ACTA ARITHMETICA
V (1959)

59/1

Some results on diophantine approximation

by

P. ERDÖS (Haifa, Israel)

Denote by $\varphi(n, \varepsilon, C)$ the set in $a, 0 < a < 1$, for which the inequality

(1) $$\left|a-\frac{p}{q}\right| < \frac{\varepsilon}{q^2}, \quad n < q < Cn, \quad (p, q) = 1$$

is not solvable. In a recent paper Szüsz, Turán and I (see [1]) have obtained various inequalities for $m[\varphi(n, \varepsilon, C)]$ ($m(\varphi)$ denotes the Lebesgue measure of φ). We have conjectured that for every ε and C

$$\lim_{n\to\infty} m[\varphi(n, \varepsilon, C)]$$

exists. So far we have not yet been able to prove this conjecture. At the end of our paper we state without proof the following

THEOREM 1. *For every ε and η, there exists $C = C(\varepsilon, \eta)$ so that for every n*

$$m[\varphi(n, \varepsilon, C)] < \eta.$$

I have now obtained a different proof of this Theorem from the one we had in mind at the time of writing our triple paper; the new proof has also other applications, and thus it seems worth while to give it in detail.

By the same method we can prove the following Theorem, which contains Theorem 1 as a special case.

THEOREM 2. *Let $h(n) > 0$ be a non-decreasing function for which $\sum_{n=1}^{\infty}\left(1/n\,h(n)\right)$ diverges. Then for every $\eta > 0$ there exists a $C_1(\eta)$ so that if*

$$\sum_{n<q<k(n)} \frac{1}{q\,h(q)} > C_1(\eta),$$

then the measure of the set in a for which the inequality

$$\left| a - \frac{p}{q} \right| < \frac{1}{q^2 h(q)}, \quad (p, q) = 1, \quad n < q < k(n),$$

is not solvable is less than η.

We omit the proof of Theorem 2 since it is very similar to that of Theorem 1. We obtain an interesting special case of Theorem 2 by putting $h(n) = \log n$; here $k(n) = n^{e^{C_1(n)}}$.

Finally we shall outline the proof of the following

THEOREM 3. *Let* $l(n) > 0$ *be a non-decreasing function and assume that* $\sum_{n=1}^{\infty} \big(1/l(n)\big)$ *diverges. Denote by* $N(l, a, n)$ *the number of solutions of the equation*

$$ma - [ma] < \frac{1}{l(m)}, \quad 1 \leqslant m \leqslant n.$$

Then for almost all a

$$\lim_{n\to\infty} N(l, a, n) \Big(\sum_{m=1}^{n} \frac{1}{l(m)} \Big)^{-1} = 1.$$

By the same method we can prove the following

THEOREM 3'. *Denote by* $N'(l, a, n)$ *the number of solutions of*

$$0 < qa - p < \frac{1}{l(q)}, \quad (p, q) = 1, \quad 0 < q < n.$$

Then for almost all a

$$\lim_{n\to\infty} N'(l, a, n) \Big(\sum_{q=1}^{n} \frac{1}{l(q)} \Big)^{-1} = \frac{12}{\pi^2}.$$

We omit the proof of Theorem 3' since it is similar to that of Theorem 3. Theorems 3 and 3' should be compared with a recent result by Leveque(1) — Leveque's result is much stronger than ours but applies to a more restricted class of functions.

Throughout this paper $m, n, p, q, r, s, t, \ldots$ will denote integers, Greek letters will denote real numbers, $\varepsilon, \delta_1, \delta_2, \delta_3, \delta_4, \eta$ will denote suitably chosen positive, sufficiently small numbers, Θ will denote a number satisfying $|\Theta| \leqslant 1$, $C_1, C_2, \ldots$ will denote positive constants, C will denote a suitably chosen large constant ($C = C(\varepsilon, \eta, \delta_i)$). We will always

(1) See [2]; through the kindness of Professor Leveque I saw the manuscript of another paper on the same subject, which helped me in writing some parts of this paper.

have $(p,q)=(p_i,q_i)=1$, $0<p<q$, $0<p_i<q_i$. $I_{p,q}$ will denote the interval

$$\left(\frac{p}{q}-\frac{\varepsilon}{q^2},\frac{p}{q}+\frac{\varepsilon}{q^2}\right).$$

Define $f_q(a)$, $0<a<1$, as follows:

$$f_q(a)=\begin{cases}1 & \text{if for some } p \quad |a-p/q|<\varepsilon/q^2,\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 1 will be proved if we show that the measure of the set in a for which $(C=C(\varepsilon,\eta))$

$$\sum_{n<q<Cn} f_q(a)=0$$

is less than η. In fact we shall prove considerably more. Put (clearly $\int_0^1 f_q(a)\,da=2\varepsilon\varphi(q)/q^2$)

$$E_C=\sum_{n<q<Cn}\int_0^1 f_q(a)\,da=2\varepsilon\sum_{n<q<Cn}\frac{\varphi(q)}{q^2}.$$

By partial summation we easily obtain (as $n\to\infty$)

$$E_C=\big(1+o(1)\big)\frac{12\varepsilon}{\pi^2}\log C.$$

We are going to prove that for every η and sufficiently large C

$$\text{(2)}\qquad I=\int_0^1\Big(\sum_{n<q<Cn} f_q(a)-E_C\Big)^2da<\eta E_C^2.$$

From (2) we immediately find by Tchebycheff's inequality that the measure of the set in a for which

$$\Big|\sum_{n<q<Cn} f_q(a)-E_C\Big|>\beta E_C$$

holds is less than η/β^2, and thus the measure of the set with $\sum_{n<q<Cn} f_q(a)=0$ is less than η (here $\beta=1$), which proves Theorem 1.

Thus we only have to prove (2). Clearly by $f_q(a)=f_q^2(a)$ we have for sufficiently large $C=C(\varepsilon,\eta)$ (we omit da since there is no danger of confusion)

$$(3)\qquad I=\int_0^1\Big(\sum_{n<q_1,q_2<Cn} f_{q_1}(a)f_{q_2}(a)\Big)-2E_C\int_0^1\Big(\sum_{n<q<Cn} f_q(a)\Big)+E_C^2$$

$$=2\int_0^1\sum_{n<q_1<q_2<Cn} f_{q_1}(a)f_{q_2}(a)-E_C^2+\int_0^1\sum_{n<q<Cn} f_q^2(a)$$

$$=2\int_0^1\Big(\sum_{n<q_1<q_2<Cn} f_{q_1}(a)f_{q_2}(a)+E_C-E_C^2$$

$$=2\sum+E_C-E_C^2=2\sum-E_C^2+\frac{1}{4}\eta\Theta E_C^2.$$

To estimate

$$\sum=\int_0^1\sum_{n<q_1<q_2<Cn} f_{q_1}(a)f_{q_2}(a)$$

we shall need several lemmas.

LEMMA 1. $\int_0^1 f_{q_1}(a)f_{q_2}(a)<8\varepsilon^2/q_1q_2$.

$f_{q_1}(a)f_{q_2}(a)>0$ holds if and only if for some p_1 and p_2

$$\left|a-\frac{p_1}{q_1}\right|<\frac{\varepsilon}{q_1^2},\qquad \left|a-\frac{p_2}{q_2}\right|<\frac{\varepsilon}{q_2^2}$$

(i. e. if I_{p_1,q_1} and I_{p_2,q_2} overlap). But then

$$(4)\qquad \left|\frac{p_1}{q_1}-\frac{p_2}{q_2}\right|<\varepsilon\left(\frac{1}{q_1^2}+\frac{1}{q_2^2}\right)<\frac{2\varepsilon}{q_1^2},\quad \text{or}\quad |p_1q_2-p_2q_1|<2\varepsilon\frac{q_2}{q_1}.$$

Put $d=(q_1,q_2)$. The number of solutions of $p_1q_2-p_2q_1=a$ is 0 if $a\not\equiv 0 \pmod d$ or $a=0$, and is at most d otherwise. Thus the number of solutions of (4) (in p_1 and p_2) is at most $4\varepsilon q_2/q_1$. Thus the Lemma follows immediately since the intervals I_{p_1,q_1} and I_{p_2,q_2} overlap in an interval of length at most $2\varepsilon/q_2^2$ (i. e. the length of I_{p_2,q_2}).

Now write

$$(5)\qquad \sum=\sum\nolimits_1+\sum\nolimits_2$$

where in $\sum_1$ the summation is extended over the q_1 and q_2 satisfying $n<q_1<q_2<Cn$ satisfying every one of the following three conditions:

a. $(q_1,q_2)>\delta_1^{-1}$,

b. $q_1<q_2<q_1\delta_2^{-1}$,

c. $\sum\frac{1}{r}>\delta_4$,

where, in c., r runs through the primes satisfying $r|q_1q_2$, $r > \delta_3^{-1}$, δ_1, δ_2, δ_3, δ_4 are suitably small numbers, which will be determined later.

LEMMA 2. $\sum_1 < \frac{1}{10}\eta E_C^2$.

We evidently have

$$\sum_1 \leqslant \sum_a + \sum_b + \sum_c \tag{6}$$

where in $\sum_a$ condition a. is satisfied, etc. Thus by Lemma 1 (the dash in the summation indicates that $n < q_1 < q_2 < Cn$, $(q_1, q_2) > \delta_1^{-1}$)

$$\sum_a < 8\varepsilon^2 \sum{}' \frac{1}{q_1q_2} < 8\varepsilon^2 \sum_{d>\delta_1^{-1}} \frac{1}{d^2} \sum_{n/d<q_1<q_2<Cn/d} \frac{1}{q_1q_2} \tag{7}$$

$$< 16\varepsilon^2(\log C)^2 \sum_{d>\delta_1^{-1}} \frac{1}{d^2} < 16\varepsilon^2\delta_1(\log C)^2 < \frac{\eta}{30} E_C^2$$

if $\delta_1 = \delta_1(\varepsilon, \eta)$ is sufficiently small.

Further by Lemma 1

$$\sum_b < 8\varepsilon^2 \sum_{n<q_1<Cn} \frac{1}{q_1} \sum_{q_1<q_2<q_1\delta_1^{-1}} \frac{1}{q_2} < 16\varepsilon^2 \log C \log \delta_2^{-1} < \frac{1}{30}\eta E_C^2 \tag{8}$$

if $C = C(\varepsilon, \eta, \delta_2)$ is large enough.

Next we estimate $\sum_c$. Clearly c. implies that for at least one of the numbers q_1 or q_2 we have

$$\sum_{\substack{r|q \\ r>\delta_3^{-1}}} \frac{1}{r} > \frac{1}{2}\delta_4. \tag{9}$$

From (9) and Lemma 1 we obtain

$$\sum_c < 8\varepsilon^2 \sum_{n<q'<Cn} \frac{1}{q'} \sum{}' \frac{1}{q} < 16\varepsilon^2 \log C \sum{}' \frac{1}{q}, \tag{10}$$

where in $\sum'$ the summation is extended over the $n < q < Cn$ satisfying (9). We have

$$\sum_{q=1}^{x} \sum_{\substack{r|q \\ r>\delta_3^{-1}}} \frac{1}{r} < \sum_{r>\delta_3^{-1}} \left[\frac{x}{r}\right] \frac{1}{r} < x \sum_{r>\delta_3^{-1}} \frac{1}{r^2} < x\delta_3.$$

Thus the number of integers $q \leqslant x$ satisfying (9) is less than

$$2\delta_3 x/\delta_4 < \delta_5 x \quad \text{if} \quad \delta_3 < \tfrac{1}{2}\delta_5\delta_4.$$

Hence by partial summation

$$\sum{}' \frac{1}{q} < 4\delta_5 \log C. \tag{11}$$

From (10) and (11) we obtain

$$\sum\nolimits_{c} < 64\varepsilon^2\delta_5 \log C < \tfrac{1}{30}\eta E_c^2 \tag{12}$$

for sufficiently small δ_5. Lemma 2 follows form (6), (7), (8) and (12).

Now we estimate $\sum_2$. First we prove

LEMMA 3. *Assume that $n < q_1 < q_2 < Cn$ and that the pair q_1, q_2 does not satisfy,* a., b. *or* c. *Then for some $|\Theta| < 1$*

$$\int_0^1 f_{q_1}(a) f_{q_2}(a) = \left(1 + \frac{\eta\Theta}{20}\right)\frac{4\varepsilon^2\varphi(q_1)\varphi(q_2)}{q_1^2 q_2^2}.$$

The Lemma implies that $\int_0^1 f_{q_1}(a) f_{q_2}(a)$ nearly equals $\int_0^1 f_{q_1}(a)\int_0^1 f_{q_2}(a)$, or the $f_q(a)$ behave in some respects as if they were independent functions.

The intervals I_{p_1,q_1} and I_{p_2,q_2} overlap if and only if (4) holds. Clearly if

$$\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| < \varepsilon\left(\frac{1}{q_1^2} - \frac{1}{q_2^2}\right), \quad \text{or} \quad |p_1 q_2 - p_2 q_1| < \varepsilon\left(\frac{q_2}{q_1} - \frac{q_1}{q_2}\right)$$

then I_{p_2,q_2} is contained in I_{p_1,q_1}. Thus

$$\frac{2\varepsilon}{q_2^2} \sum_{|a| < \varepsilon\left(\frac{q_2}{q_1} - \frac{q_1}{q_2}\right)} g(a) < \int_0^1 f_{q_1}(a) f_{q_2}(a) < \frac{2\varepsilon}{q_2^2} \sum_{|a| < \varepsilon\left(\frac{q_2}{q_1} + \frac{q_1}{q_2}\right)} g(a) \tag{13}$$

where $g(a)$ denotes the number of solutions in p_1 and p_2 of

$$a = p_1 q_2 - p_2 q_1. \tag{14}$$

Put $(q_1, q_2) = d \leqslant \delta_1^{-1}$. Clearly $g(a) \leqslant d \leqslant \delta_1^{-1}$ (by a.), and since by b. there is at most one integer in the interval

$$\left(\varepsilon\left(\frac{q_2}{q_1} - \frac{q_1}{q_2}\right), \varepsilon\left(\frac{q_2}{q_1} + \frac{q_1}{q_2}\right)\right)$$

the right and left sides of (13) differ by less than $2\varepsilon d/q_2^2 \leqslant 2\varepsilon/q_2^2\delta_1$. Thus we have

$$\int_0^1 f_{q_1}(a)f_{q_2}(a) = \frac{2\varepsilon}{q_2^2}\sum_{|a|<\varepsilon q_2/q_1} g(a)+\frac{2\varepsilon\Theta}{q_2^2\delta_1}, \tag{15}$$

for some $|\Theta| \leqslant 1$. (Θ will always denote a real number satisfying $|\Theta| \leqslant 1$, but it will necessarily be the same number.) Clearly $g(a) = 0$ if $a \not\equiv 0 \pmod d$. Put $a' = a/d$, $q_1' = q_1/d$, $q_2' = q_2/d$. Clearly $g(a) = 0$ unless

$$a \equiv 0 \pmod d, \quad (a', q_1'q_2') = 1. \tag{16}$$

LEMMA 4. *Assume that a satisfies* (16). *Then*

$$g(a) = d\prod\left(1-\frac{2}{r}\right)\prod\left(1-\frac{1}{s}\right),$$

where the r's are the prime factors of d for which $r \nmid a'q_1'q_2'$ and the s run through all the other prime factors of d.

To prove the Lemma observe that clearly

$$a' = p_1q_2' - p_2q_1'$$

has a unique solution in

$$0 < p_1 < q_1', \quad 0 < p_2 < q_2', \quad (p_1, q_1') = (p_2, q_2') = 1.$$

We obtain $g(a)$ by determining the number of integers u satisfying

$$(p_1+uq_1', d) = (p_2+uq_2', d) = 1, \quad 0 \leqslant u < d. \tag{17}$$

Clearly every solution of (17) satisfies (14), and (14) can have no other solutions. Thus we have to determine the number of solutions of (17). Let t be a prime factor of d. By $(q_1', q_2') = 1$, $t|q_1'$ and $t|q_2'$ cannot both hold. If $t|q_1'$ then (17) implies $u \not\equiv -p_2/q_2' \pmod t$, if $t|q_2'$ then (17) implies $u \not\equiv -p_1/q_1' \pmod t$. If $t \nmid q_1'q_2'$ then $u \not\equiv -p_1/q_1' \pmod t$, $u \not\equiv -p_2/q_2' \pmod t$. These two residues coincide if and only if $t|a'$. Thus Lemma 4 follows by a simple sieve process.

Now we return to the proof of Lemma 3. Let $u_1, u_2, \ldots, u_d$ run through a complete set of residues $(\bmod d)$ where we further assume that $p \nmid u_i$ for every prime factor of $q_1'q_2'$ which is not also a prime factor of d. (In fact unless $(u_i, q_1'q_2') = 1$ we find from (16) that $g(u_id) = 0$, but if we did not exclude the prime factors of d in the condition $p \nmid u_i$, the u's could not run through a complete set of residues $(\mathrm{mod}\, d)$). From Lemma 4 and (16) we obtain by a simple argument

$$\sum_{i=1}^{d} g(du_i) = d^2 \prod \left(1 - \frac{1}{t}\right)^2 \tag{18}$$

where t runs through all the prime factors of d.

Denote by $N(z, u)$, $z = [q_2 \varepsilon / q_1]$ the number of integers m satisfying

$$1 \leqslant m \leqslant z/d, \quad m \equiv u (\mathrm{mod}\, d), \quad (m, t') = 1$$

where t' runs through all prime factors of $q_1' q_2'$ which do not divide d. A simple sieve process (the details of which can be left to the reader) shows that for some $|\Theta| \leqslant 1$

$$N(z, u) = \frac{z}{d^2} \left(1 + \frac{\Theta \eta}{40}\right) \prod \left(1 - \frac{1}{t'}\right) \tag{19}$$

if z/d^2 is sufficiently large (i. e. $\delta_2 = \delta_2(\eta, \delta_1, \delta_3, \delta_4)$ is sufficiently small).

From (18) and (19) we easily find that (since as a' runs from 1 to z/d through the integers relatively prime to $q_1' q_2'$, (19) shows that it runs through at least

$$\frac{z}{d^2} \left(1 - \frac{\eta}{40}\right) \prod \left(1 - \frac{1}{t'}\right)$$

and at most through

$$\frac{z}{d^2} \left(1 + \frac{\eta}{40}\right) \prod \left(1 - \frac{1}{t'}\right)$$

complete set of residues $\mathrm{mod}\, d$)

$$\sum_{a=1}^{z} g(a) = \sum_{a'=1}^{z/d} g(a') = z \left(1 + \frac{\eta \Theta}{40}\right) \prod \left(1 - \frac{1}{t}\right)^2 \prod \left(1 - \frac{1}{t'}\right) \tag{20}$$

$$= z \left(1 + \frac{\eta \Theta}{40}\right) \frac{\varphi(q_1) \varphi(q_2)}{q_1 q_2}.$$

Thus finally by (15) and (20) we have

$$\int_0^1 f_{q_1}(a) f_{q_2}(a) = \left(1 + \frac{\eta \Theta}{40}\right) \frac{4 \varepsilon^2 \varphi(q_1) \varphi(q_2)}{q_1^2 q_2^2} + \frac{2 \varepsilon \Theta}{q_2^2 \delta_1}. \tag{21}$$

Now we have by a simple computation for sufficiently small $\delta_2 = \delta_2(\varepsilon, \eta, \delta_1, \delta_3, \delta_4)$

$$(22) \qquad \frac{1}{q_2^2 \delta_1} < \frac{\eta\varepsilon^2}{40} \cdot \frac{\varphi(q_1)\varphi(q_2)}{q_1^2 q_2^2}$$

(21) and (22) clearly implies Lemma 3.

From Lemma 3 we have

$$(23) \qquad \sum\nolimits_2 = \left(1 + \frac{\eta\Theta}{20}\right) \sum\nolimits_2 \frac{4\varepsilon^2 \varphi(q_1)\varphi(q_2)}{q_1^2 q_2^2}$$

and from the proof of Lemma 2 we have

$$(24) \qquad \sum\nolimits_1 \frac{4\varepsilon^2 \varphi(q_1)\varphi(q_2)}{q_1^2 q_2^2} < \frac{\eta}{10} E_C^2.$$

Thus from (23), (24), and (5) we have

$$(25) \qquad \sum = \sum\nolimits_1 + \sum\nolimits_2 = \left(1 + \frac{\eta\Theta}{20}\right) \sum_{n < q_1 < q_2 < Cn} \frac{4\varepsilon^2 \varphi(q_1)\varphi(q_2)}{q_1^2 q_2^2} + \frac{\eta\Theta}{5} E_C^2$$

$$= \frac{1}{2}\left(1 + \frac{\eta\Theta}{10}\right) E_C^2 + \frac{\eta\Theta}{5} E_C^2 = \frac{1}{2} E_C^2 + \frac{\eta\Theta}{4} E_C^2.$$

(25) and (3) imply (2), and thus the proof of Theorem 1 is complete.

Now we outline the proof of Theorem 3. The most interesting special case is $l(n) = n$ and to save complications we will only prove our Theorem in this case. Thus we have to prove that the number of solutions $N_a(n)$ of

$$0 < ta - [ta] < \frac{1}{t}, \quad 0 < t < n,$$

satisfies for almost all a the relation

$$(26) \qquad N_a(n)/\log n \to 1.$$

Now define

$$F_q(a) = \begin{cases} k & \text{if for some} \quad p, \qquad \dfrac{1}{q^2(k+1)^2} < a - \dfrac{p}{q} \leqslant \dfrac{1}{q^2 k^2}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\sum_{q=1}^{n} F_q(a) \geqslant N_a(n)$. Define further

$$F_q'(a) = \begin{cases} F_q(a) & \text{if} \quad F_q(a) < (\log q)^2, \\ 0 & \text{otherwise.} \end{cases}$$

A well-known theorem of Khintchine asserts that for almost all α the inequality

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^2(\log q)^2}$$

has only a finite number of solutions. Thus for almost all α

$$\sum_{q=1}^{n}\big(F_q(\alpha)-F_q'(\alpha)\big)=O(1). \tag{27}$$

Also a simple argument shows that

$$\sum_{q=1}^{n}F_q'(\alpha)\leqslant N_\alpha\big(n(\log n)^2\big).$$

Thus to prove Theorem 3 it will suffice to prove that for almost all α

$$\frac{1}{\log n}\sum_{q=1}^{n}F_q'(\alpha)\to 1. \tag{28}$$

As in the proof of Theorem 1, put

$$I=\int_0^1\Big(\sum_{q=1}^{n}F_q'(\alpha)-E_n\Big)^2 d\alpha \tag{29}$$

where

$$E_n=\int_0^1\Big(\sum_{q=1}^{n}F_q'(\alpha)\Big)=\sum_{q=1}^{n}\frac{\varphi(q)}{q^2}\Big(1+\frac{1}{2^2}+\ldots+\frac{1}{[(\log q)^2]^2}\Big) \tag{30}$$

$$=\sum_{q=1}^{n}\frac{\varphi(q)}{q^2}\Big(\frac{\pi^2}{6}-\frac{\Theta}{(\log q)^2}\Big)=\log n+O(\log\log n).$$

Further a simple computation shows that

$$\int_0^1\big(F_q'(\alpha)\big)^2<\frac{c_1\varphi(q)\log\log q}{q^2}. \tag{31}$$

Thus from (29), (30) and (31) we obtain as in (3)

$$I=2\sum-E_n^2+O(\log n\log\log n), \tag{32}$$

where

$$\sum=\int_0^1\sum_{1\leqslant q_1<q_2\leqslant n}F_{q_1}(\alpha)F_{q_2}(\alpha). \tag{33}$$

Now we write

$$\sum = \sum\nolimits_1 + \sum\nolimits_2 \tag{34}$$

where in $\sum_1$

$$q_2 \leqslant q_1 \exp\left((\log n)^{1/2}\right) \quad (\exp z = e^z)$$

and in $\sum_2$

$$q_2 > q_1 \exp\left((\log n)^{1/2}\right).$$

As in Lemma 1, we can prove that

$$\int_0^1 F_{q_1}(a) F_{q_2}(a) < \frac{c}{q_1 q_2} \tag{35}$$

if $q_1 < q_2 \leqslant q_1 \exp\left((\log n)^{1/2}\right)$. Further as in Lemma 3 for $q_2 > q_1 \times \times \exp\left((\log n)^{1/2}\right)$

$$\int_0^1 F'_{q_1}(a) F'_{q_2}(a) = \left(1 + \frac{\Theta}{(\log n)^{1/10}}\right) \frac{1}{q_1 q_2}. \tag{36}$$

Thus from (32), (33), (34), (35), and (36) we finally obtain

$$I < c(\log n)^{2-1/10}. \tag{37}$$

From (37) we infer by Tchebycheff's inequality that the measure of the set (in a) for which

$$\left|\sum_{q=1}^{n} F'_q(a) - \log n\right| > \varepsilon \log n$$

is less than $\frac{c}{\varepsilon^2}(\log n)^{-1/10}$, and the proof of (28) proceeds by well-known arguments.

The factor $(\log n)^{-1/10}$ in (36) could easily be improved to say $(\log n)^{-2}$ but the q_1 and q_2 in Σ_1 cause considerable difficulties and because of these I have found it impossible to obtain a result analogous to the central limit theorem which would generalize and strengthen the results of Leveque.

References

[1] P. Erdös, P. Szüsz, P. Turán, *Remarks on the theory of diophantine approximation*, Coll. Math. 6 (1958), p. 119-126.

[2] W. J. Leveque, *On the frequency of small fractional parts in certain real sequences*, Trans. Amer. Math. Soc. 87 (1958), p. 237-260.

TECHNION, HAIFA, ISRAEL

Reçu par la Rédaction le 15. 12. 1958