INTERSECTION THEOREMS FOR SYSTEMS OF FINITE SETS

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1. Introduction

E. Sperner (**1**) has proved that *every system of subsets a_ν of a set of finite cardinal m, such that $a_\mu \not\subset a_\nu$ for $\mu \neq \nu$ contains at most $\binom{m}{p}$ elements, where $p = [\frac{1}{2}m]$.* This note concerns analogues of this result. We shall impose an upper limitation on the cardinals of the a_ν and a lower limitation on the cardinals of the intersection of any two sets a_ν, and we shall deduce upper estimates, in many cases best-possible, for the number of elements of such a system of sets a_ν.

2. Notation

The letters a, b, c, d, x, y, z denote finite sets of non-negative integers, all other lower-case letters denote non-negative integers. If $k \leqslant l$, then $[k, l)$ denotes the set

$$\{k, k+1, k+2,..., l-1\} = \{t: k \leqslant t < l\}.$$

The *obliteration operator* ^ serves to remove from any system of elements the element above which it is placed. Thus $[k,l) = \{k, k+1,..., \hat{l}\}$. The *cardinal* of a is $|a|$; *inclusion* (in the wide sense), *union*, *difference*, and *intersection* of sets are denoted by $a \subset b$, $a+b$, $a-b$, ab respectively, and $a-b = a-ab$ for all a, b.

By $S(k, l, m)$ we denote the set of all systems $(a_0, a_1,..., \hat{a}_n)$ such that

$$a_\nu \subset [0, m);\ |a_\nu| \leqslant l \quad (\nu < n),$$

$$a_\mu \not\subset a_\nu \not\subset a_\mu;\ |a_\mu a_\nu| \geqslant k \quad (\mu < \nu < n).$$

3. Results

Theorem 1. *If $1 \leqslant l \leqslant \frac{1}{2}m$; $(a_0,..., \hat{a}_n) \in S(1, l, m)$, then $n \leqslant \binom{m-1}{l-1}$. If, in addition, $|a_\nu| < l$ for some ν, then $n < \binom{m-1}{l-1}$.*

Theorem 2. *Let $k \leqslant l \leqslant m$, $n \geqslant 2$, $(a_0,..., \hat{a}_n) \in S(k, l, m)$. Suppose that either*

$$2l \leqslant k+m, \qquad |a_\nu| = l \quad (\nu < n) \tag{1}$$

or†

$$2l \leqslant 1+m, \qquad |a_\nu| \leqslant l \quad (\nu < n). \tag{2}$$

† The condition $|a_\nu| \leqslant l$ is in fact implied by $(a_0,..., \hat{a}_n) \in S(k, l, m)$.

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Then (*a*) *either* (i)

$$|a_0 \ldots \hat{a}_n| \geqslant k, \qquad n \leqslant \binom{m-k}{l-k},$$

or (ii) $$|a_0 \ldots \hat{a}_n| < k < l < m, \qquad n \leqslant \binom{m-k-1}{l-k-1}\binom{l}{k}^3;$$

(*b*) *if* $$m \geqslant k+(l-k)\binom{l}{k}^3, \quad \textit{then} \quad n \leqslant \binom{m-k}{l-k}.$$

Remark. Obviously, if $|a_\nu| = l$ for $\nu < n$, then the upper estimates for n in Theorem 1 and in Theorem 2 (*a*) (i) and (*b*) are best-possible. For, if $k \leqslant l \leqslant m$ and if $a_0, \ldots, \hat{a}_n$ are the distinct sets a such that

$$[0,k) \subset a \subset [0,m), \qquad |a| = l,$$

then $$(a_0, \ldots, \hat{a}_n) \in S(k,l,m), \qquad n = \binom{m-k}{l-k}.$$

4. The following lemma is due to Sperner (**1**). We give the proof since it is extremely short.

LEMMA. *If*

$$n_0 \geqslant 1, \quad a_\nu \subset [0,m), \quad |a_\nu| = l_0 \quad (\nu < n_0),$$

then there are at least $n_0(m-l_0)(l_0+1)^{-1}$ *sets* b *such that, for some* ν,

$$\nu < n_0, \quad a_\nu \subset b \subset [0,m), \quad |b| = l_0+1. \tag{3}$$

Proof. Let n_1 be the number of sets b defined above. Then, by counting in two different ways the number of pairs (ν, b) satisfying (3), we obtain $n_0(m-l_0) \leqslant n_1(l_0+1)$, which proves the lemma.

5. Proof of Theorem 1

Case 1. Let $|a_\nu| = l$ $(\nu < n)$. We have $m \geqslant 2$. If $m = 2$, then $l = 1$; $n = 1 \leqslant \binom{m-1}{l-1}$. Now let $m \geqslant 3$ and use induction over m. Choose, for fixed l, m, n, the a_ν in such a way that the hypothesis holds and, in addition, the number

$$f(a_0, \ldots, \hat{a}_n) = s_0 + \ldots + \hat{s}_n$$

is minimal, where s_ν is the sum of the elements of a_ν. Put $A = \{a_\nu : \nu < n\}$. If $2l = m$, then $[0,m) - a_\nu \notin A$ and, hence

$$n \leqslant \frac{1}{2}\binom{m}{l} = \binom{m-1}{l-1}.$$

Now let $2l < m$.

Case 1 *a*. Suppose that whenever

$$m-1 \in a \in A, \qquad \lambda \in [0,m)-a,$$

then

$$a-\{m-1\}+\{\lambda\} \in A.$$

We may assume that, for some $n_0 < n$,

$$m-1 \in a_\nu \quad (\nu < n_0), \qquad m-1 \notin a_\nu \quad (n_0 \leqslant \nu < n).$$

Put

$$b_\nu = a_\nu - \{m-1\} \quad (\nu < n_0).$$

Let $\mu < \nu < n_0$. Then

$$|a_\mu + a_\nu| < 2l < m,$$

and there is

$$\lambda \in [0,m)-a_\mu-a_\nu.$$

Then $\quad b_\mu+\{\lambda\} \in A, \qquad b_\mu b_\nu = (b_\mu+\{\lambda\})b_\nu = (b_\mu+\{\lambda\})a_\nu \neq \varnothing$

and therefore

$$l-1 \geqslant 1, \qquad (b_0,\ldots,b_{n_0}) \in S(1,l-1,m-1).$$

Since $2(l-1) < m-2 < m-1$ we obtain, by the induction hypothesis, $n_0 \leqslant \binom{m-2}{l-2}$. Similarly, since

$$(a_{n_0},\ldots,\hat{a}_n) \in S(1,l,m-1), \qquad 2l \leqslant m-1,$$

we have $n-n_0 \leqslant \binom{m-2}{l-1}$. Thus

$$n = n_0+(n-n_0) \leqslant \binom{m-2}{l-2}+\binom{m-2}{l-1} = \binom{m-1}{l-1}.$$

Case 1 *b*. Suppose that there are $a \in A$, $\lambda \in [0,m)-a$ such that

$$m-1 \in a, \qquad a-\{m-1\}+\{\lambda\} \notin A.$$

Then $\lambda < m-1$. We may assume that

$$\begin{aligned}
&m-1 \in a_\nu, \quad \lambda \notin a_\nu, \quad b_\nu = a_\nu-\{m-1\}+\{\lambda\} \notin A \quad (\nu < n_0),\\
&m-1 \in a_\nu, \quad \lambda \notin a_\nu, \quad c_\nu = a_\nu-\{m-1\}+\{\lambda\} \in A \quad (n_0 \leqslant \nu < n_1),\\
&m-1 \in a_\nu, \quad \lambda \in a_\nu \quad (n_1 \leqslant \nu < n_2),\\
&m-1 \notin a_\nu \qquad\qquad (n_2 \leqslant \nu < n).
\end{aligned}$$

Here $1 \leqslant n_0 \leqslant n_1 \leqslant n_2 \leqslant n$. Put $b_\nu = a_\nu$ $(n_0 \leqslant \nu < n)$. We now show that

$$(b_0,\ldots,\hat{b}_n) \in S(1,l,m). \tag{4}$$

Let $\mu < \nu < n$. We have to prove that

$$b_\mu \neq b_\nu, \qquad b_\mu b_\nu \neq \varnothing. \tag{5}$$

If $\mu < \nu < n_0$ or $n_0 \leqslant \mu < \nu$, then (5) clearly holds. Now let $\mu < n_0 \leqslant \nu$. Then $b_\mu \notin A$, $b_\nu = a_\nu \in A$, and hence $b_\mu \neq b_\nu$. If $n_0 \leqslant \nu < n_1$, then $c_\nu \in A$, and there is $\sigma \in a_\mu c_\nu$. Then $\sigma \neq \lambda$, $\sigma \neq m-1$, and $\sigma \in b_\mu b_\nu$. If $n_1 \leqslant \nu < n_2$, then $\lambda \in b_\mu b_\nu$. If, finally, $n_2 \leqslant \nu < n$, then there is

$\rho \in a_\mu a_\nu$. Then

$$\rho \in a_\nu, \qquad \rho < m-1, \qquad \rho \in b_\mu b_\nu.$$

This proves (5) and therefore (4). However, we have

$$f(b_0,\dots,\hat{b}_n)-f(a_0,\dots,\hat{a}_n) = n_0(-[m-1]+\lambda) < 0,$$

which contradicts the minimum property of $(a_0,\dots,\hat{a}_n)$. This shows that Case 1 b cannot occur.

Case 2: $\min(\nu < n)\ |a_\nu| = l_0 \leqslant l$.

If $l_0 = l$, then we have Case 1. Now let $l_0 < l$ and use induction over $l-l_0$. We may assume that

$$|a_\nu| = l_0 \quad (\nu < n_0), \qquad |a_\nu| > l_0 \quad (n_0 \leqslant \nu < n), \tag{6}$$

where $1 \leqslant n_0 \leqslant n$. Let $b_0,\dots,\hat{b}_{n_1}$ be the distinct sets b such that (3) holds for some ν. Then

$$(l_0+1)-(m-l_0) \leqslant 2(l-1)+1-m \leqslant 0, \tag{7}$$

and hence, by the lemma, $n_1 \geqslant n_0$. Also,

$$(b_0,\dots,\hat{b}_{n_1}, a_{n_0},\dots,\hat{a}_n) \in S(1,l,m).$$

Hence, from our induction hypothesis,

$$n \leqslant n_1+(n-n_0) \leqslant \binom{m-1}{l-1},$$

and Theorem 1 follows.

6. Proof of Theorem 2

Case 1: $k = 0$. Then

$$2l \leqslant 1+m, \qquad |a_\nu| \leqslant l \quad (\nu < n), \qquad (a_0,\dots,\hat{a}_n) \in S(0,l,m).$$

Now (a) (ii) is impossible, and (a) (i) is identical with (b), so that all we have to prove is that $n \leqslant \binom{m}{l}$. Again, we may assume (6), where $l_0 \leqslant l$, $1 \leqslant n_0 \leqslant n$. If $l_0 = l$ then

$$|a_\nu| = l \quad (\nu < n), \qquad n \leqslant \binom{m}{l}.$$

Now let $l_0 < l$ and use induction over $l-l_0$. Let $b_0,\dots,\hat{b}_{n_1}$ be the distinct sets b such that (3) holds for some ν. Again (7) holds and, by the lemma, $n_1 \geqslant n_0$. We have

$$(b_0,\dots,\hat{b}_{n_1}, a_{n_0},\dots,\hat{a}_n) \in S(0,l,m)$$

and, by our induction hypothesis,

$$n \leqslant n_1+(n-n_0) \leqslant \binom{m}{l}.$$

This proves the assertion.

Case 2: $k > 0$. We separate this into two cases.

Case 2a. Suppose that (1) holds. Put $|a_0 ... \hat{a}_n| = r$. We now show that, if $r \geqslant k$, then (i) follows. We may assume that

$$a_0 ... \hat{a}_n = [m-r, m).$$

Put
$$a_\nu[0, m-r) = c_\nu \quad (\nu < n).$$
Then
$$(c_0, ..., \hat{c}_n) \in S(0, l-r, m-r),$$
$$2(l-r)-(m-r) = 2l-r-m \leqslant 2l-k-m \leqslant 0.$$

Hence, by Case 1,

$$n \leqslant \binom{m-r}{l-r} = \binom{m-r}{m-l} \leqslant \binom{m-k}{m-l} = \binom{m-k}{l-k},$$

so that (i) holds. We now suppose that (i) is false, and we deduce (ii). We have
$$|a_0 ... \hat{a}_n| = r < k \leqslant |a_0 a_1| < |a_0| \leqslant l,$$
and therefore
$$2l \leqslant k+m < l+m, \qquad k < l < m.$$

There is a maximal number $p \geqslant n$ such that there exist $p-n$ sets $a_n, ..., \hat{a}_p$ satisfying
$$(a_0, ..., \hat{a}_p) \in S(k, l, m). \tag{8}$$

Put $A' = \{a_\nu : \nu < p\}$. We assert that

$$(a_0, ..., \hat{a}_p) \notin S(k+1, l, m). \tag{9}$$

For otherwise $|a_\mu a_\nu| > k$ $(\mu < \nu < n)$. Let $a \in A'$. Then we can choose $a' \subset [0, m)$ such that

$$|a'| = l, \qquad |aa'| = l-1.$$

Then, for every $b \in A'$, we have

$$|a'b| \geqslant |ab|-1 \geqslant k$$

and hence, since p is maximal, $a' \in A'$. By repeated application of this result we find that

$$[0, l), [m-l, m) \in A', \qquad k < |[0, l)[m-l, m)| = l-(m-l) \leqslant k,$$

which is the desired contradiction. This proves (9), and hence there are sets a, $b \in A'$ such that $|ab| = k$. Since $|a_0 ... \hat{a}_p| \leqslant |a_0 ... \hat{a}_n| < k$, there is $c \in A'$ such that $|abc| < k$. Denote by T the set of all triples (x, y, z) such that $x \subset a$, $y \subset b$, $z \subset c$, $|x| = |y| = |z| = k$, $|x+y+z| \leqslant l$. Put $\phi(x, y, z) = \{d : x+y+z \subset d \in A'\}$. Then, by (8),

$$A' = \sum ((x, y, z) \in T)\ \phi(x, y, z).$$

If $(x, y, z) \in T$ and $s = |x+y+z|$, then $s > k$ since otherwise we obtain the contradiction

$$k > |abc| \geqslant |xyz| = |x| = k.$$

Hence

$$|\phi(x,y,z)| \leqslant \binom{m-s}{l-s} = \binom{m-s}{m-l} \leqslant \binom{m-k-1}{m-l} = \binom{m-k-1}{l-k-1},$$

$$n \leqslant p = |A'| \leqslant \binom{m-k-1}{l-k-1}\binom{l}{k}^3,$$

which proves (ii).

Case $2b$. Suppose that (2) holds. We may assume (6), where $l_0 \leqslant l$; $1 \leqslant n_0 \leqslant n$. If $l_0 = l$, then Case $2a$ applies. Now let $l_0 < l$ and use induction over $l-l_0$. Let $b_0,\ldots, \hat{b}_{n_1}$ be the distinct sets b satisfying, for some ν, the relations (3). Then (7) holds and hence, by the lemma, $n_1 \geqslant n_0$. Also, since $l_0 < l < m$, so that $m-l_0 \geqslant 2$, we have, by definition of the b_μ,

$$b_0 \ldots \hat{b}_{n_1} = a_0 \ldots \hat{a}_{n_0}, \qquad |b_0 \ldots \hat{b}_{n_1} a_{n_0} \ldots \hat{a}_n| = |a_0 \ldots \hat{a}_n| < k.$$

Since

$$(b_0,\ldots, \hat{b}_{n_1}, a_{n_0},\ldots, \hat{a}_n) \in S(k,l,m),$$

it follows from our induction hypothesis that

$$n \leqslant n_1+(n-n_0) \leqslant \binom{m-k-1}{l-k-1}\binom{l}{k}^3.$$

It remains to prove (b) in Case 2. If $k = l$, then (b) is trivial. If $k < l$ and $m \geqslant k+(l-k)\binom{l}{k}^3$, then

$$\binom{m-k}{l-k} = \binom{m-k-1}{l-k-1}\frac{m-k}{l-k} \geqslant \binom{m-k-1}{l-k-1}\binom{l}{k}^3,$$

so that (b) follows from (a). This completes the proof of Theorem 2.

7. Concluding remarks

(i) In Theorem 2 (b) the condition

$$m \geqslant k+(l-k)\binom{l}{k}^3,$$

though certainly not best-possible, cannot be omitted. It is possible for

$$(a_0,\ldots, \hat{a}_n) \in S(k,l,m), \qquad k \leqslant l \leqslant m$$

to hold and, at the same time, $n > \binom{m-k}{l-k}$. This is shown by the following example due to S. H. Min and kindly communicated to the authors. Let $a_0,\ldots, \hat{a}_n$ be the distinct sets a such that

$$a \subset [0,8), \qquad |a| = 4, \qquad |a[0,4)| = 3.$$

Then

$$n = 16, \qquad (a_0,\ldots, a_{15}) \in S(2,4,8), \qquad \binom{m-k}{l-k} = \binom{6}{2} = 15 < n.$$

A more general example is the following. Let $r > 0$ and denote by $a_0, \ldots, \hat{a}_n$ the distinct sets a such that

$$a \subset [0, 4r), \qquad |a| = 2r, \qquad |a[0, 2r)| > r.$$

Then
$$(a_0, \ldots, \hat{a}_n) \in S(2, 2r, 4r),$$
and we have

$$n = \sum (r < \lambda \leqslant 2r) \binom{2r}{\lambda}\binom{2r}{2r-\lambda} = \tfrac{1}{2} \sum (\lambda \leqslant 2r) \binom{2r}{\lambda}\binom{2r}{2r-\lambda} - \frac{1}{2}\binom{2r}{r}^2$$
$$= \frac{1}{2}\binom{4r}{2r} - \frac{1}{2}\binom{2r}{r}^2.$$

In this case
$$\binom{m-k}{l-k} = \binom{4r-2}{2r-2},$$
and, for every large r, possibly for every $r > 2$, we have $\binom{m-k}{l-k} < n$. We put forward the conjecture that, for our special values of k, l, m, this represents a case with maximal n, i.e.

If
$$r > 0, \qquad (a_0, \ldots, \hat{a}_n) \in S(2, 2r, 4r),$$
then
$$n \leqslant \frac{1}{2}\binom{4r}{2r} - \frac{1}{2}\binom{2r}{r}^2.$$

(ii) If in the definition of $S(1, l, m)$ in § 2, the condition $a_\mu \not\subset a_\nu \not\subset a_\mu$ is replaced by $a_\mu \neq a_\nu$ and if no restriction is placed upon $|a_\nu|$, then the problem of estimating n becomes trivial, and we have the result:

Let $m > 0$ and $a_\nu \subset [0, m)$ for $\nu < n$, and $a_\mu \neq a_\nu$, $a_\mu a_\nu \neq \varnothing$ for $\mu < \nu < n$. Then $n \leqslant 2^{m-1}$, and there are $2^{m-1} - n$ subsets $a_n, \ldots, \hat{a}_{2^{m-1}}$ of $[0, m)$ such that $a_\mu \neq a_\nu$, $a_\mu a_\nu \neq \varnothing$ for $\mu < \nu < 2^{m-1}$.

To prove this we note that of two sets which are complementary in $[0, m)$ at most one occurs among $a_0, \ldots, \hat{a}_n$, and, if $n < 2^{m-1}$, then there is a pair of complementary sets a, b neither of which occurs among $a_0, \ldots, \hat{a}_n$. It follows that at least one of a, b intersects every a_ν, so that this set can be taken as a_n.

(iii) Let $l \geqslant 3$, $2l \leqslant m$, and suppose that

$$a_\nu \subset [0, m), \qquad |a_\nu| = l \quad \text{for} \quad \nu < n,$$

and

$$a_\mu \neq a_\nu, \qquad a_\mu a_\nu \neq \varnothing \quad \text{for} \quad \mu < \nu < n, \quad \text{and} \quad a_0 \ldots \hat{a}_n = \varnothing.$$

We conjecture that the maximum value of n for which such sets a_ν can be found is n_0, where

$$n_0 = 3\binom{m-3}{l-2} + \binom{m-3}{l-3}.$$

A system of n_0 sets with the required properties is obtained by taking all sets a such that

$$a \subset [0, m), \qquad |a[0,3)| \geqslant 2, \qquad |a| = l.$$

(iv) The following problem may be of interest. Let $k \leqslant m$. Determine the largest number n such that there is a system of n sets a_ν satisfying the conditions

$$a_\mu \neq a_\nu, \qquad |a_\mu a_\nu| \geqslant k \quad (\mu < \nu < n).$$

If $m+k$ is even, then the system consisting of the a such that

$$a \subset [0, m), \qquad |a| \geqslant \tfrac{1}{2}(m+k)$$

has the required properties. We suspect that this system contains the maximum possible number of sets for fixed m and k such that $m+k$ is even.

(v) If in (ii) the condition $a_\mu a_\nu \neq \varnothing$ $(\mu < \nu < n)$ is replaced by $a_\mu a_\nu a_\rho \neq \varnothing$ $(\mu < \nu < \rho < n)$, then the structure of the system a_ν is largely determined by the result:

Let $m \geqslant 2$, $a_\nu \subset [0, m)$ for $\nu < n$, $a_\mu \neq a_\nu$ for $\mu < \nu < n$, and $a_\mu a_\nu a_\rho \neq \varnothing$ for $\mu < \nu < \rho < n$. Then $n \leqslant 2^{m-1}$, and, if $n = 2^{m-1}$, then $a_0 a_1 \ldots \hat{a}_n \neq \varnothing$, so that the a_ν are all 2^{m-1} sets $a \subset [0, m)$ which contain some fixed number t $(t < m)$.

For there is a largest p $(1 \leqslant p \leqslant n)$ such that

$$a_0 a_1 \ldots \hat{a}_p \in \{a_0, a_1, \ldots, \hat{a}_n\}.$$

If $p = n$, then $a_0 \ldots \hat{a}_n = a_\nu \neq \varnothing$ for some $\nu < n$. If $p < n$, then any two of the $n+1$ distinct sets

$$a_0 a_1 \ldots a_p,\ a_0,\ a_1, \ldots,\ \hat{a}_n$$

have a non-empty intersection and hence, by (ii), $n+1 \leqslant 2^{m-1}$. Different proofs of (v) have been found by L. Pósa, G. Hajós, G. Pollák, and M. Simonovits.

REFERENCE

1. E. Sperner, *Math. Z.* 27 (1928) 544–8.