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## On the structure of inner set mappings

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SZEGEDI TUDOMÁNYEGYETEM BOLYAI-INTÉZETE

# On the structure of inner set mappings

By P. ERDŐS in Haifa, G. FODOR in Szeged, and A. HAJNAL in Budapest

Let $S$ be a given set of power $\mathfrak{m}$, $I_1$ and $I_2$ two arbitrary classes of subsets of $S$. A function $G(X)$ is called a set mapping if $G(X)$ is defined on $I_1$ and such that, for each $X \in I_1$, $G(X) \in I_2$. We say that $G(X)$ is an *inner set mapping* if, for each $X \in I_1$, $G(X) \subset X$. Let further $X_0 \in I_2$, we define the inverse of $X_0$ in two different ways, first as the set

$$\bigcup_{G(X)=X_0} X = X_0^{-1}$$

and second as the set

$$\{X : G(X) = X_0\} = X_0^{*-1}.$$

The set of all subsets of power $\mathfrak{n}$ and the set of all subsets of power $<\mathfrak{n}$ of $S$ are denoted by $[S]^{\mathfrak{n}}$ and $[S]^{<\mathfrak{n}}$, respectively. If $I_1 = [S]^{\mathfrak{n}}$ or $I_1 = [S]^{<\mathfrak{n}}$, then a set mapping defined on $I_1 = [S]^{\mathfrak{n}}$ or $I_1 = [S]^{<\mathfrak{n}}$ is called a set mapping of type $\mathfrak{n}$ or type $<\mathfrak{n}$, respectively. If for a set mapping $G(X)$ is $I_2 = [S]^{\mathfrak{n}}$ or $I_2 = [S]^{<\mathfrak{n}}$, then $G(X)$ is called a set mapping of range $\mathfrak{n}$ or range $<\mathfrak{n}$, respectively.

We introduce now the symbols $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{r}$. These symbols indicate that for every set mapping of the type $\mathfrak{q}$ and range $\mathfrak{p}$, defined on the set $S$ of power $\mathfrak{m}$, there exists an element $X_0 \in [S]^{\mathfrak{p}}$ for which $\overline{\overline{X_0^{-1}}} = \mathfrak{r}$ or $\overline{\overline{X_0^{*-1}}} = \mathfrak{r}$, respectively. The symbol $((\mathfrak{m}, <\mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$ has an analogous meaning. The same symbols, with $\to$ replaced by $\not\to$, indicate the negation of the corresponding statement.

It is obvious, that we have to suppose $\mathfrak{m} \geqq \mathfrak{q} \geqq \mathfrak{p}$. We prove in this paper the following results:

a) *negative results* ($\mathfrak{q} \geqq \aleph_0$):

1) if $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{q}^{\mathfrak{p}}$, then $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \not\to \mathfrak{q}^+$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \not\to 2$,

2) if $\mathfrak{p} = \mathfrak{q}$, then $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \not\to \mathfrak{q}^+$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \not\to 2$.

b) *positive results* ($\mathfrak{q} \geqq \aleph_0$):

1) $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$ if $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$,

2) $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$ if $\mathfrak{q}^{\mathfrak{p}} < (\mathfrak{m}^{\mathfrak{q}})^*$ and $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{q}}$.

These results make possible with the aid of the generalized continuum hypothesis, the discussion in almost every case. We can obviously assume, that $\mathfrak{p} < \mathfrak{q}$ and $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^{\mathfrak{q}}$. Thus we can state:

c) $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$, if $\mathfrak{q}^{\mathfrak{p}} \neq \mathfrak{m}^*$ or $\mathfrak{q} \geqq \mathfrak{m}^*$. Thus the only open question is the following:

Is it true, that $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$ or $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$ if $\mathfrak{m} = \aleph_\alpha$, $\alpha$ is of second kind, $\mathfrak{q} = \aleph_{cf(\alpha)-1}$, $cf(\alpha)-1$ is of second kind and $\mathfrak{p} = \aleph_\beta$ with $\beta \geqq cf(cf(\alpha)-1)$?; for instance in the simplest case:

$$((\aleph_{\omega_{\omega+1}}, \aleph_0, \aleph_\omega)) \to \aleph_{\omega_{\omega+1}}\,?$$

or

$$((\aleph_{\omega_{\omega+1}}, \aleph_0, \aleph_\omega))^* \to \aleph_{\omega_{\omega+1}}^{\aleph_\omega} = \aleph_{\omega_{\omega+1}}\,?$$

d) if $0 < k < l < \infty$, then $((\aleph_{\alpha+k}, k, l)) \to \aleph_\alpha$;
if $0 < k < l < \infty$, then $((\aleph_{\alpha+k}, k, l)) \not\to \aleph_{\alpha+1}$.

e) Finally we deal with the symbol $((\mathfrak{m}, <\mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$. If $\mathfrak{p} < \mathfrak{q}$, then the validity of the symbol $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$ implies the validity of $((\mathfrak{m}, <\mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$. This holds in the case too, if $\mathfrak{p} = \mathfrak{q}$ and $\mathfrak{q} = \aleph_\alpha$ has an index of first kind. If $\mathfrak{q}$ is regular, $\mathfrak{q} \geqq \aleph_0$, and $\mathfrak{r}^{\mathfrak{n}} < \mathfrak{m}^*$ for every $\mathfrak{r} < \mathfrak{q}$ and $\mathfrak{n} < \mathfrak{q}$, then $((\mathfrak{m}, <\mathfrak{q}, \mathfrak{q})) \to \mathfrak{m}$; thus in particular $((\mathfrak{m}, <\aleph_0, \aleph_0)) \to \mathfrak{m}$. The simplest unsolved problem with respect to the symbol $((\mathfrak{m}, <\mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$ is the following:

$$((\aleph_{\omega+2}, <\aleph_\omega, \aleph_\omega)) \to \aleph_{\omega+1} \text{ or } \aleph_{\omega+2}\,?$$

Set mappings of type 1 and range $\mathfrak{n}$ or $<\mathfrak{n}$ have been investigated previously in [1], [2], [3], [4].

Notations and definitions. Throughout this paper, the symbols $\overline{\overline{S}}$ and $\overline{\beta}$ denote the cardinal number of $S$ and the ordinal number $\beta$, respectively. For any cardinal number $\mathfrak{r} (= \aleph_\alpha)$ we denote by $\varphi_{\mathfrak{r}}$ the initial number of $\mathfrak{r}$, by $\mathfrak{r}^*$ the smallest cardinal number for which $\mathfrak{r}$ is the sum of $\mathfrak{r}^*$ cardinal numbers each of which is smaller than $\mathfrak{r}$, by $cf(\alpha)$ the index $\beta$ of the initial number $\omega_\beta$ of $\mathfrak{r}^*$, by $\mathfrak{r}^+$ the cardinal number immediately following $\mathfrak{r}$.

## I.

We prove now negative results with respect to the symbols $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{r}$. First we prove the following:

Theorem 1. *Let* $\mathfrak{p}$, $\mathfrak{q}$ *and* $\mathfrak{m}$ *be cardinal numbers such that* $\mathfrak{m} \geqq \mathfrak{q} \geqq \mathfrak{p} \geqq \aleph_0$. *If* $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{q}^{\mathfrak{p}} = \mathfrak{r}$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \not\to \mathfrak{q}^+$.

Proof. Let $\overline{\overline{S}} = \mathfrak{m}$. We define on $S$ a one to one set mapping $G(X)$ of type $\mathfrak{q}$ and range $\mathfrak{p}$ which shows that the theorem is true. By the hypothesis

$$\overline{\overline{[S]^{\mathfrak{q}}}} = \mathfrak{r}.$$

Let

$$X_0, X_1, \ldots, X_\omega, X_{\omega+1}, \ldots, X_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{r}})$$

be a well-ordering of the set $[S]^{\mathfrak{q}}$ of the type $\varphi_{\mathfrak{r}}$. We define $G(X)$ by transfinite induction as follows. Let $G(X_0)$ be an arbitrary subset of $X_0$ of power $\mathfrak{p}$, and $\nu$ a given ordinal number, $0 < \nu < \varphi_{\mathfrak{r}}$. Suppose that all sets $G(X_\mu)$, where $0 \leqq \mu < \nu$, have been already defined such that

1) $\overline{\overline{G(X_\mu)}} = \mathfrak{p}$, for $\mu < \nu$,
2) $G(X_\mu) \subset X_\mu$, for $\mu < \nu$,
3) $G(X_{\mu_1}) \neq G(X_{\mu_2})$ for $\mu_1 < \mu_2 < \nu$.

Since the power of the set $[X_\nu]^{\mathfrak{p}}$ is $\mathfrak{r}$ too, there exists a subset of $X_\nu$ of power $\mathfrak{p}$ which is distinct from each $G(X_\mu)$ with index $\mu < \nu$, because $\nu < \varphi_{\mathfrak{r}}$. Let $G(X_\nu)$ be such a subset of $X_\nu$. Then $\overline{\overline{G(X_\nu)}} = \mathfrak{p}$, $G(X_\nu) \subset X_\nu$ and $G(X_\mu) \neq G(X_\nu)$ for $\mu < \nu$. Thus $G(X)$ is defined for every element of $[S]^{\mathfrak{q}}$ and it is a one to one inner set mapping of type $\mathfrak{q}$ and range $\mathfrak{p}$. The theorem is proved.

Corollary 1. *If $2^{\aleph_\beta} = \aleph_{\beta+1}$ for every $\beta$, then* $((\aleph_{\omega_\alpha+1}, \aleph_\alpha, \aleph_{\omega_\alpha})) \not\rightarrow \aleph_{\omega_\alpha+1}$.

It follows from the proof of Theorem 1 also the following

Theorem 2. *Let $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{m}$ be cardinal numbers such that $\mathfrak{m} \geqq \mathfrak{q} \geqq \mathfrak{p} \geqq \aleph_0$. If $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{q}^{\mathfrak{p}}$, then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \not\rightarrow 2$.

Theorem 3. *If $\mathfrak{q} \geqq \aleph_0$, then $((\mathfrak{m}, \mathfrak{q}, \mathfrak{q})) \not\rightarrow \mathfrak{q}^+$ for every cardinal number $\mathfrak{m} > \mathfrak{q}$.*

Instead of Theorem 3 we prove the following stronger result:

Theorem 4. *Let $S$ be a set of power $\mathfrak{m} > \mathfrak{q}$. There exists a function $G(X)$ defined on $[S]^{\mathfrak{q}}$ with the following properties:*

(1) *$G(X) \subset X$ and $X - G(X) \neq 0$ for every $X \in [S]^{\mathfrak{q}}$*

(2) *$G(X) \in [S]^{\mathfrak{q}}$ for every $X \in [S]^{\mathfrak{q}}$;*

(3) *$G(X) \neq G(Y)$ if $X$ and $Y$ are two distinct elements of $[S]^{\mathfrak{q}}$;*

(4) *for every $Y \in [S]^{\mathfrak{q}}$ there exists an element $X \in [S]^{\mathfrak{q}}$ such that $Y = G(X)$.*[1]

Proof. Let $E$ be a set of power $\mathfrak{n} \geqq \mathfrak{q}$; we prove that there exists a a function $F(X)$ defined on $[E]^{\mathfrak{q}}$ which satisfies the conditions (1), (2), and (3).

We consider two cases: (i) $\overline{\overline{E}} = \mathfrak{q}$, and (ii) $\overline{\overline{E}} > \mathfrak{q}$.

*Ad* (i). Let

$$X_0, X_1, \ldots, X_\omega, \ldots, X_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{r}})$$

[1]) For the proof of Theorem 1 it is sufficient that $G(X)$ satisfy the conditions (1), (2), and (3). This theorem is proved in [5].

be a well-ordering of $[E]^{\mathfrak{q}}$ of the type $\varphi_{\mathfrak{r}}$, where $\mathfrak{r}=2^{\mathfrak{q}}$. We define $F(X)$ by transfinite induction as follows. Let $F(X_0)$ be an arbitrary proper subset of $X_0$ of power $\mathfrak{q}$, and $\beta$ a given ordinal number, $0<\beta<\varphi_{\mathfrak{r}}$. Suppose that all sets $F(X_\xi)$, where $0\leqq\xi<\beta$, have been already defined such that the conditions (1), (2), (3) are satisfied. Since the power of the set $[X_\beta]^{\mathfrak{q}}$ is $2^{\mathfrak{q}}$, and $\overline{\beta}<2^{\mathfrak{q}}$, there is a subset $Y$ of $X_\beta$, of power $\mathfrak{q}$, such that $X_\beta - Y \neq 0$ and $Y$ is distinct from each $F(X_\xi)$ with index $\xi<\beta$. Let $F(X_\beta)=Y$. Thus $F(X)$ is defined for every element of $[E]^{\mathfrak{q}}$ such that the conditions (1), (2), and (3) are satisfied.

*Ad* (ii) Consider the set $\mathbf{M}$ of all subsets $M$ of $[E]^{\mathfrak{q}}$ such that if $X$ and $Y$ are two distinct elements of $M$ then $\overline{\overline{X\cap Y}}<\mathfrak{q}$. By ZORN's Lemma there is a maximal element $M_0$ of $\mathbf{M}$. Let

$$Z_0, Z_1, \ldots, Z_\omega, Z_{\omega+1}, \ldots, Z_\xi, \ldots \qquad (\xi<\varphi_{\mathfrak{i}})$$

be a well-ordering of $M_0$ of the type $\varphi_{\mathfrak{i}}$, where $\mathfrak{i}=\overline{\overline{M_0}}$. Since $\overline{\overline{Z_\xi}}=\mathfrak{q}$ for every $\xi<\varphi_{\mathfrak{i}}$, there exists a function $F_\xi(Z)$ on $[Z_\xi]^{\mathfrak{q}}$ which satisfies the conditions (1), (2), and (3). Let now $X\in[E]^{\mathfrak{q}}$. By the definition of $M_0$ there is a smallest ordinal number $\nu=\nu(X)$ for which $\overline{\overline{X\cap Z_\nu}}=\mathfrak{q}$. Let

$$F(X)=F_{\nu(X)}(X\cap Z_{\nu(X)})\cup(X-Z_{\nu(X)}).$$

It is obvious that $F(X)$ satisfies the conditions (1) and (2). For the proof of (3) let $Y\neq X$ be another element of $[E]^{\mathfrak{q}}$ Then

$$F(Y)=F_{\nu(Y)}(Y\cap Z_{\nu(Y)})\cup(Y-Z_{\nu(Y)}).$$

There are two cases: 1) $\nu(X)=\nu(Y)$, 2) $\nu(X)\neq\nu(Y)$.

*Ad 1.* If $X\cap Z_{\nu(X)}\neq Y\cap Z_{\nu(X)}$, then by the definition of $F_{\nu(X)}$ $F_{\nu(X)}(X\cap Z_{\nu(X)})\neq F_{\nu(X)}(Y\cap Z_{\nu(X)})$. We may assume that $F_{\nu(X)}(Y\cap Z_{\nu(X)})$ does not contain $F_{\nu(X)}(X\cap Z_{\nu(X)})$. Let $x_0\in F_{\nu(X)}(X\cap Z_{\nu(X)})$ such that $x_0\notin F_{\nu(X)}(Y\cap Z_{\nu(X)})$. By the condition $F_{\nu(X)}(Z)\subset Z$, we have that $x_0\in X\cap Z_{\nu(X)}$. It follows that $x_0\notin Y-Z_{\nu(X)}$; consequently $x_0\notin F_{\nu(X)}(Y\cap Z_{\nu(X)})\cup(Y-Z_{\nu(X)})$ i.e. $F(X)\neq F(Y)$.

If $X\cap Z_{\nu(X)}=Y\cap Z_{\nu(X)}$, then, since $X\neq Y$, $X-Z_{\nu(X)}\neq Y-Z_{\nu(X)}$; consequently, by the definition of $F$, $F(X)\neq F(Y)$.

*Ad 2.* We may suppose that $\nu(X)<\nu(Y)$. By the definition of $M_0$, $\overline{\overline{Z_{\nu(X)}\cap Z_{\nu(Y)}}}<\mathfrak{q}$, i.e. $\overline{\overline{(X\cap Z_{\nu(X)})\cap(Y\cap Z_{\nu(Y)})}}<\mathfrak{q}$ consequently $F(X)\neq F(Y)$. Thus $F(X)$ statisfies the condition (3) too.

Let now $F$ be a set of power $\mathfrak{r}>\mathfrak{q}$. It is easy see that there exists a function $H(X)$ on $[F]^{\mathfrak{q}}$ such that

a) $X\subset H(X)$ and $H(X)-X\neq 0$,

b) $\overline{\overline{H(X)}}=\mathfrak{q}$,

c) $H(X)\neq H(Y)$ if $X\neq Y$.

We apply now the following theorem of BANACH [6]: If the function $\varphi$ maps the set $A$ one to one onto a subset of $B$ and the function $\psi$ maps the set $B$ one to one onto a subset of $A$, then there exists a decomposition $A = A_1 \cup A_2$ of $A$ and a decomposition $B = B_1 \cup B_2$ of $B$ such that $A_1 \cap A_2 = B_1 \cap B_2 = 0$, $\varphi(A_1) = B_1$ and $\psi(B_2) = A_2$.

Let now $A = B = [S]^{\mathfrak{q}}$ ($\overline{S} = \mathfrak{m} > \mathfrak{q}$). Let further $\varphi$ be a function on $[S]^{\mathfrak{q}}$ such that the conditions (1), (2), (3), and $\psi$ a function on $[S]^{\mathfrak{q}}$ such that the conditions a), b), c) hold respectively. Then there exist two decompositions $[S]^{\mathfrak{q}} = A^1 \cup A_2 = B_1 \cup B_2$ such that $A_1 \cap A_2 = B_1 \cup B_2 = 0$, $\varphi(A_1) = B_1$ and $\psi(B_2) = A_2$. We define $G(X)$ on $[S]^{\mathfrak{q}}$ as follows. Let

$$G(X) = \begin{cases} \varphi(X), & \text{if } X \in A_1, \\ \psi^{-1}(X), & \text{if } X \in A_2. \end{cases}$$

Obviously $G(X)$ satisfies the conditions (1), (2), (3) and (4).

The proof of Theorem 4 gives also the following

Theorem 5. *If* $\mathfrak{q} \geqq \aleph_0$, *then* $((\mathfrak{m}, \mathfrak{q}, \mathfrak{q}))^* \not\to 2$.

## II.

We assume in this chapter that $\mathfrak{p} < \mathfrak{q}$, $\mathfrak{q} \geqq \aleph_0$ and $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^{\mathfrak{q}}$ and prove:

Theorem 6. *If* $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$.

Proof. Suppose that the theorem is not true, i. e. for every subset $P$ of power $\mathfrak{p}$

$$\overline{\overline{\bigcup_{G(Q)=P} Q}} < \mathfrak{m}.$$

By the condition,

$$\overline{\overline{\bigcup_{P' \subseteq P} \bigcup_{G(Q)=P'} Q}} < \mathfrak{m}$$

for every subset $P$ of $S$ of power $\mathfrak{p}$.

We define now by transfinite induction a sequence $\{P_\xi\}_{\xi < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{q}+}}$ of the type $\varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}$ of the subsets of $S$ of power $\mathfrak{p}$ as follows. Let $P_0$ be an arbitrary subset of $S$ of power $\mathfrak{p}$ and $\beta$ a given ordinal number, $0 < \beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}$. Suppose that all sets $P_\xi$, where $0 \leqq \xi < \beta$, have been already defined, and let $A_\beta = \bigcup_{\xi < \beta} P_\xi$. Since $\beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}$ and $\overline{P}_\xi = \mathfrak{p} < \mathfrak{q}$ we have $\overline{\overline{A}}_\beta \leqq \mathfrak{q}$. It follows by the hypothesis $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$ that

$$\overline{\overline{\bigcup_{P \subseteq A_\beta} \bigcup_{G(Q)=P} Q}} < \mathfrak{m}.$$

We define the set $P_\beta$ as a subset of power $\mathfrak{p}$, of the set

$$S - \bigcup_{\xi < \beta} P_\xi - \bigcup_{P \subseteq A_\beta} \bigcup_{G(Q)=P} Q.$$

Put

(1)
$$H=\bigcup_{\xi<\varphi_{\mathfrak q}+\varphi_{\mathfrak p+}} P_\xi .$$

It is obvious that $\overline{\overline{H}}=\mathfrak q$. It follows that there exists a subset $P$ of $H$ of power $\mathfrak p$ such that $G(H)=P$. Since $\mathfrak p^+$ is regular there exists an ordinal number $\beta<\varphi_{\mathfrak q}+\varphi_{\mathfrak p+}$ such that

$$P\subseteq\bigcup_{\xi<\beta}P_\xi=A_\beta .$$

But then clearly by the definition of $P_\beta$, $P_\beta \not\subseteq H$, which contradicts (1).

Corollary 2. *If* $2^{\aleph_\beta}=\aleph_{\beta+1}$ *for every* $\beta$, *then* $((\aleph_{\omega_\alpha+2},\aleph_\alpha,\aleph_{\omega_\alpha}))\to \to\aleph_{\omega_\alpha+2}$.

Theorem 7. *If* $\mathfrak p<\mathfrak q^*$ *and* $\mathfrak r^{\mathfrak p}<\mathfrak m^*$ *for every* $\mathfrak r<\mathfrak q$, *then* $((\mathfrak m,\mathfrak p,\mathfrak q))\to\mathfrak m$.

The proof of Theorem 7 is similar to the proof of Theorem 6.

Remark. If $\mathfrak q<\mathfrak m^*$, then $\mathfrak q^{\mathfrak p}<\mathfrak m^*$, because if $\mathfrak q=\aleph_\alpha$ with index $\alpha$ of second kind or $\aleph_{\alpha+1}=\mathfrak q$, then

$$\sum_{\mathfrak r<\mathfrak q}\mathfrak r^{\mathfrak p}=\mathfrak q^{\mathfrak p}\quad\text{or}\quad\sum_{\mathfrak r<\mathfrak q}\mathfrak r^{\mathfrak p}=\aleph_\alpha^{\mathfrak p},$$

respectively, i. e. in this case Theorem 7 is a particular case of Theorem 6.

Corollary 2. *If* $\mathfrak q=\mathfrak m^*$ *and* $\mathfrak r^{\mathfrak p}<\mathfrak m^*$ *for every* $\mathfrak r<\mathfrak q$, *then* $((\mathfrak m,\mathfrak p,\mathfrak q))\to\mathfrak m$.

Corollary 3. *If* $2^{\aleph_\beta}=\aleph_{\beta+1}$ *for every* $\beta$, $\mathfrak m^*=\mathfrak q=\aleph_{\alpha+1}$ *and* $\mathfrak p<(\aleph_\alpha)^*$, *then* $(\mathfrak m,\mathfrak p,\mathfrak q)\to\mathfrak m$.

Theorem 8. *Let* $\mathfrak p$, $\mathfrak q$ *and* $\mathfrak m$ *be cardinal numbers such that* $\mathfrak m\geqq\mathfrak q$. *If* $\mathfrak m^{\mathfrak q}=\mathfrak m^{\mathfrak p}$ *and* $\mathfrak q^{\mathfrak p}<(\mathfrak m^{\mathfrak q})^*$, *then* $((\mathfrak m,\mathfrak p,\mathfrak q))^*\to\mathfrak m^{\mathfrak q}$.

Proof. The proof of this theorem is similar to the proof of Theorem 6. Suppose that the theorem is not true, i. e. for every subset $P$ of $S$ of power $\mathfrak p$, the power of the set

$$P^{*^{-1}}=\{Q: G(Q)=P\}$$

is smaller than $\mathfrak m$. Let $\Gamma(P)$ be the set of all sets $P'\in[S]^{\mathfrak p}$ for which there exists a set $Q\in[S]^{\mathfrak q}$ such that $G(Q)=P_0$ for some $P_0\subseteq P$ and $P'\subset Q$. Then by the condition

$$\overline{\overline{\Gamma(P)}}<\mathfrak m^{\mathfrak q}$$

for every subset $P$ of $S$ of power $\mathfrak p$.

We define now by transfinite induction a sequence $\{P_\xi\}_{\xi<\varphi_{\mathfrak q}+\varphi_{\mathfrak p+}}$ of the type $\varphi_{\mathfrak q}+\varphi_{\mathfrak p+}$ of the sets $\in[S]^{\mathfrak p}$ as follows. Let $P_0$ be an arbitrary element of $[S]^{\mathfrak p}$ and $\beta$ a given ordinal number, $0<\beta<\varphi_{\mathfrak q}+\varphi_{\mathfrak p+}$. Suppose that all sets $P_\xi$, where $0\leqq\xi<\beta$, have been already defined, and let $A_\beta=\bigcup_{\xi<\beta}P_\xi$. Since

$\beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}$ and $\overline{\overline{P_\xi}} = \mathfrak{p} < \mathfrak{q}$, we have $\overline{\overline{A_\beta}} \leqq \mathfrak{q}$. It follows by the hypothesis $\mathfrak{q}^{\mathfrak{p}} < (\mathfrak{m}^{\mathfrak{q}})^*$ that

$$\overline{\overline{\bigcup_{P \subseteq A_\beta} \Gamma(P)}} < \mathfrak{m}^{\mathfrak{q}}.$$

We define the set $P_\beta$ as a subset of power $\mathfrak{p}$, of the set

$$[S]^{\mathfrak{p}} - \{P_\xi\}_{\xi < \beta} - \bigcup_{P \subseteq A_\beta} \Gamma(P).$$

Since $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{q}}$, there exists such an element of $[S]^{\mathfrak{p}}$. Put

$$H = \bigcup_{\xi < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}} P_\xi. \tag{2}$$

It is obvious that $\overline{\overline{H}} = \mathfrak{q}$. It follows that there exists a subset $P$ of $H$ of power $\mathfrak{p}$ such that $G(H) = P$. Since $\mathfrak{p}^+$ is regular there exists an ordinal number $\beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}$ such that

$$P \subseteq \bigcup_{\xi < \beta} P_\xi = A_\xi.$$

But then clearly, by the definition of $P_\beta$, $P_\beta \subseteq\!\!\!\!/\; H$, which contradicts (2).

## III.

We assume in this chapter that $\mathfrak{p} < \mathfrak{q}$, $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^{\mathfrak{q}}$ and the generalized continuum hypothesis holds, i. e. $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal number $\alpha$.

Lemma. *If* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}$. We omit the proof.

Theorem 9. *If* $\mathfrak{q}^{\mathfrak{p}} \neq \mathfrak{m}^*$ *or* $\mathfrak{q} \geqq \mathfrak{m}^*$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$ *and* $(\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$.

Proof. Suppose first, that $\mathfrak{q}^{\mathfrak{p}} \neq \mathfrak{m}^*$. Thus if $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$, then $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$ by Theorem 6 and $(\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$ by the Lemma and Theorem 6, because in this case $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{m}$.

If $\mathfrak{q}^{\mathfrak{p}} > \mathfrak{m}^*$, then we consider two cases: a) $\mathfrak{p} < \mathfrak{m}^*$ and b) $\mathfrak{p} \geqq \mathfrak{m}^*$.

*Ad a.* We have in this case that $\mathfrak{q} \geqq \mathfrak{m}^*$. It follows that $\mathfrak{m} = \mathfrak{m}^{\mathfrak{p}} < \mathfrak{m}^{\mathfrak{q}} = \mathfrak{m}^+$; therefore there exists a set $P_0$ in $[S]^{\mathfrak{p}}$ for which $\overline{\overline{P_0^{*-1}}} = \mathfrak{m}^{\mathfrak{p}}$ and consequently $\overline{\overline{P_0^{*-1}}} = \mathfrak{m}$.

*Ad b.* We have in this case that $\mathfrak{q} \geqq \mathfrak{m}^*$; consequently $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}$. It follows that $\mathfrak{m}^{\mathfrak{q}} = (\mathfrak{m}^{\mathfrak{q}})^*$. Since the assumptions of Theorem 8 hold, there exists a set $P_0$ in $[S]^{\mathfrak{p}}$ such that $\overline{\overline{P_0^{*-1}}} = \mathfrak{m}^+$ i. e. $\overline{\overline{P_0^{-1}}} = \mathfrak{m}$.

Finally if $\mathfrak{q}^{\mathfrak{p}} = \mathfrak{m}^*$, then $\mathfrak{q} \geqq \mathfrak{m}^*$ by the assumption, and if in this case $\mathfrak{p} < \mathfrak{m}^*$, then the proof is the same as in the case a) while if $\mathfrak{p} \geqq \mathfrak{m}^*$, then our statement follows from Theorem 8.

## IV.

We assume now that $\mathfrak{p}$ and $\mathfrak{q}$ are finite cardinal numbers and we prove

Theorem 10. *If $k$ and $l$ are two natural numbers such that $0<k<l$, then $((\aleph_{\alpha+k}, k, l)) \rightarrow \aleph_\alpha$ for every ordinal number $\alpha$.*

Proof. We use induction with respect to $k$. Let $k=1$ and $l>1$. Suppose that the theorem is false, i. e., for every element

$$\overline{\overline{\bigcup_{G(P)=\{x\}} P}} < \aleph_\alpha. \tag{3}$$

Let $F$ be a subset of $S$ of the power $\aleph_\alpha$ and omit from the set the elements of the set

$$H = \bigcup_{x\in F} \bigcup_{G(P)=\{x\}} P.$$

Since $\overline{\overline{F}}=\aleph_\alpha$, it follows from (3) that $\overline{\overline{S-H}}=\aleph_{\alpha+1}$. Let $x_0$ be an arbitrary element of $S-H$. If $\{x_0, y_1, \dots, y_{l-1}\}$ is a set of $l$ elements such that $\{y_1, y_2, \dots, y_{l-1}\} \subset F$, then $G(\{x_0, y_1, \dots, y_{l-1})\} = \{x_0\}$, for if not then $G(\{x_0, y_1, \dots, y_{l-1}\}) = \{y_n\}$ for some $n$, $1 \leqq n \leqq l-1$. In this case $x_0 \in H$, which is a contradiction. Thus, since $\overline{\overline{H}}=\aleph_\alpha$,

$$\overline{\overline{\bigcup_{G(P)=\{x_0\}} P}} = \aleph_\alpha,$$

which contradicts (3). The theorem is proved in the case $k=1$.

Suppose now that $k>1$ and the theorem is true for $k-1$. Let $F$ be a subset of $S$, of power $\aleph_{\alpha+k-1}$. Let $\mathfrak{L}$ be the set of all subsets $L$ of $S$, of $l$ elements, such that

$$\overline{\overline{L \cap (S-F)}} = 1.$$

We have two cases:

1) $\mathfrak{L}$ has a subset $\mathfrak{L}'$ of power $\aleph_{\alpha+k}$ such that $G(L) \subset F$ for every $L \in \mathfrak{L}'$.

2) For every subset $L$ of $[F]^{l-1}$ the power of the set of the elements $x \in S-F$ for which $G(L \cup \{x\}) \subset F$, is smaller than $\aleph_{\alpha+k}$.

*Ad 1.* Since the power of the set $[F]^{l-1}$ is $\aleph_{\alpha+k-1}$ there exists an element $L_0$ of $[F]^{l-1}$ and a subset $B$ of $S-F$ of power $\aleph_{\alpha+k}$ such that

$$G(L_0 \cup \{x\}) \subset L_0$$

for every $x \in B$. It follows that there exists a subset $K_0$ of $k$ elements and a subset $B'$ of $B$ of power $\aleph_{\alpha+k}$ such that

$$G(L_0 \cup \{x\}) = K_0$$

for every $x_0 \in B'$. But then

$$\overline{\overline{\bigcup_{G(L)=K_0} L}} = \aleph_{\alpha+k}.$$

*Ad 2.* Since $\aleph_{\alpha+k}$ is regular $S-F$ has an element $x_0$ such that

$$x_0 \in G(L \cup \{x_0\})$$

for every element $L$ of $[F]^{l-1}$. We define now an inner set mapping $F(X)$ on $[F]^{l-1}$ into $[F]^{k-1}$ as follows. Let

$$F(L) = G(L \cup \{x_0\}) - \{x_0\}$$

for every $L \in [F]^{l-1}$. It is obvious that $F(L) \subset L$. By the induction hypothesis for $k-1$ the theorem is true, i. e. there is an element $K$ of $[F]^{k-1}$ such that

$$\overline{\overline{\bigcup_{F(L)=K} L}} = \aleph_\alpha.$$

It follows from the definition of $F(X)$ that

$$\overline{\overline{\bigcup_{G(L)=K\cup\{x_0\}} L}} = \aleph_\alpha.$$

which proves the theorem.

Next we show that Theorem 5 cannot be improved.

Theorem 11. *If $k$ and $l$ natural numbers, $0 < k < l$, then $((\aleph_{\alpha+k}, k, l)) \not\rightarrow \aleph_{\alpha+1}$.*

Proof. Let $S$ be a set of power $\aleph_{\alpha+k}$ and

(4) $$x_0, x_1, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\xi, \ldots \qquad (\xi < \omega_{\alpha+k})$$

a well-ordering of $S$ of type $\omega_{\alpha+k}$. We define now an inner set mapping $G(X)$ of type $k$ and range $l$ as follows. Let $L$ be an arbitrary element of $[S]^l$, and $x_{\xi_1}$ the greatest element of $L$ in the series (4). Let further

(5) $$x_0^{\xi_1}, x_1^{\xi_1}, \ldots, x_\omega^{\xi_1}, \ldots, x_{\omega+1}^{\xi_1}, \ldots, x_\xi^{\xi_1}, \ldots, \qquad (\xi < \omega(\xi_1))$$

be a well-ordering of the set $\{x_\mu\}_{\mu<\xi_1}$, where $\omega(\xi_1)$ is the initial number of $\bar{\xi}_1$. Let now $x_{\xi_2}^{\xi_1}$ be the greatest element of $L-\{x_{\xi_1}\}$ in the series (5) and let $\{x_\xi^{\xi_1,\xi_2}\}_{\xi<\omega(\xi_2)}$ be a well-ordering of the subset $\{x_\xi^{\xi_1}\}_{\xi<\xi_2}$ of (5), where $\omega(\xi_2)$ is the initial number of $\bar{\xi}_2$. Suppose that the element $x_{\xi_n}^{\xi_1,\ldots,\xi_{n-1}}$ and the series $\{x_\xi^{\xi_1,\ldots,\xi_n}\}_{\xi<\omega(\xi_n)}$ are defined for every $n$, $1 < n \leqq m < k$. We define now the element $x_{\xi_{m+1}}^{\xi_1,\xi_2,\ldots,\xi_m}$ as the greatest element of $L-\{x_{\xi_1}, x_{\xi_2}^{\xi_1}, x_{\xi_3}^{\xi_1,\xi_2}, \ldots, x_{\xi_m}^{\xi_1,\ldots,\xi_{m-1}}\}$ in the series $\{x_\xi^{\xi_1,\ldots,\xi_m}\}_{\xi<\omega(\xi_m)}$, where $\omega(\xi_m)$ is the initial number of $\bar{\xi}_m$. We define $G(L)$ as the set $\{x_{\xi_1}, x_{\xi_2}^{\xi_1}, \ldots, x_{\xi_k}^{\xi_1,\ldots,\xi_{k-1}}\}$. It is easy to see that for every element of $[S]^k$ the inverse has power $\leqq \aleph_\alpha$, which proves Theorem 9.

## V.

We deal in this chapter with the symbol $((\mathfrak{m}, <\mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{r}$.

Theorem 12. *Let $\mathfrak{q}$ and $\mathfrak{m}$ be two cardinal numbers such that $\mathfrak{q}$ is regular and $\mathfrak{q} \geqq \aleph_0$. If $\mathfrak{r}^{\mathfrak{n}} < \mathfrak{m}$ for every $\mathfrak{r} < \mathfrak{q}$ and $\mathfrak{n} < \mathfrak{q}$, then $((\mathfrak{m}, <\mathfrak{q}, \mathfrak{q})) \rightarrow \mathfrak{m}$.*

The proof of Theorem 12 is similar to the proof of Theorem 6.

Corollary 4. *If* $\mathfrak{q} = \aleph_0$ *or* $\mathfrak{q} > \aleph_0$ *is strongly inaccessible and* $\mathfrak{q} \leqq \mathfrak{m}^*$, *then* $((\mathfrak{m}, < \mathfrak{q}, \mathfrak{q})) \rightarrow \mathfrak{m}$.

Corollary 5. *Let* $2^{\aleph_\beta} = \aleph_{\beta+1}$ *for every* $\beta$. *If* $\aleph_\alpha$ *is regular and either* $\mathfrak{m} = \aleph_\alpha$ *or* $\aleph_\alpha < \mathfrak{m}^*$, *then* $((\mathfrak{m}, < \aleph_\alpha, \aleph_\alpha)) \rightarrow \mathfrak{m}$.

*We can not prove that* $((\mathfrak{m}, < \aleph_\omega, \aleph_\omega)) \rightarrow \mathfrak{n}$ *for some* $\mathfrak{m}$, *if* $\mathfrak{n} > \aleph_\omega$. If the generalized continuum hypothesis is true, then $((\aleph_{\omega+1}, < \aleph_\omega, \aleph_\omega)) \not\rightarrow \aleph_{\omega+1}$ (this is a consequence of Theorem 1).

Furthermore we are as yet not able to prove if $((\aleph_{\omega+2}, < \aleph_\omega, \aleph_\omega)) \rightarrow \aleph_{\omega+1}$ or if even $((\aleph_{\omega+2}, < \aleph_\omega, \aleph_\omega)) \rightarrow \aleph_{\omega+2}$?

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