xi} \bigcup_{X\in K_\zeta} X \quad \text{for every limit ordinal } \xi.$$

Hence we conclude:

(15) $\lambda \sim \bigcup_{X\in K_\xi} X = \bigcup_{\zeta<\xi} (\bigcup_{X\in K_\zeta} X \sim \bigcup_{X\in K_{\zeta+1}} X)$ for every ordinal ξ.

From (4) and (7) we see that

$$\bigcup_{X\in K_\zeta} X \sim \bigcup_{X\in K_{\zeta+1}} X = \{\mu(X) | X \in K_\zeta\},$$

and therefore, by (14),

(16) $\kappa(\bigcup_{X\in K_\zeta} X \sim \bigcup_{X\in K_{\zeta+1}} X) < \lambda$ for every $\zeta < \lambda$.

Since the cardinal λ is inaccessible and hence *a fortiori* regular, the formulas (15) and (16) imply:

$$\kappa(\lambda \sim \bigcup_{X\in K_\xi} X) < \lambda \text{ for every } \xi < \lambda.$$

From this we conclude at once that $K_\xi \neq 0$ for every $\xi < \lambda$ and consequently, by (13),

(17) the ramification system $\langle \bigcup_{\xi<\lambda} K_\xi, \supseteq \rangle$ is of order λ.

Consider now any family of sets L which is included in $\bigcup_{\xi<\lambda} K_\xi$ and is well ordered by the relation $\supseteq$. By (10), L has at most one set in common with each of the families K_ξ for $\xi < \lambda$. Hence we infer that there is an ordinal $\nu \leqq \lambda$, a sequence $H \in L^\nu$, and a sequence $\sigma \in \lambda^\nu$ which satisfy the following conditions:

(18) $L = \{H_\xi \,|\, \xi < \nu\}$, $H_\xi \in K_{\sigma_\xi}$ for every $\xi < \nu$;

(19) if $\xi < \zeta < \nu$, then $H_\xi \supset H_\zeta$ and $\sigma_\xi < \sigma_\zeta$.

Let

(20) $\qquad L_i = \{H_\xi \mid \xi + 1 < \nu \text{ and } H_{\xi+1} \subseteq F_i(H_\xi\} \text{ for } i = 1, 2.$

With the help of (11) and (18)–(20) we obtain:

(21) $L = L_1 \cup L_2 \cup \{H_{\nu-1}\}$ if ν has an immediate predecessor, and $L = L_1 \cup L_2$ otherwise.

If $\xi < \zeta$ and $H_\xi, H_\zeta \in L_i$ ($i = 1,2$), we conclude by (11) and (18)–(20):

$$\mu(H_\xi) < \mu(H_\zeta) \text{ and } \{\mu(H_\xi), \mu(H_\zeta)\} \in G_i.$$

Hence, by putting

$$M_i = \{\mu(H_\xi) \mid \xi < \nu \text{ and } H_\xi \in L_i\}, \ i = 1, 2,$$

we derive the conclusions:

$$\kappa(L_i) = \kappa(M_i) \text{ and } G(M_i) \subseteq G_i;$$

and, by applying (2), we arrive at the formula

$$\kappa(L_i) < \lambda \text{ for } i = 1, 2.$$

By (21), this implies that $\kappa(L) < \lambda$. Thus we have established:

(22) every family of sets $L \subseteq \bigcup_{\xi<\lambda} K_\xi$ which is well ordered by $\supseteq$ has power $< \lambda$.

By (13), (14), (17), and (22), the system $\langle \bigcup_{\xi<\lambda} K_\xi, \supseteq \rangle$ satisfies all the conditions listed in Q. Hence the cardinal λ has the property Q, and the proof is complete.

THEOREM 4.2. *Every infinite cardinal λ which has the property Q also has the property P_3.*

PROOF. By hypothesis, there is a ramification system $\langle \lambda, \preccurlyeq \rangle$ with the following properties:

(1) for every $\xi < \lambda$, the set R_ξ of all elements of λ of order ξ has power $< \lambda$;

(2) every subset A of λ well ordered by $\preccurlyeq$ has power $< \lambda$.

We want to show that λ has the property P_3. Suppose, on the contrary, that there is a λ-complete prime ideal I in the set algebra of all subsets of λ which is not a principal ideal. As a non-principal prime ideal, I must contain all one-element subsets of λ; since, in addition, I is λ-complete, it must also contain all subsets of λ with power $< \lambda$. Hence, by (1),

(3) $$R_\xi \in I \text{ for every } \xi < \lambda.$$

Let

(4) $$F(\xi) = \{\zeta \mid \zeta < \lambda,\ \xi \preccurlyeq \zeta, \text{ and } \xi \neq \zeta\} \text{ for every } \xi < \lambda,$$

and

(5) $$J = \{\xi \mid \xi < \lambda \text{ and } F(\xi) \in I\}.$$

Notice the following formula which easily follows from (4) and the definition of a ramification system:

(6) $$\bigcup_{\xi \leqq \zeta} R_\zeta = R_\xi \cup \bigcup_{\eta \in R_\xi} F(\eta) \text{ for every } \xi < \lambda.$$

whence, in particular, for $\xi = 0$

(7) $$\lambda = R_0 \cup \bigcup_{\eta \in R_0} F(\eta).$$

We now define by transfinite recursion a sequence $T \in [S(\lambda)]^\lambda$ satisfying the formulas:

(8) $$T_0 = R_0 \sim J,$$

(9) $$T_\xi = [R_\xi \cap \bigcap_{\zeta < \xi} F(\mu(T_\zeta))] \sim J \text{ for every } \xi \text{ such that } 0 < \xi < \lambda.$$

We shall prove by transfinite induction that

(10) $$T_\xi \neq 0 \text{ for every } \xi < \lambda.$$

Consider first the case of T_0. By (3), $R_0 \in I$. If $R_0 \subseteq J$, then, by (5), $F(\xi) \in I$ for every $\xi \in R_0$. Since, by (1), $\kappa(R_0) < \lambda$ and the ideal I is λ-complete, this implies

$$\bigcup_{\eta \in R_0} F(\eta) \in I.$$

Hence, by (7), $\lambda \in I$, which obviously contradicts the assumption that I is a prime ideal. Thus $R_0 \sim J \neq 0$, i.e., by (8),

$$(11) \qquad T_0 \neq 0.$$

Assume now that

$$(12) \qquad 0 < \theta < \lambda, \text{ and } T_\xi \neq 0 \text{ for every } \xi < \theta.$$

Then $\mu(T_\xi) \in T_\xi$ whence, by (7) and (5), $\mu(T_\xi) \notin J$ and $F(\mu(T_\xi)) \notin I$, for every $\xi < \theta$. Since I is a λ-complete prime ideal, we conclude that

$$(13) \qquad \bigcap_{\xi<\theta} F(\mu(T_\xi)) \notin I.$$

Furthermore we have

$$(14) \quad \bigcap_{\xi<\theta} F(\mu(T_\xi)) \subseteq R_\theta \cup \bigcup_{\eta \in B} F(\eta), \text{ where } B = R_\theta \cap \bigcap_{\xi<\theta} F(\mu(T_\xi)).$$

In fact, if

$$(15) \qquad \zeta \in \bigcap_{\xi<\theta} F(\mu(T_\xi)),$$

then, by (4), $\mu(T_\xi) \preccurlyeq \zeta$ and $\mu(T_\xi) \neq \zeta$ for every $\xi < \theta$. Since, by (9), $\mu(T_\xi) \in T_\xi \subseteq R_\xi$, we infer that $\zeta \notin R_\xi$ for any $\xi < \theta$ and hence $\zeta \in R_\sigma$ for some $\sigma \geqq \theta$. Therefore, by (6), either $\zeta \in R_\theta$ or else there is an $\eta \in R_\theta$ such that $\zeta \in F(\eta)$; in the latter case we easily infer from (4), (15) and the definition of a ramification system that

$$\eta \in \bigcap_{\xi<\theta} F(\mu(T_\xi)).$$

The inclusion (14) has thus been established. If now

$$B = R_\theta \cap \bigcap_{\xi<\theta} F(\mu(T_\xi)) \subseteq J,$$

we obtain, by (5), $F(\eta) \in I$ for every $\eta \in B$. Since, by (1), $\kappa(B) < \lambda$ and I is λ-complete, we conclude that

$$\bigcup_{\eta \in B} F(\eta) \in I.$$

Hence, by (3) and (14),

$$\bigcap_{\xi<\theta} F(\mu(T_\xi)) \in I,$$

in contradiction to (13). Therefore $B \sim J \neq 0$, i.e., by (9),

$$(16) \qquad T_\theta \neq 0.$$

Thus, (11) holds, and (12) always implies (16). Consequently, condition (10) has been shown to hold.

By (10) we have $\mu(T_\xi) \in T_\xi$ for every $\xi < \lambda$. Hence, by (9) and (4), $\mu(T_\xi) \in F(\mu(T_\zeta))$, i.e., $\mu(T_\zeta) \preccurlyeq \mu(T_\xi)$ and $\mu(T_\zeta) \neq \mu(T_\xi)$ for all ξ and ζ such that $\zeta < \xi < \lambda$. In consequence, the set

$$A = \{\mu(T_\xi) \mid \xi < \lambda\}$$

is well ordered by the relation $\preccurlyeq$ and has power λ. Since this contradicts (2), we have to reject our initial supposition and assume that λ has the Property P_3, and this is what was to be proved.

As was mentioned in §**2**, the property P_3 fails for the cardinal $\lambda = \omega$. Hence, by Theorem 4.2, *the property Q fails for $\lambda = \omega$ as well.* This can easily be proved directly; the result is known as König's theorem (or as the theorem on the passage from finiteness to infinity); see [8].

Except for the case $\lambda = \omega$, the results obtained so far in this paper do not help us in answering the question whether the property Q applies to a given (so to speak, "constructively" defined) cardinal λ. In particular, we do not know whether Theorem 4.1 can be extended to accessible cardinals, nor do we know whether the converse of Theorem 4.2 holds; hence we cannot use Theorem 1.3 or Theorem 2.2 in order to show that every accessible cardinal has the property Q. Thus, the problem whether Q applies to accessible cardinals must be attacked by direct methods. It was shown a long time ago by Aronszajn that *Q holds for $\lambda = \omega^+$* (i.e., $\lambda = \omega_1$); for the proof see [9]. Recently it has turned out that, *under the assumption of the generalized continuum hypothesis ($\kappa^+ = 2^{\cdot\kappa}$ for every infinite cardinal κ), this result can be extended to all cardinals λ of the form $\lambda = \nu^+$ where ν is a regular cardinal.* The proof of this result is given by Specker in [23]; it is rather involved and will not be repeated here.(10) On the other hand, it can easily be shown that *Q applies to all singular cardinals*; see [14]. Thus, assuming again the generalized con-

(10) Donald Monk has pointed out to us that, by analysing the argument in [23], one obtains the following result independently of the general continuum hypothesis: Q applies to all cardinals λ of the form $\lambda = \nu^+$ provided $\nu = \nu{\cdot}\pi$ for every π such that $0 < \pi < \nu$ (and hence, in particular, it applies to all cardinals $\lambda = \nu^+$ where ν is inaccessible).

tinuum hypothesis, Q proves to hold for all accessible cardinals λ except those of the form $\lambda = \nu^+$ with a singular ν. It is still an open problem (even under the assumption of the generalized continuum hypothesis) whether Q applies to those exceptional cardinals as well.[11]

Since we do not know whether the property P_2 applies to any inaccessible cardinal $\lambda > \omega$, Theorem 4.1 is of no help in discussing the problem whether the property Q applies to any such cardinal. We shall return to this problem in a later part of the present section.

We agree to say that a cardinal λ has the *property* R if there is a λ-distributive Boolean algebra which is λ-generated by a set of power λ and which is not isomorphic to any λ-complete set algebra. As is easily seen, in case λ is inaccessible, every λ-distributive and, more generally, every λ-complete Boolean algebra which is λ-generated by a set of power λ is itself of power λ (and conversely); under the generalized continuum hypothesis, this remark extends to all regular cardinals λ.

THEOREM 4.3. *Every inaccessible cardinal λ which has the property R also has the property Q.*

PROOF. By hypothesis, there is a λ-distributive Boolean algebra $\mathfrak{A}$ of power λ which is not isomorphic to any λ-complete set algebra. Hence some proper principal ideal of $\mathfrak{A}$ cannot be extended to any λ-complete prime ideal; in other words, there is an element $a \in A$ such that $a \neq 1$ and $a \notin I$ for every λ-complete prime ideal I of $\mathfrak{A}$. Let f be a function which maps λ onto A in a one-one way and for which $f(0) = a$.

We put

(1) $G = \{g \mid$ there is a $\xi < \lambda$ such that $g \in A^\xi, g(0) = a$, in case $0 < \xi$, $g(\zeta) = f(\zeta)$ or $g(\zeta) = -f(\zeta)$ for every $\zeta < \xi$, and $\sum_{\zeta<\xi} g(\zeta) \neq 1\}$.

As is easily seen, $\langle G, \subseteq \rangle$ is a ramification system (functions being considered as sets of ordered couples). If $g \in G$, then, by (1), there is a uniquely determined ordinal $\xi < \lambda$ such that $g \in A^\xi$; clearly ξ is the order of g. For every $\xi < \lambda$ there are elements $g \in G$ of order ξ. This is obvious in case $\xi = 0$. If $\xi > 0$, define $h \in A^{\xi+2}$ by putting $h(0,0) = h(0,1) = -a$, and $h(\zeta, 0) = f(\zeta)$, $h(\zeta, 1) = -f(\zeta)$ for $0 < \zeta < \xi$. Then

$$\prod_{\zeta<\xi} \sum_{\eta<2} h(\zeta, \eta) = -a \neq 0.$$

(11) In [3], p. 328, it was erroneously stated that, with the help of the generalized continuum hypothesis, the property Q was shown to hold for all accessible cardinals λ; the result was ascribed to Aronszajn, who actually obtained it only for $\lambda = \omega^+$ (without the help of the continuum hypothesis).

Hence, by λ-distributivity, there is a $k \in 2^{\xi}$ such that

$$\prod_{\zeta<\xi} h(\zeta, k(\zeta)) \neq 0, \text{ i.e., } \sum_{\zeta<\xi} -h(\zeta, k(\zeta)) \neq 1.$$

We now define a function $g \in A^{\xi}$ by letting

$$g(\zeta) = -h(\zeta, k(\zeta)) \text{ for every } \zeta < \xi,$$

and using (1) we easily check that g is an element of G with order ξ. Thus our ramification system $\langle G, \subseteq \rangle$ is of order λ. For each $\xi < \lambda$ the set of all elements $g \in G$ of order ξ has power $\leqq 2^{\cdot k(\xi)}$ and hence $< \lambda$ (since λ is inaccessible).

Suppose now that G has a subset H of power λ which is well ordered by $\subseteq$. Let

(2) $I = \{x \mid \text{ for some } g \text{ and } \xi,\ g \in H,\ \xi < \lambda, g \in A^{\xi}, \text{ and } x \leqq \sum_{\zeta<\xi} g(\zeta)\}.$

Clearly, I is an ideal in $\mathfrak{A}$. By (1) and (2) we have $a \in I$ and $1 \notin I$. Let x be any element of A. Since f maps λ onto A, there is a $\xi < \lambda$ for which $f(\xi) = x$. Since H has power λ, H must contain an element g of some order $\theta > \xi$. By (1), $g(\xi) = x$ or $g(\xi) = -x$ whence

$$x \leqq \sum_{\zeta<\theta} g(\zeta) \text{ or } -x \leqq \sum_{\zeta<\theta} g(\zeta).$$

Therefore, by (2), $x \in I$ or $-x \in I$. Consequently, I is a prime ideal. Finally, from the facts that λ is inaccessible and hence regular, and $\kappa(H) = \lambda$, we conclude that for every set $K \subseteq H$ with $\kappa(K) < \lambda$ there is an $h \in H$ such that

$$\bigcup_{g \in K} g \subseteq h.$$

This clearly implies by (2) that the ideal I is λ-complete. Thus our supposition leads to a contradiction, for the element a does not belong to any λ-complete prime ideal.

Consequently, the ramification system $\langle G, \subseteq \rangle$ has all the properties listed for Q, and the proof is complete.

It is shown in [14] that the converse of Theorem 4.3 also holds. Thus, the properties Q and R turn out to be equivalent for every inaccessible cardinal λ.

As an immediate corollary of Theorems 2.2, 4.2, and 4.3 we obtain

THEOREM 4.4. *Every infinite cardinal λ which has the property R also has the property P_3.*

We turn to the problem of determining those individual cardinals and classes of cardinals to which the property R applies. The results obtained so far in this direction have a rather scattered character.

First, it is well known that *R fails for $\lambda = \omega$*; this result is a special case of Stone's representation theorem. Secondly, we have the following

THEOREM 4.5. *If λ is a singular cardinal satisfying the formula $2^{\cdot\nu} < \lambda$ for every cardinal $\nu < \lambda$, then λ does not have the property R.*

PROOF. Let $\mathfrak{A}$ be a λ-distributive Boolean algebra which is λ-generated by a set $G \subseteq A$ of power λ. Hence $\mathfrak{A}$ is λ-complete and therefore, λ being singular, $\mathfrak{A}$ is λ^+-complete. Making now use of the fact that $2^{\cdot\nu} < \lambda$ for every $\nu < \lambda$, we easily conclude that $\mathfrak{A}$ is λ^+-distributive. From this we infer that $\mathfrak{A}$ is atomistic; in fact the set

$$\{y \mid \text{ for some } f,\ f \in \mathsf{P}_{x\in G}\{x, -x\} \text{ and } y = \textstyle\prod_{x\in G} f(x) \neq 0\}$$

is the set of all atoms of $\mathfrak{A}$. Thus the Boolean algebra $\mathfrak{A}$ is λ-complete and atomistic; hence, as is well known, it is isomorphic to a λ-complete set algebra. Since this applies to every λ-distributive Boolean algebra which is λ-generated by λ elements, the property R does not hold for λ. (For an elaboration of some details in this proof compare [22], pp. 239–240.)

Many cardinals can be constructed which satisfy the hypothesis of Theorem 4.5; the smallest of them is the cardinal λ determined by the formula

$$\lambda = \textstyle\sum_{\xi<\omega} \nu_\xi,$$

where $\nu_0 = \omega$, and $\nu_{\xi+1} = 2^{\cdot\nu_\xi}$ for every $\xi < \omega$.

The generalized continuum hypothesis obviously implies that every singular cardinal satisfies the hypothesis of Theorem 4.5. Hence we see that, *under the assumption of the generalized continuum hypothesis, no singular cardinal λ has the property R.*

As was mentioned before, Q holds for all singular λ's. Thus, in general, the properties Q and R are not equivalent; at any rate, the equivalence does not hold for singular λ's. By comparing Theorem 4.5 with Theorems 1.1, 1.3, 2.2, and 3.1, we see that R is not equivalent for singular λ's to any of the properties $P_1 - P_4$.

On the other hand, we do not know whether R is equivalent to any of the properties $P_1 - P_4$ and Q for all regular λ's. The problem whether R holds for all, or at least for some, regular accessible cardinals is still fully open (even under the assumption of the generalized continuum hypothesis).

In this situation the following fact appears to be especially interesting: it has recently turned out that *R applies to a very comprehensive class of inaccessible cardinals $\lambda > \omega$*; see [28]. By virtue of Theorems 4.3, 4.4, and 3.2, *this result automatically extends to properties Q, P_3 and P_4*. At present we cannot define "constructively" a single inaccessible cardinal $\lambda > \omega$ of which we could not show that it possesses all these properties. Nevertheless we are still unable to prove that properties P_3, P_4, Q, and R apply to all inaccessible cardinals $> \omega$ without exception; and from the remark which concludes §1 it is seen that the new results have not yet been extended to the properties P_1 and P_2.

Concluding remarks

In addition to the problems treated in this paper, numerous outstanding problems in several branches of abstract mathematics which involve in some way the notion of an inaccessible number have been discussed in the literature; many of them exhibit exactly the pattern described at the beginning of this paper. For further problems in the general theory of sets see [4], [6], [12], [2] (graphs); and [25], [26], [29] (covering theorems and ideals in set algebras). Various related problems in the theory of Boolean algebras are studied in [22], section 3. For problems in the theory of Abelian groups see [10], [17], and [33]. Some problems in abstract analysis are discussed in [11] and [32] (the measure problem); problems in topology are discussed in [13] and [20] (compactness of Cartesian products). Finally, we refer to [5], [7], [15], and [18] for problems in metamathematics (logics with infinitely long formulas, model theory).

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