A REMARK ON THE ITERATION OF ENTIRE FUNCTIONS*

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Let $F(z)$ be an entire function. Denote

$$M(F(z),r) = \max_{|z|=r} |F(z)|.$$

In a recent interesting paper on the iteration of entire functions I. N. Baker[1] proved (among many others) the following result: Let $u(r)$ be a real function satisfying $u(r)\to\infty$ as $r\to\infty$. Then to every $0<\alpha<1$, $0<\beta<1$ there exist two entire functions $f(z)$ and $g(z)$ of orders[2] α and β respectively so that for all sufficiently large r

$$(1) \qquad M(f(g(z)),r) < \exp(r^{u(r)}), \quad (\exp z = e^z).$$

An old result of Pólya[3] stated that there exist a constant $c>0$ so that

$$(2) \qquad M(f(g(z)),r) > M(f(z),R) \text{ where } R = cM(g(z),\tfrac{r}{2}).$$

It is easy to see that (2) implies that if $g(z)$ is not a polynomial and the order of $f(z)$ is positive then the order of $f(g(z))$ must be infinite and Bakers result shows that at least if the orders of $f(z)$ and $g(z)$ are less than 1 Pólyas result can not be strengthened, since $u(r)$ can tend to infinity as slowly as we please. In the present note we are going to strengthen the result of Baker, in fact we shall prove the following:

THEOREM. Let $u(r)\to\infty$ be an increasing function satisfying $u(r^2) < c_1 u(r)$ for some constant $c_1 > 1$ and let $v(r)$ be an increasing function satisfying $v(r)\to\infty$, $v(r)/u(r)\to 0$. Then there exists an entire function $f(z)$ for which

$$(3) \qquad M(f(z),r_n) \geq \exp(r_n^{v(r_r)})$$

holds for an infinite sequence $r_n\to\infty$ (i. e. $f(z)$ is certainly of infinite order) and for which

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1) I. N. Baker, Math. Zeitschrift 69 (1958), 121-163. The theorem in question is Theorem 5, p. 133.

2) The order of the entire function $f(z)$ is defined as $\limsup_{r=\infty} \frac{\log\log M(f(z),r)}{\log r}$.

3) G. Pólya, Journal London Math. Soc. 1 (1926), 12-15.

(4) $$M(f_t(z);r) < \exp(r^{u(r)})$$

for all $r>r_t$. Here $f_t(z) = f(f_{t-1}(z))$ denotes the t-th iterate of $f(z)$.

If $u(r)\to\infty$ and $u(r)$ does not satisfy $u(r^2) < c_1u(r)$, it clearly is possible to construct a function $u_1(r)$ satisfying $u_1(r^2)<c_1u_1(r)$ and $u_1(r)/u(r)\to 0$ (thus our condition $u(r^2)<c_1u(r)$ permits $u(r)$ to tend to infinity as slowly as we please).

Let r_k tend to infinity very fast. Put

(5) $$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \text{ where } a_k=r_k^{-r_k^{v(r_k)}}, \quad n_k=2[r_k^{v(r_k)}]+1$$

Clearly

$$f(r_k) > a_k r_k^{n_k} > r_k^{r_k^{v(r_k)}} > \exp(r_k^{v(r_k)}),$$

thus (3) is satisfied.

We shall only prove (4) for t=2, it will be clear from our proof that it holds for all t. Since the coefficients of $f(z)$ are all non negative it will suffice to show that for all sufficiently large r

(6) $$f(f(r)) < \exp(r^{u(r)})$$

To prove (6) we can assume $r_{k-1}\leq r<r_k$. First we assume

(7) $$r_k^{1/n_{k-1}^2} \leq r < r_k$$

A simple computation shows that if the r_k tend to infinity fast enough then for

(8) $$r_k^{1/n_{k-1}^2} \leq r \leq r_k^{n_k}$$

We have

(9) $$f(r) < r^{n_k}$$

(the r_k will of course depend on the function $v(r)$). (9) is easy to see since if the r_k tend to infinity fast enough we have for all $|z|\leq r^{n_k}$ $|\sum_{l=k+1}^{\infty} a_l z^{n_l}|<1$).

Thus from (8) and (9) we have that for the r's satisfying (7)

$$f(r) < r^{n_k} \text{ and } f(f(r)) < r^{n_k^2}$$

Thus to prove (6) for the r's satisfying (7) we only have to show that

$$\exp(r^{u(r)}) > r^{n_k^2},$$

or by taking logarithms twice we have to show that

$$u(r) \log r > 2 \log n_k + \log\log r$$

Now by (5)

$$\log n_k < 2v(r_k) \log r_k$$

thus it will suffice to prove (since $\log\log r < \log r < \log r_k$).

$$u(r) \log r > 5\, v(r_k) \log r_k$$

or by (7)

(10) $$u(r) > 5\, n_{k-1}^2\, v(r_k)$$

From $u(r^2) < c_1 u(r)$ we have for the $u(r)$ satisfying (7)

(11) $$u(r) > n_{k-1}^{-c_2}\, u(r_k).$$

Thus by (10) and (11) we have to show that

(12) $$u(r_k)/v(r_k) > 5\, n_{k-1}^{c_2+2}.$$

But (12) clearly follows from $u(r)/v(r) \to \infty$ if the r_k tend to infinity fast enough. Thus (6) is proved for the r's satisfying (7).

Next we assume

(13) $$r_{k-1} \leq r < r_k^{1/n_{k-1}^2}.$$

We have for the r's satisfying $r \leq r_k^{1/n_{k-1}}$

$$a_k r^{n_k} \leq a_k r_k^{n_k n_{k-1}^{-1}} = r_k^{-r_k^{v(r_k)}}\, r_k^{(2[r_k^{v(r_k)}]+1) n_{k-1}^{-1}} < 1.$$

Thus we have for the r's satisfying (13), if the r_k tend to infinity fast enough

$$f(r) < 2a_{k-1} r^{n_{k-1}} < r^{n_{k-1}}$$

and

$$ff(r) < f(r^{n_{k-1}}) < 2a_{k-1} r^{n_{k-1}^2} < r^{n_{k-1}^2}.$$

Thus to complete our proof we only have to show that

$$\exp(r^{u(r)}) > r^{n_{k-1}^2}.$$

Taking logarithms twice we obtain

$$u(r) \log r > 2 \log n_{k-1} + \log\log r,$$

or by (5) it will suffice to show that

(14) $$u(r) \log r > 4v(r_{k-1}) \log r_{k-1}.$$

But (14) immediately follows from (13) and $u(r)/v(r) \to \infty$, hence the proof of our theorem is complete.

It is clear from our construction that for every $\alpha>0$ and $\beta>0$ we can find two entire functions $f(z)$ and $g(z)$ of orders α and β so that

$$M(f(g(z)),r) < \exp r^{u(r)}$$

for all sufficiently large r.

Further it is clear that by the same argument we can prove the following theorem: Let $u(r^2)<cu(r)$, $u(r)\to\infty$, $u(r)/v(r)\to\infty$ as $r\to\infty$. Let $f(z)=\sum_{k=1}^{\infty}a_k z^k$ and assume that $M(f(z),r) < \exp(r^{v(r)})$ for all sufficiently large r. Then by omitting sufficiently many terms from the power series development of $f(z)$ we obtain

$$f_1(z) = \sum_{i=1}^{\infty} a_{n_i} z^{n_i}$$

and

$$M(f_1(f_1(z)),r) < \exp (r^{u(r)}).$$

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