ASYMPTOTIC FORMULAS FOR SOME ARITHMETIC FUNCTIONS

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Let $f(x)$ be an increasing function. Recently [1)] there have been several papers which proved that under fairly general conditions on $f(x)$ the density of integers n for which $(n, f(n)) = 1$ is $6/\pi^2$ and that ($d(n)$ denotes the number of divisors of n)

$$\sum_{n=1}^{x} d(n, [f(n)]) = \big((1 + o(1)\big) \pi^2 x/6 .$$

In particular both of these results hold if $f(x) = x^{\alpha}$, $0 < \alpha < 1$ and the first holds if $f(x) = [\alpha x]$, α irrational.

In this note we are going to prove the following:

THEOREM 1. The necessary and sufficient condition that for an irrational α we should have

$$(1) \sum_{n=1}^{x} d(n, [\alpha n]) = (1 + o(1)) \pi^2 x/6$$

is that for every $c > 0$ the number of solutions of

$$(2) \quad \alpha < a/b < \alpha + 1/(1+c)^{b}$$

should be finite in positive integers a and b.

Denote $\sigma(n) = \sum_{d|n} d$. It is easy to see that for $0 < \alpha < \frac{1}{2}$

$$(3) \sum_{n=1}^{x} \sigma(n, [n^{\alpha}]) = (1 + o(1)) x \log x$$

Very likely (3) also holds for $1/2 < \alpha < 1$ but I have not yet been able to show this. By more complicated arguments I can show

THEOREM 2. The necessary and sufficient condition that for an irrational α we should have

$$(4) \quad \sum_{n=1}^{x} \sigma(n, [n\alpha]) = (1/2 + o(1))\, x \log x$$

is that for every $\varepsilon > 0$ the number of solutions in positive integers a and b of

$$(5) \quad \left| \alpha - a/b \right| < \frac{1}{b^{2+\varepsilon}}$$

and of

$$(6) \quad \alpha < a/b < \alpha + \varepsilon b^{-2}/\log b$$

should be finite.

It is easy to see that conditions (5) and (6) are equivalent to the following: Put $\alpha = a_0 + \frac{1}{a_1 +}\,\frac{1}{a_2 +} \ldots$, then

$$(1/n) \log a_n \to 0, \quad (1/n)\, a_{2n+1} \to 0.$$

In the present note we will not prove Theorem 2 since the proof is similar to that of Theorem 1, but is rather more complicated.

Similarly one could try to obtain an asymptotic formula for

$$\sum_{n=1}^{x} \sigma(n, [f(n)])$$

for more general functions $f(x)$, but I have not succeeded in obtaining any interesting results.

Now we prove Theorem 1. Denote by $N(y, 1/k)$ the number of integers $1 < n < y$ for which

$$0 < n\alpha - [n\alpha] < 1/k.$$

$(n, [n\alpha]) \equiv 0 \pmod k$ holds if and only if $n = vk$ and

$$vk\alpha = uk + \theta, \quad 0 < \theta < 1,$$

that is $(n, [n\alpha]) \equiv 0 \pmod k$ holds if and only if

$$0 < v\alpha - [v\alpha] < 1/k.$$

Thus the number of integers $n < x$ satisfying $(n, [n\alpha]) \equiv 0 \pmod k$ equals $N(x/k, 1/k)$, (since $n = vk$ implies $v < x/k$). Thus by interchanging the order of summation

$$(7)\quad \sum_{n=1}^{x} d(n, [n\alpha]) = \sum_{k=1}^{x} N(x/k, 1/k) .$$

Since $n\alpha - [n\alpha]$ is equidistributed (mod 1) we evidently have

$$(8)\quad N(x/k, 1/k) = (1 + o(1))\ (x/k^2) ,$$

for fixed k as x tends to infinity. Thus from (7) and (8) for every irrational α

$$(9)\quad \sum_{n=1}^{x} d(n, [n\alpha]) \geq (1 + o(1)) \sum_{k=1}^{\infty} x/k^2 = (1 + o(1)) \pi^2 x/6$$

Assume now that (2) is not satisfied. Then there is a fixed $c > 0$ and arbitrarily large values of b for which

$$(10)\quad \alpha < a/b < \alpha + 1/(1+c)^b.$$

Put $(1+c)^b = x$. Write

$$(11)\quad \sum_{n=1}^{x} d(n, [n\alpha]) = \Sigma_1 + \Sigma_2$$

where in Σ_1, $n \not\equiv 0 \pmod b$ and in Σ_2, $n \equiv 0 \pmod b$. From the equidistribution of $n\alpha - [n\alpha]$ it follows that for fixed k the number of integers satisfying

$$1 < n < x, \quad n \not\equiv 0 \pmod b,\ 0 < n\alpha - [n\alpha] < 1/k$$

is not less than

$$(12)\quad N(x/k, 1/k) - x/b = (1 + o(1))\ x/k^2 - x/b .$$

Thus from (7) and (12) we have for every fixed t

$$(13)\quad \Sigma_1 > \sum_{k=1}^{t} ((1 + o(1))\ x/k^2) - tx/b = (1 + o(1))\ \pi^2 x/6.$$

In Σ_2, $n = vb \leq x$. Thus from (10) and $vb \leq x$, $(1+c)^b = x$ we have

$$[n\alpha] = [vb\alpha] = [va + \theta\, vb/(1+c)^b] = va\ (0 < \theta < 1)$$

Thus $(vb, [vb\alpha]) \equiv 0 \pmod v$ for all $1 \leq v < x/b$. Hence

$$(14)\quad \Sigma_2 \geq \sum_{1 \leq v < x/b} d(v) = (1+o(1))(x/b)\log(x/b)$$
$$= (1+o(1))x\log(1+c)$$

Now (11), (13) and (14) show that (1) does not hold. Thus (2) is a necessary condition for the validity of (1).

To show that (2) is sufficient we need an upper estimation for $N(x/k, 1/k)$ for large k. Put $x/k = y$: it is well known that there exists an a/b satisfying

$$(15)\quad |\alpha - a/b| < 1/(2by)\ ,\ b < 2y,\ (a,b) = 1.$$

Now we distinguish two cases. First assume $b \geq k/2$. Clearly for $1 \leq n \leq y$

$$(16)\quad n\alpha - [n\alpha] = u/b + \theta/b, \qquad |\theta| < 1/2\ .$$

Thus $0 < n\alpha - [n\alpha] < 1/k$ can only hold if $u = 0, 1, \dots, z+1$ where

$$(17)\quad z/b \leq 1/k < (z+1)/k\ ,\ \text{or } z \leq b/k\ .$$

The number of n's not exceeding y for which u has a given value is clearly less than $2y/b + 1$. Thus from (17) and $b \geq k/2$ we have

$$(18)\quad N(x/k, 1/k) < (b/k+1)(2y/b+1) \leq (3b/k)(4y/b) = 12x/k^2\ .$$

Next assume $b < k/2$. If $a/b < \alpha$ then $N(x/k, 1/k) = 0$ since in (16) $\theta \leq 0$, thus for $u = 0$ $n\alpha - [n\alpha]$ is not in $(0, 1/k)$ and for $u = 1$ $n\alpha - [n\alpha] > 1/2b > 1/k$.

Thus $a/b > \alpha$. Clearly $0 < n\alpha - [n\alpha] < 1/k$ is only possible if $u = 0$, that is if $n \equiv 0 \pmod b$. Thus

$$(19)\quad N(x/k, 1/k) \leq (x/(bk).$$

If $N(x/k, 1/k) > 0$, then (since all the $n < x/k$ for which $0 < n\alpha - [n\alpha] < 1/k$ are multiples of b) we have by (15)

$$b\alpha - [b\alpha] < \min(k/x, 1/k) \leq x^{-1/2}\ ,$$

but this implies by (2) that

$$(20)\quad b/\log x \longrightarrow \infty.$$

Thus finally from (7), (8), (18) and (19) we have for every fixed t

$$\sum_{n=1}^{x} d(n, \quad n\alpha \quad) \leq (1+o(1))\,\pi^2 x/6 + 12\, x \sum_{k>t} (1/k^2) + (x/b) \sum_{k<x} \frac{1}{k}$$

hence by (20)

$$\sum_{n=1}^{x} d(n, [n\alpha]) \leq (1+o(1))\,\pi^2 x/6 \,. \tag{21}$$

From (9) and (21) we have that if (2) is satisfied, then

$$\sum_{n=1}^{x} d(n, [n\alpha]) = (1+o(1))\,\pi^2 x/6 \,.$$

Thus condition (2) is sufficient, which completes the proof of our Theorem.

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1) See G.L. Watson, Canadian Journal of Math. 5(1953), 451-455, T. Estermann, ibid 5(1953), 456-459 and J. Lambek and L. Moser, ibid 7(1955), 155-158. See also a forthcoming paper by P. Erdös and G.G. Lorentz in Acta Arithmetica.

CORRECTION

In the paper "On an elementary problem in number theory" by Paul Erdös in Vol. 1, no. 1 of this Bulletin, P. 5, line 5 should read

$$0 \leq u, v < f(x) \text{ and } (x+u,\, y+v) \neq 1.$$