ON THE NUMBER OF ZEROS OF SUCCESSIVE DERIVATIVES OF ENTIRE FUNCTIONS OF FINITE ORDER

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In our joint paper [1][1] published recently, we have proved among other results the following

THEOREM 2'. *If $f(z)$ is an arbitrary entire function,* $M(r)=\operatorname*{Max}_{|z|=r}|f(z)|$, *and* $x=H(y)$ *denotes the inverse function of* $y=\log M(r)$, *then we have*

$$\liminf_{k\to\infty}\frac{N_k(f(z),1)\,H(k)}{k}\leqq e^2. \tag{1}$$

Here $N_k(f(z),1)$ denotes the number of zeros of $f^{(k)}(z)$ in the unit circle.

The aim of the present note is to prove an improvement of this theorem for entire functions of finite order $\geqq 1$, contained in the following

THEOREM A. *If $f(z)$ is an arbitrary entire function of finite order* $\alpha\geqq 1$, $M(r)=\operatorname*{Max}_{|z|=r}|f(z)|$, *and* $x=H(y)$ *denotes the inverse function of* $y=\log M(x)$, *further if* $N_k(f(z),1)$ *denotes the number of zeros of* $f^{(k)}(z)$ *in the unit circle, then we have*

$$\liminf_{k\to\infty}\frac{N_k(f(z),1)\,H(k)}{k}\leqq e^{2-\frac{1}{\alpha}}. \tag{2}$$

[1] We use this occasion to point out that the condition

$$\liminf_{r\to\infty}\frac{\log M(r)}{g(r)}<1$$

in Theorem 2 of [1] can be replaced by the somewhat weaker condition: there exists a sequence $r_n\to+\infty$ such that $\log M(r_n)\leqq g(r_n)$. It is clear from the proof that only this is actually used. Thus the following assertion is true:

THEOREM B. *Let $g(r)$ denote an arbitrary increasing function, defined in* $0<r<+\infty$, *tending to* $+\infty$ *for* $r\to+\infty$. *Let* $x=h(y)$ *denote the inverse function of* $y=g(x)$. *Let us suppose that $f(z)$ is an entire function for which, putting* $M(r)=\operatorname*{Max}_{|z|=r}|f(z)|$, *we have*

$$\log M(r_n)\leqq g(r_n)\qquad (n=1,2,\dots)$$

where r_n is some sequence of positive numbers, tending to $+\infty$ *for* $n\to\infty$. *Then we have*

$$\liminf_{k\to\infty}\frac{N_k(f(z),1)\,h(k)}{k}\leqq e^2.$$

PROOF. It has been shown in [1] (formula (30), p. 132) that if $\nu(r)$ denotes the central index of the power series of $f(z)$ for $|z|=r$, then

$$N_{\nu(r)}(f(z),1) \leqq (\nu(r)+1)\log\frac{1}{1-\frac{e}{r}}. \tag{3}$$

It follows from (3) that

$$\limsup_{r\to\infty}\frac{N_{\nu(r)}(f(z),1)\,r}{\nu(r)} \leqq e. \tag{4}$$

Now we may suppose without loss of generality that $f(0)=1$. In that case if $\mu(r)$ denotes the absolute value of the maximal term of the power series of $f(z)$ on the circle $|z|=r$, the following well-known formula is valid (see [2], Vol. II, p. 5, Problem IV. 33):

$$\log\mu(r)=\int_0^r\frac{\nu(t)}{t}\,dt. \tag{5}$$

It follows from (5) that if $c>1$, taking into account that $\nu(t)$ is non-decreasing (see [2], Vol. I, p. 21, Problem I. 120), we have

$$\log\mu(rc)-\log\mu(r)=\int_r^{rc}\frac{\nu(t)}{t}\,dt \geqq \nu(r)\log c. \tag{6}$$

On the other hand, it is known (see [2], Vol. II, p. 9, Problem IV. 60) that

$$\liminf_{r\to\infty}\frac{\nu(r)}{\log\mu(r)} \leqq \alpha. \tag{7}$$

Thus to any $\varepsilon>0$ there can be found a sequence r_n $(n=1,2,\ldots)$ for which $r_n\to\infty$ and $\nu(r_n)\leqq(\alpha+\varepsilon)\log\mu(r_n)$. Applying (6) for $r=r_n$ we obtain

$$\nu(r_n)\left(\log c+\frac{1}{\alpha+\varepsilon}\right) \leqq \log\mu(r_n c). \tag{8}$$

Choosing $c=e^{1-\frac{1}{\alpha+\varepsilon}}$, it follows that

$$\nu(r_n) \leqq \log\mu\left(r_n e^{1-\frac{1}{\alpha+\varepsilon}}\right). \tag{9}$$

As $\mu(r)\leqq M(r)$, (9) implies

$$\nu(r_n) \leqq \log M\left(r_n e^{1-\frac{1}{\alpha+\varepsilon}}\right) \tag{10}$$

and thus

$$H(\nu(r_n)) \leqq r_n e^{1-\frac{1}{\alpha+\varepsilon}}. \tag{11}$$

As by (4)

$$\limsup_{n\to\infty} \frac{N_{\nu(r_n)}(f(z),1)\, r_n}{\nu(r_n)} \leqq e \tag{12}$$

and with respect to (11), we obtain

$$\limsup_{n\to\infty} \frac{N_{\nu(r_n)}(f(z),1)\, H(\nu(r_n))}{\nu(r_n)} \leqq e^{2-\frac{1}{\alpha+\varepsilon}}. \tag{13}$$

But (13) clearly implies

$$\liminf_{k\to\infty} \frac{N_k(f(z),1)\, H(k)}{k} \leqq e^{2-\frac{1}{\alpha+\varepsilon}}. \tag{14}$$

As (14) is valid for any $\varepsilon > 0$, the assertion of Theorem A is proved. Especially[2] we have for entire functions of exponential type, with type A,

$$\liminf_{k\to\infty} N_k(f(z),1) \leqq Ae. \tag{15}$$

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References

[1] P. Erdős and A. Rényi, On the number of zeros of successive derivatives of analytic functions, *Acta Math. Acad. Sci. Hung.*, **7** (1956), pp. 125—144.

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[3] S. S. Macintyre, On the zeros of successive derivatives of integral functions, *Trans. Amer. Math. Soc.*, **67** (1949), pp. 241—251.

[4] N. Levinson, The Gontcharoff polynomials, *Duke Math. J.*, **11** (1944), pp. 729—733; and **12** (1944), p. 335.

[5] S. S. Macintyre, On the bound for the Whittaker constant, *Journ. London Math. Soc.*, **22** (1947), pp. 305—311.

[2] Let W (Whittaker's constant) denote the greatest number such that if $f(z)$ is of exponential type $A < W$, then an infinity of derivatives of $f(z)$ have no zeros in the unit circle. The exact value of W is not known. It follows from (15) that $\frac{1}{e} \leqq W$. This estimate is, however, much weaker than the estimate $0{,}7259 \leqq W$, proved by Sheila Scott Macintyre [3]. (In footnote [4] of [1] we mentioned only the weaker estimate $0{,}7199 \leqq W$, due to N. Levinson [4].) It has been shown also by S. S. Macintyre [5], that $W \leqq 0{,}7378$. $\left(\text{It has been conjectured (see [4]) that } W = \frac{2}{e}.\right)$