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ON A PERFECT SET

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(From a letter of P. Erdös to E. Marczewski)

... Enclosed I send you our promised solution to your problem[1]). The problem is this: A linear set S is said to have *property* (S_n) if there exists an η_n such that if $x_1 < x_2 < \ldots < x_n$, $x_n - x_1 < \eta_n$ are any n real numbers, there exist n elements $y_1, y_2, \ldots, y_n$ of S, congruent to $x_1, x_2, \ldots, x_n$. You ask: Does there exist a perfect set S of measure 0 having property (S_3)?

Kakutani and I have constructed a perfect set S of measure 0 having property (S_n) for all $n \geqslant 2$. Our set S is defined as the set of non-negative numbers

$$\sum_{k=2}^{\infty} \frac{a_k}{k!}, \quad 0 \leqslant a_k \leqslant k-2.$$

It is easy to see that the measure of S is 0 (every number x, $0 \leqslant x \leqslant 1$, is uniquely of the form

$$\sum_{k=2}^{\infty} \frac{a_k}{k!}, \quad 0 \leqslant a_k \leqslant k-1).$$

Thus we only have to prove that S has property (S_n) for all $n \geqslant 2$.

To show that S has property (S_n) it clearly suffices to show that if we put $x_2 - x_1 = z_1, x_3 - x_1 = z_2, \ldots, x_n - x_1 = z_{n-1}, z_{n-1} < \eta_n$, there exists a number z_0 in S such that all the numbers $z_0 + z_i$, $1 \leqslant i \leqslant n-1$, are also in S. Assume $\eta_n < 1/(m-1)!$ where m will be determined later. Then clearly

$$z_i = \sum_{k=m}^{\infty} \frac{b_k^{(i)}}{k!}, \quad 0 \leqslant b_k^{(i)} \leqslant k-1, \quad 1 \leqslant i \leqslant n-1.$$

[1]) E. Marczewski, P 125, Colloquium Mathematicum 3.1 (1954), p. 75.

Now we have to determine

$$z_0 = \sum_{k=2}^{\infty} \frac{b_k^{(0)}}{k!}, \quad 0 \leqslant b_k^{(i)} \leqslant k-2,$$

so that all the z_0+z_i are in S. To do this put $b_k^{(0)} = 0$, $2 \leqslant k \leqslant m-1$, and further for $k \geqslant m$, $1 \leqslant i \leqslant n-1$,

$$(1) \qquad b_k^{(0)}+b_k^{(i)} \neq k-1,\, k-2,\, 2k-2,\, 2k-3.$$

If $m > 4n$ such a choice of $b_k^{(0)}$ is always possible since for each i (1) excludes at most 4 values of $b_0^{(i)}$ and there are $k-1 \geqslant m-1$ possible values for $b_k^{(0)}$ (*i. e.* $0 \leqslant b_k^{(0)} \leqslant k-2$ and $k \geqslant m$).

If $b_0^{(k)}$ satisfies (1) for all $k \geqslant m$ then z_0+z_i is clearly in S since the k-th digit of z_0+z_i is $\leqslant k-2$, *i. e.*

$$z_0+z_i = \sum_{k=2}^{\infty} \frac{c_k}{k!}, \quad 0 \leqslant c_k \leqslant k-2. \quad \ldots$$

Budapest, October 4, 1955