ON THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION

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1. Some time ago Littlewood and Offord† gave estimates of the number of real roots that an equation of degree n selected at random might be expected to have for various classes of equations in which the coefficients were selected on some probability basis. They found that, when each coefficient was treated on the same basis, the results were practically the same in all cases considered and agreed with those found for the family of equations

$$f_n(x) = 1+\epsilon_1 x+\epsilon_2 x^2+\ldots+\epsilon_n x^n = 0 \tag{1.1}$$

in which each ϵ_ν, $\nu = 1, 2,\ldots, n$, is $+1$ or -1 with equal probability.

The object of this paper is to give a refinement of their result. We shall prove

Theorem. *The number of real roots of most of the equations*

$$f_n(x) = \sum_0^n \epsilon_\nu x^\nu = 0$$

is

$$\frac{2}{\pi}\log n+o\{(\log n)^{\frac{2}{3}}\log(\log n)\}. \tag{1.2}$$

The exceptional set does not exceed a proportion

$$o\{(\log\log n)^{-\frac{1}{3}}\}$$

of the total number of equations.

Dr. and Mrs. A. D. Booth‡ have kindly worked out the number of roots of the 256 equations

$$1\pm x\pm x^2\pm\ldots\pm x^8 = 0$$

of degree 8. They find that 58 have no real roots, 190 have 2 real roots, 8 have 4 real roots, and none has more than 4. The average number of roots is thus 1·609, but if we treat those with 4 roots as exceptional then the average number of roots for the remainder is 1·532. $\dfrac{2}{\pi}\log n$ is 1·324 for $n = 8$. Thus there is some reasonable agreement with our result even for $n = 8$, although the number of roots would appear to be slightly in excess of our estimate.

† *Proc. Cambridge Phil. Soc.* 35 (1939), 133–48.

‡ K. H. V. Booth, 'An investigation into the real roots of certain polynomials,' *Math. Tables and Aids to Computation*, 8 (1954), 47.

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Broadly the idea of our proof is the following. In the first place it is sufficient to prove that the number of roots of $f_n(x)$ in $(\frac{1}{2}, 1)$ is $\dfrac{1}{2\pi}\log n$, plus the error term given. For all the roots must lie in $\frac{1}{2} < |x| < 2$, and to each root of $f_n(x)$ in $(\frac{1}{2}, 1)$ there corresponds a root of $f_n(-x)$ in $(-1, -\frac{1}{2})$ and conversely. Also if $f_n(x)$ has a root in $(1, 2)$ then $x^n f_n(y)$ where $y = 1/x$ has a root in $(\frac{1}{2}, 1)$.

Suppose now that $\alpha \leqslant x \leqslant \beta$ is an interval in $(\frac{1}{2}, 1)$ and that $f_n(\alpha) \geqslant 0$ and $f_n(\beta) \leqslant 0$. It follows that $f_n(x)$ has at least one root in (α, β). Our procedure is then to divide $(\frac{1}{2}, 1)$ into a carefully chosen number of intervals, and then (i) to estimate the probability that the number of changes of sign of $f(x)$ at the end-points of these intervals differs from $\dfrac{2}{\pi}\log n$ by more than the error term in (1.2), and (ii) to show that the number of changes of sign corresponds closely to the number of zeros. Stage (ii) is carried out in § 2. In § 3 we calculate the probability that $f_n(\alpha)f_n(\beta) \leqslant 0$ for given α and β and in § 4 the probability that we have simultaneously $f_n(\alpha)f_n(\beta) \leqslant 0$ and $f_n(\alpha')f_n(\beta') \leqslant 0$ for intervals (α, β) and (α', β') which are not too close. With this information we are able in § 5 to find both the average and the standard deviation of the number of changes of sign at the end-points of our set of intervals.

2. We write
$$f(x, t) = \sum_0^n r_\nu(t)x^\nu,$$
where
$$r_0(t) = \begin{cases} 1, & 0 \leqslant t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leqslant t < 1, \end{cases}$$
$$r_0(t+1) = r_0(t), \qquad r_n(t) = r_0(2^n t).$$

We denote by $N(t) = N(t; \alpha, \beta)$ the number of zeros of $f(x, t)$ in the interval $\alpha \leqslant x \leqslant \beta$ reckoned according to multiplicity except for zeros at α and β which are reckoned according to half their multiplicity; and further write
$$N^*(t) = N^*(t; \alpha, \beta) = \begin{cases} 1 & \text{if } f(\alpha, t)f(\beta, t) < 0, \\ \frac{1}{2} & \text{if } f(\alpha, t)f(\beta, t) = 0, \\ 0 & \text{if } f(\alpha, t)f(\beta, t) > 0. \end{cases}$$

It is clear that if $N^*(t) > 0$, $f(x, t)$ must have at least one zero in $\alpha \leqslant x \leqslant \beta$, so that
$$N(t) - N^*(t) \geqslant 0.$$

In this section we shall show that
$$\operatorname*{av}_t\{N(t) - N^*(t)\} \leqslant C\gamma^2\{\log(1/\gamma)\}^{\frac{1}{2}}$$

where $\gamma = (\beta-\alpha)\min\{n; (1-\beta)^{-1}\}$. This result is contained in Lemma 4. It enables us to replace the function $N(t)$ by $N^*(t)$ when estimating the numbers of zeros.

We shall suppose that $\frac{1}{2} \leqslant \alpha < \beta \leqslant 1$ and that $\gamma < 1$.

LEMMA 1. *If $f(x,t)$ has k zeros in $\alpha \leqslant x \leqslant \beta$, then outside a set of measure at most γ^4*

$$\sup_{\alpha \leqslant x \leqslant \beta} |f(x,t)| \leqslant C(k!)\gamma^k\{\log 1/\gamma\}^{\frac{1}{2}} \min\{\sqrt{n}, (1-\beta)^{-\frac{1}{2}}\},$$

where C is an absolute constant.

Proof. If $f(x)$ has k zeros in $\alpha \leqslant x \leqslant \beta$, then $f^{(\nu)}(x)$ has $k-\nu$ zeros for $\nu = 0, 1,..., k$. Let t_ν be a zero of $f^\nu(x)$, $\nu = 0, 1,..., k-1$. Then

$$f^{k-1}(x) = \int_{t_{k-1}}^{x} f^{(k)}(u)\, du,$$

so

$$|f^{k-1}(x)| \leqslant |x-t_{k-1}|^{\frac{1}{2}}\left(\left|\int_{t_{k-1}}^{x} |f^{(k)}(u)|^2\, du\right|\right)^{\frac{1}{2}}$$

$$\leqslant (\beta-\alpha)^{\frac{1}{2}}\left(\int_{\alpha}^{\beta} |f^{(k)}(u)|^2\, du\right)^{\frac{1}{2}}.$$

Similarly

$$|f^{k-2}(x)| = \left|\int_{t_{k-2}}^{x} f^{(k-1)}(u)\, du\right|$$

$$\leqslant (\beta-\alpha) \sup_{\alpha \leqslant x \leqslant \beta} |f^{(k-1)}(x)|$$

$$\leqslant (\beta-\alpha)^{\frac{3}{2}}\left(\int_{\alpha}^{\beta} |f^{(k)}(u)|^2\, du\right)^{\frac{1}{2}},$$

and consequently

$$\sup_{\alpha \leqslant x \leqslant \beta} |f(x)| \leqslant (\beta-\alpha)^{k-\frac{1}{2}}\left(\int_{\alpha}^{\beta} |f^{(k)}(u)|^2\, du\right)^{\frac{1}{2}}. \tag{2.1}$$

Now write, for shortness,

$$f^{(k)}(u,t) = \sum_{0}^{n} r_\nu(t) a_\nu$$

and let $\mathfrak{E}$ be an arbitrary set of values of t. Let E_1 be the set of t for which

$$|f^{(k)}(u,t)|^2 \leqslant \Lambda\left(\sum_{0}^{n} a_\nu^2\right)$$

and E_s the set for which

$$2^{s-2}\Lambda\left(\sum_{0}^{n} a_\nu^2\right) < |f^{(k)}(u,t)|^2 \leqslant 2^{s-1}\Lambda\left(\sum_{0}^{n} a_\nu^2\right).$$

Then for a given u

$$\int_{\mathfrak{E}} |f^{(k)}(u,t)|^2\,dt = \sum_{s=1}^{\infty} \int_{E_s\cap\mathfrak{E}} |f^{(k)}(u,t)|^2\,dt$$
$$\leqslant \Lambda\left\{m(\mathfrak{E}) + \sum_{2}^{\infty} 2^{s-1} m(E_s)\right\}\left(\sum_{0}^{n} a_\nu^2\right).$$

Now by Khintchine's lemma† the set for which

$$\left|\sum_{0}^{n} r_\nu(t) a_\nu\right|^2 > \Lambda\left(\sum_{0}^{n} a_\nu^2\right)$$

has measure at most $Ce^{-\frac{1}{2}\Lambda}$, where C is an absolute constant, and so

$$m(E_s) \leqslant C\exp(-2^{s-3}\Lambda).$$

Hence
$$\sum_{2}^{\infty} 2^{s-1} m(E_s) \leqslant Ce^{-\frac{1}{4}\Lambda},$$

and so, taking $\Lambda = -2\log m(\mathfrak{E})$, we get

$$\int_{\mathfrak{E}} |f^{(k)}(u,t)|^2\,dt \leqslant C\left(\sum_{0}^{n} a_\nu^2\right) m(\mathfrak{E}) \log\{1/m(\mathfrak{E})\}.$$

Now, by a simple calculation,

$$\sum_{0}^{n} a_\nu^2 \leqslant \left(\frac{1}{2}\right)^{2k} \frac{(2k)!}{x(1-x)^{2k+1}}$$

provided that $x < 1$, and in any case

$$\sum_{0}^{n} a_\nu^2 < n^{2k+1}.$$

Hence
$$\int_{\alpha}^{\beta} du \int_{\mathfrak{E}} |f^{(k)}(u,t)|^2\,dt \leqslant C\mathfrak{A} m(\mathfrak{E}) \log\{1/m(\mathfrak{E})\},$$

where
$$\mathfrak{A} = \min\left\{n^{2k+1};\quad \frac{(2k)!}{2^{2k}(1-x)^{2k+1}}\right\}.$$

Let $\mathfrak{E}$ be the set for which

$$\int_{\alpha}^{\beta} |f^{(k)}(u,t)|^2\,dt > CK\mathfrak{A}.$$

Then
$$K\mathfrak{A} m(\mathfrak{E}) < \mathfrak{A} m(\mathfrak{E}) \log\{1/m(\mathfrak{E})\},$$

and so
$$\log m(\mathfrak{E}) < -K,$$

or
$$m(\mathfrak{E}) < e^{-K}.$$

Hence outside a set of measure at most e^{-K} we have, by (2.1),

$$\sup_{\alpha\leqslant x\leqslant\beta} |f(x,t)| \leqslant (\beta-\alpha)^{k-\frac{1}{2}}\{CK\mathfrak{A}\}^{\frac{1}{2}}$$
$$\leqslant CK^{\frac{1}{2}} 2^{-k}\{(2k)!\}^{\frac{1}{2}} \gamma^k \min\{n^{\frac{1}{2}}, (1-\beta)^{-\frac{1}{2}}\},$$

† A. Khintchine, 'Über dyadische Brücke', *Math. Zeit.* 18 (1923), 109–11.

where $$\gamma = (\beta-\alpha)\min\{n, (1-\beta)^{-1}\}.$$

Now put $K = -4\log\gamma$; we get

$$\sup|f(x,t)| \leqslant C\, 2^{-k}\{(2k)!\}^{\frac{1}{2}}\gamma^k(\log 1/\gamma)^{\frac{1}{2}}\min\{\surd n, (1-\beta)^{-\frac{1}{2}}\}$$

outside a set of measure at most γ^4. This completes the proof of the lemma.

LEMMA 2. *For given* x

$$|f(x,t)| \geqslant \kappa\min\{\surd n; (1-x)^{-\frac{1}{2}}\},$$

except for a set of t of measure at most 10κ.

Proof. By a theorem of Erdős,† for any given number C,

$$|f(x,t)-C| \geqslant x^m,$$

except for a set of t of measure at most $\surd(2/\pi m)$. If we choose m so that $x \geqslant 1-1/m$, and consequently

$$x^m > e^{-1},$$

we deduce that $$|f(x,t)-C| \geqslant e^{-1}$$

except for a set of t of measure at most $\frac{1}{2}\pi/\surd m$. Giving C the values

$$0, \pm 2e^{-1}, \pm 4e^{-1}, \pm ..., \pm[\kappa\surd m]e^{-1}$$

we infer that $$|f(x,t)| \geqslant \kappa\surd m$$

except for a set of measure at most 10κ. Furthermore, m can be chosen so that $m \geqslant \min\{n, (1-x)^{-1}\}$ and so the result follows.

LEMMA 3. *The set of values of t for which $f(x, t)$ has k or more zeros in $\alpha \leqslant x \leqslant \beta$ has measure at most $C\gamma^2(\log 1/\gamma)^{\frac{1}{2}}$ if $k = 2$, and at most*

$$C\gamma^3 k^{-2}\{\log(k/\gamma)\}^{\frac{1}{2}} \quad \textit{if } k > 2.$$

Proof. We apply Lemmas 1 and 2 in the cases $k = 2$ and 3. In Lemma 2 take $\kappa = C(k!)\gamma^k(\log 1/\gamma)^{\frac{1}{2}}$, and we shall then have

$$|f(\beta,t)| \geqslant C(k!)\gamma^k(\log 1/\gamma)^{\frac{1}{2}}\min\{\surd n, (1-\beta)^{-\frac{1}{2}}\},$$

except for a set of measure at most $C(k!)\gamma^k(\log 1/\gamma)^{\frac{1}{2}}$. Hence by Lemma 1, if $f(x,t)$ has two or three zeros in $\alpha \leqslant x \leqslant \beta$, then $f(x,t)$ must belong to a set of measure at most $\gamma^4+C(k!)\gamma^k(\log 1/\gamma)^{\frac{1}{2}}$, where $k = 2, 3$. This proves the lemma in the cases when k is 2 or 3. If $k > 3$ we choose p so that

$$2^p < k \leqslant 2^{p+1}$$

and divide the interval (α,β) into 2^{p-1} equal parts. Then one of these intervals must contain 3 zeros. Denote this interval by (α_p, β_p) and let

$$\gamma_p = (\beta_p-\alpha_p)\min\{n, (1-\beta_p)^{-1}\}.$$

Then, by the above result, the chance of this interval containing 3 zeros is at most

$$C\gamma_p^3(\log 1/\gamma_p)^{\frac{1}{2}} < C\left(\frac{\gamma}{2^{p-1}}\right)^3\left(\log\frac{2^{p-1}}{\gamma}\right)^{\frac{1}{2}}.$$

† P. Erdős, 'On a lemma of Littlewood and Offord', *Bull. American Math. Soc.* 51 (1945), 898–902. Cf. Littlewood and Offord, *Mat. Sbornik*, N.S. 12 (1943), 277–86.

Hence the chance of one or other of the 2^{p-1} intervals containing 3 zeros is at most

$$C\frac{\gamma^3}{4^{p-1}}\left(\log\frac{2^{p-1}}{\gamma}\right)^{\frac{1}{2}} < 4C\frac{\gamma^3}{k^2}\left(\log\frac{k}{\gamma}\right)^{\frac{1}{2}},$$

and this completes the proof of the lemma.

LEMMA 4. *For* $\alpha \leqslant x \leqslant \beta$

$$\operatorname*{av}_t\{N(t)-N^*(t)\} \leqslant C\gamma^2(\log 1/\gamma)^{\frac{1}{2}},$$

where C is an absolute constant.

Proof. Write $N^{(2)}(t)$ = number of zeros in $\alpha \leqslant x \leqslant \beta$, reckoned according to multiplicity, if this number exceeds one, and 0 otherwise.

Then
$$N(t)-N^*(t) \leqslant N^{(2)}(t).$$

But
$$\operatorname*{av}_t N^{(2)}(t) = \sum_{p=1}^{\infty} 2^{p-1}m(E_p),$$

where E_p denotes the set of values of t for which $f(x,t)$ has at least 2^p zeros in $\alpha \leqslant x \leqslant \beta$. Hence using the result of Lemma 3 we get, after a simple calculation,

$$\operatorname*{av}_t N^{(2)}(t) \leqslant C\gamma^2(\log 1/\gamma)^{\frac{1}{2}},$$

as desired.

We shall now apply the above results to obtain an estimate for the error made by replacing $N(t)$ by $N^*(t)$ in estimating the zeros of $f(x,t)$ in $(0,1)$. Since for $|x| \leqslant \frac{1}{2}$

$$|f(x,t)| \geqslant 1-\sum_1^N 2^{-n} = 2^{-N},$$

all zeros of $f(x,t)$ in $0 \leqslant x \leqslant 1$ lie in $\frac{1}{2} < x \leqslant 1$ and so we confine our attention to this range. We choose a positive number δ and define p_0 and p_1 by

$$(1+\delta)^{-p_0} \leqslant \tfrac{1}{2} < (1+\delta)^{-p_0+1},$$

and
$$(1+\delta)^{-p_1} \leqslant 1/2n < (1+\delta)^{-p_1+1},$$

and α_p and β_p by

$$1-\alpha_{p_0} = \tfrac{1}{2}, \qquad 1-\alpha_p = (1+\delta)^{-p} \quad (p_0 < p \leqslant p_1),$$
$$1-\beta_{p_1} = 0, \qquad 1-\beta_p = (1+\delta)^{-p-1} \quad (p_0 \leqslant p < p_1).$$

Then it is clear that the intervals (α_p, β_p) defined for $p_0 \leqslant p \leqslant p_1$ together cover the interval $\frac{1}{2} \leqslant x \leqslant 1$. Clearly

$$\gamma_p = \frac{\beta_p-\alpha_p}{1-\beta_p} = \delta \quad (p_0 < p < p_1),$$

while
$$\gamma_{p_0} < \delta,$$

and
$$\gamma_{p_1} = (1-\alpha_{p_1})n \leqslant \tfrac{1}{2}.$$

We denote by $N_p(t)$ and $N_p^*(t)$ the functions $N(t)$ and $N^*(t)$ for the ranges $\alpha_p \leqslant x \leqslant \beta_p$. We have

LEMMA 5.

$$\operatorname*{av}_t\Big[\sum_{p=p_0}^{p_1}\{N_p(t)-N_p^*(t)\}\Big] \leqslant C\log n\,.\,\delta(\log 1/\delta)^{\frac{1}{2}},$$

where C is a numerical constant.

Proof. By Lemma 4,

$$\operatorname*{av}_t\{N_p(t)-N_p^*(t)\} \leqslant C\gamma_p^2(\log 1/\gamma_p)^{\frac{1}{2}}.$$

Hence
$$\operatorname{av}\sum_{p=p_0}^{p_1}\{N_p(t)-N_p^*(t)\} \leqslant C(p_1-p_0)\delta^2(\log 1/\delta)^{\frac{1}{2}}+C,$$

and the desired result follows.

3. In this section we estimate the averages of the function $N^*(t)$ defined in § 2. We shall give our results in a somewhat more general form than in the preceding paragraph because many have interest of their own. These results deal with the sums $\sum_0^n a_\nu r_\nu(t)$ and $\sum_0^n b_\nu r_\nu(t)$ in which the coefficients satisfy $|a_\nu| \leqslant 1$, $|b_\nu| \leqslant 1$, and in certain of the lemmas $a_\nu b_\nu \geqslant 0$. This latter condition is equivalent to assuming $a_\nu \geqslant 0$, $b_\nu \geqslant 0$. We introduce the function

$$\mu(t) = \begin{cases} 1, & \Big\{\sum_0^n a_\nu r_\nu(t)\Big\}\Big\{\sum_0^n b_\nu r_\nu(t)\Big\} < 0, \\ \frac{1}{2}, & \Big\{\sum_0^n a_\nu r_\nu(t)\Big\}\Big\{\sum_0^n b_\nu r_\nu(t)\Big\} = 0, \\ 0, & \Big\{\sum_0^n a_\nu r_\nu(t)\Big\}\Big\{\sum_0^n b_\nu r_\nu(t)\Big\} > 0, \end{cases} \tag{3.1}$$

and the main object of the section is to obtain the evaluation of $\int_0^1 \mu(t)\,dt$ given in Lemma 12.

LEMMA 6. *If $\mu(t)$ is defined as in (3.1), then*

$$\int_0^1 \mu(t)\,dt = \frac{1}{2}+\frac{1}{\pi^2}\int\int_0^\infty \frac{\phi(x,y)-\phi(x,-y)}{xy}\,dxdy,$$

where
$$\phi(x,y) = \prod_0^n \cos(a_\nu x+b_\nu y). \tag{3.2}$$

Proof. This result follows from a standard theorem on the characteristic function. It may be proved directly as follows. There are 2^{n+1} distinct sums $\sum_0^n a_\nu r_\nu(t)$. Writing

$$A_k = \sum_0^n a_\nu r_\nu\Big(\frac{k-\frac{1}{2}}{2^{n+1}}\Big), \qquad k = 1, 2,..., 2^{n+1},$$

and denoting the corresponding expression in which a_ν is replaced by b_ν by B_k, we have

$$\phi(x,y)-\phi(x,-y) = -\frac{1}{2^n}\sum_{k=1}^{2^{n+1}} 2\sin(A_k x)\sin(B_k y); \tag{3.3}$$

thence

$$\frac{1}{\pi^2}\int_0^\infty\!\!\int \frac{\phi(x,y)-\phi(x,-y)}{xy}\,dxdy$$

$$= -\frac{2}{2^{n+1}\pi^2}\sum_{k=1}^{2^{n+1}}\int_0^\infty \frac{\sin A_k x}{x}\,dx\int_0^\infty \frac{\sin B_k y}{y}\,dy$$

$$= -\frac{1}{2}+\frac{1}{2^{n+1}}\sum_{k=1}^{2^{n+1}}\mu\left(\frac{k-\frac{1}{2}}{2^{n+1}}\right),$$

and this is the desired result.

LEMMA 7. *If the numbers a_ν are real and satisfy $|a_\nu| \leqslant 1$, and if*

$$A^2 = \sum_0^n a_\nu^2,$$

then

$$\operatorname{av}\left\{\frac{1}{\left(\max 1,\left|\sum_0^n r_\nu(t)a_\nu\right|\right)}\right\} \leqslant \frac{\log^+ A}{A}+\frac{5}{A}.$$

Proof. Without loss of generality we may assume $A > 1$, for if $A \leqslant 1$ the conclusion is trivial since the first member cannot exceed unity. We denote by $F(x)$ the distribution function of $\sum_0^n r_\nu(t)a_\nu$. Then

$$\operatorname{av}\left\{\frac{1}{\max\left(1,\left|\sum_0^n r_\nu(t)a_\nu\right|\right)}\right\} = \int_{-\infty}^{\infty}\min\left(1,\frac{1}{|x|}\right)dF(x). \tag{3.4}$$

Write

$$G(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt;$$

then by a theorem of Berry,† in view of our condition that $|a_\nu| \leqslant 1$,

$$\left|F(x)-G\left(\frac{x}{A}\right)\right| \leqslant \frac{2}{A}.$$

The second member of (3.4) is then the sum of

$$I_1 = \int_{-\infty}^{\infty}\min\left(1,\frac{1}{|x|}\right)d\left\{F(x)-G\left(\frac{x}{A}\right)\right\},$$

† A. C. Berry, 'The accuracy of the Gaussian approximation to the sum of independent variates', *Trans. American Math. Soc.* 49 (1941), 122–36.

and
$$I_2 = \int_{-\infty}^{\infty} \min\left(1, \frac{1}{|x|}\right) dG\left(\frac{x}{A}\right).$$

But, on integrating by parts, we get

$$I_1 = \int_1^{\infty} \left\{F(x)-F(-x)-G\left(\frac{x}{A}\right)+G\left(-\frac{x}{A}\right)\right\}x^{-2}\, dx \leqslant \frac{4}{A},$$

while it is easily verified that

$$I_2 \leqslant \sqrt{\left(\frac{2}{\pi}\right)}\frac{\log A}{A}+\frac{1}{A},$$

and we get the desired result on combining these two inequalities.

LEMMA 8. *If $\phi(x,y)$ is defined by* (3.2) *and if* $|a_\nu| \leqslant 1$, $|b_\nu| \leqslant 1$, *then*

$$\int_0^{\infty}\!\!\int \frac{\phi(x,y)-\phi(x,-y)}{xy}\, dxdy = \int_0^{\epsilon}\!\!\int \frac{\phi(x,y)-\phi(x,-y)}{xy}\, dxdy + $$
$$+O\left\{\frac{\log^+(\frac{1}{2}\epsilon\pi A_1)}{\frac{1}{2}\epsilon\pi A_1}\right\},$$

where
$$A_1^2 = \min\left\{\sum_0^n a_\nu^2, \sum_0^n b_\nu^2\right\}.$$

Proof. If $\epsilon > 0$, we have from (3.3)

$$I_1 = \int_{\epsilon}^{\infty} dx \int_0^{\infty} \frac{\phi(x,y)-\phi(x,-y)}{xy}\, dy$$
$$= -\frac{1}{2^{n+1}\pi}\sum_{k=1}^{2^{n+1}} \int_{\epsilon}^{\infty} \frac{\sin A_k x}{x}\, dx \operatorname{sgn}(B_k).$$

Hence
$$I_1 \leqslant \frac{1}{2^{n+1}\pi}\sum_{k=1}^{2^{n+1}} \int_{|\epsilon A_k|}^{\infty} \frac{\sin\theta}{\theta}\, d\theta.$$

Now
$$\left|\int_{|\epsilon A_k|}^{\infty} \frac{\sin\theta}{\theta}\, d\theta\right| \leqslant \frac{\pi}{\max(1, |\frac{1}{2}\epsilon\pi A_k|)}.$$

Hence
$$|I_1| \leqslant \operatorname{av}\frac{1}{\max(1, |\frac{1}{2}\epsilon\pi A_k|)} \leqslant \frac{\log^+(\frac{1}{2}\epsilon\pi A)}{\frac{1}{2}\epsilon\pi A}+\frac{10}{\epsilon\pi A},$$

where $A^2 = \sum_0^n a_\nu^2$, by Lemma 7.

A similar result holds for the integral $\int_0^{\infty} dx \int_{\epsilon}^{\infty} dy$ but with A replaced by B, where $B^2 = \sum_0^n b_\nu^2$. Writing $A_1 = \min(A,B)$, we get the desired result.

LEMMA 9. *If* $0 < \gamma < 1$ *and*

$$I(\gamma; A) = \int_A^\infty dx \int_0^\infty \frac{e^{-\frac{1}{2}(x^2-2\gamma xy+y^2)} - e^{-\frac{1}{2}(x^2+2\gamma xy+y^2)}}{xy}\, dy,$$

then

(i) $I(\gamma; 0) = \frac{1}{2}\pi^2 - \cos^{-1}\gamma$,

(ii) $I(\gamma; A) \leqslant C\dfrac{\sqrt{(\log A)}}{A}$ *for* $A \geqslant 2$,

where C *is an absolute constant.*

Proof. This is a matter of evaluation which we leave to the reader.

From now on we shall employ the following notation. We write $A^2 = \sum_0^n a_\nu^2$, $B^2 = \sum_0^n b_\nu^2$, $P = \sum_0^n a_\nu b_\nu$, and $\tau^2 = 1 - P^2/A^2B^2$. By Cauchy's inequality $\tau \geqslant 0$. For convenience we shall suppose that $A \leqslant B$. We shall also assume that for all ν, $a_\nu b_\nu \geqslant 0$.

LEMMA 10. *If* $\phi(x, y)$ *is defined by* (3.2) *and*

$$g(x, y) = \exp\{-\tfrac{1}{2}(A^2x^2 + 2Pxy + B^2y^2)\},$$

then for all x, y *satisfying* $0 \leqslant x \leqslant \epsilon$, $0 \leqslant y \leqslant \epsilon$, *where* $\epsilon < \frac{1}{4}$, *we have*

$$\frac{\phi(x, y) - \phi(x, -y)}{g(x, y) - g(x, -y)} = \{1 + \epsilon_1(x, y)\}e^{\epsilon_2(x,y)},$$

where $-\epsilon^2 \leqslant \epsilon_1(x, y) \leqslant 3\epsilon^2$ *and*

$$-\epsilon^2(8A^2x^2 + 9B^2y^2) \leqslant \epsilon_2(x, y) \leqslant 0.$$

Proof. We have

$$\frac{\phi(x, y)/\phi(0, y) - \phi(x, -y)/\phi(0, -y)}{e^{-\frac{1}{2}(A^2x^2+2Pxy)} - e^{-\frac{1}{2}(A^2x^2-2Pxy)}} = \frac{\partial/\partial\eta\{\phi(x, \eta)/\phi(0, \eta)\}}{-Pxe^{-\frac{1}{2}(Ax^2+2Px\eta)}}, \tag{3.5}$$

for some η satisfying $|\eta| < y$. Now

$$\frac{\partial}{\partial\eta}\left\{\frac{\phi(x, \eta)}{\phi(0, \eta)}\right\} = -\frac{\phi(x, \eta)}{\phi(0, \eta)} \sum_0^n b_\nu\{\tan(a_\nu x + b_\nu \eta) - \tan b_\nu \eta\}.$$

But
$$\tan(a_\nu x + b_\nu \eta) - \tan(b_\nu \eta) = \frac{\sin a_\nu x}{\cos(a_\nu x + b_\nu \eta)\cos(b_\nu \eta)},$$

and if $0 \leqslant |\eta| \leqslant \epsilon$, $0 \leqslant x \leqslant \epsilon$, $0 \leqslant a_\nu \leqslant 1$, $0 \leqslant b_\nu \leqslant 1$, we have

$$\frac{\sin\epsilon}{\epsilon} a_\nu x \leqslant \tan(a_\nu x + b_\nu \eta) - \tan(b_\nu \eta) \leqslant a_\nu x[(1-2\epsilon^2)(1-\tfrac{1}{2}\epsilon^2)]^{-1},$$

and so
$$\sum_0^n b_\nu\{\tan(a_\nu x + b_\nu \eta) - \tan b_\nu \eta\} = \{1 + \epsilon_1(x, y)\}Px,$$

provided that $\epsilon < \frac{1}{4}$. Hence the second member of (3.5) becomes

$$\frac{\phi(x,\eta)}{\phi(0,\eta)} e^{\frac{1}{2}(Ax^2+2Px\eta)}\{1+\epsilon_1(x,y)\}$$

and we deduce that

$$\frac{\phi(x,y)-\phi(x,-y)}{g(x,y)-g(x,-y)} = \{1+\epsilon_1(x,y)\}\frac{\phi(0,y)}{e^{-\frac{1}{2}B^2y^2}}\frac{e^{-\frac{1}{2}B^2\eta^2}}{\phi(0,\eta)}\frac{\phi(x,\eta)}{g(x,\eta)},$$

where $$-\epsilon^2 \leqslant \epsilon_1(x,y) \leqslant 3\epsilon^2.$$

We now have to estimate the ratio $\phi(x,\eta)/g(x,\eta)$. For this we require the elementary equality

$$\log\cos\theta = -\tfrac{1}{2}\theta^2 - \theta^4\psi(\theta),$$

where $\psi(\theta)$ is a positive increasing function of $|\theta|$ satisfying $\psi(\theta) \leqslant 1$ for $|\theta| \leqslant 1$. From this it follows that

$$\prod_0^n \cos\theta_\nu = \exp\Big\{-\tfrac{1}{2}\sum_0^n \theta_\nu^2 - \sum_0^n \theta_\nu^4 . \psi(\theta_\nu)\Big\},$$

and so, since $\epsilon < \frac{1}{4}$,

$$\frac{\phi(x,\eta)}{g(x,\eta)} = \exp\Big\{-\sum_0^n (a_\nu x + b_\nu \eta)^4 \psi(a_\nu x + b_\nu \eta)\Big\}.$$

But

$$\sum_0^n (a_\nu x+b_\nu \eta)^4 \leqslant (|x|+|y|)^2 \sum_0^n (a_\nu x+b_\nu\eta)^2 \leqslant 8\epsilon^2\{A^2x^2+B^2\eta^2\}.$$

Further, since $|\eta| < y$,

$$\sum_0^n (b_\nu \eta)^4\psi(b_\nu \eta) < \sum_0^n (b_\nu y)^4 \psi(b_\nu y).$$

Therefore

$$\frac{\phi(x,y)-\phi(x,-y)}{g(x,y)-g(x,-y)} = \{1+\epsilon_1(x,y)\}\frac{\phi(0,y)}{g(0,y)}\frac{g(0,\eta)}{\phi(0,\eta)}\frac{\phi(x,\eta)}{g(x,\eta)}$$
$$= \{1+\epsilon_1(x,y)\}\exp\{\epsilon_2(x,y)\},$$

where $$-\epsilon^2(8A^2x^2+9B^2y^2) \leqslant \epsilon_2(x,y) \leqslant 0,$$

as desired.

LEMMA 11. *If $|a_\nu| \leqslant 1$, $|b_\nu| \leqslant 1$, and $a_\nu b_\nu \geqslant 0$ for all ν, and $A \leqslant B$, then*

$$\int_\eta^\epsilon dx \int_0^\epsilon \frac{\phi(x,-y)-\phi(x,y)}{xy} e^{\pm\sigma(A^2x^2+B^2y^2)}\,dy$$
$$= \tfrac{1}{2}\pi^2 - \pi\sin^{-1}\tau + O\Big(\frac{\sigma+\epsilon^2}{\tau}\Big) + O\Big(\frac{\sqrt{\log^+(\epsilon A)}}{\epsilon A}\Big) + O(\eta A),$$

provided that $4\sigma+2\epsilon^2 < \tau^2$, and $\eta A < 1$.

Proof. By Lemma 10, since $\phi(x,-y)-\phi(x,y) \geqslant 0$ in the range considered,

$$\int_\eta^\epsilon \int_0^\epsilon e^{\sigma(A^2x^2+B^2y^2)} \frac{\phi(x,-y)-\phi(x,y)}{xy}\,dxdy$$

$$\leqslant (1+3\epsilon^2) \iint_0^\epsilon \frac{g(x,-y)-g(x,y)}{xy} e^{\sigma(A^2x^2+B^2y^2)}\,dxdy$$

$$= (1+3\epsilon^2) \iint_0^{\epsilon A\sqrt{(1-2\sigma)},\ \epsilon B\sqrt{(1-2\sigma)}} \frac{e^{-\frac{1}{2}(x^2-2\gamma xy+y^2)}-e^{-\frac{1}{2}(x^2+2\gamma xy+y^2)}}{xy}\,dxdy,$$

where
$$\gamma = \frac{P}{AB(1-2\sigma)}.$$

This integral therefore does not exceed

$$\iint_0^\infty \frac{e^{-\frac{1}{2}(x^2-2\gamma xy+y^2)}-e^{-\frac{1}{2}(x^2+2\gamma xy+y^2)}}{xy}\,dxdy+O(\epsilon^2).$$

By hypothesis $4\sigma < \tau^2$, so that $\gamma < 1$ and the integral converges. The value of this integral is, by Lemma 9,

$$\tfrac{1}{2}\pi^2-\pi\cos^{-1}\gamma = \tfrac{1}{2}\pi^2-\pi\sin^{-1}\sqrt{(1-\gamma^2)}.$$

But
$$1-\gamma^2 = \tau^2+O(\sigma),$$

so that
$$\sin^{-1}\sqrt{(1-\gamma^2)} = \sin^{-1}\tau+O\left(\frac{\sigma}{\tau}\right).$$

The value of the integral is therefore at most

$$\tfrac{1}{2}\pi^2-\pi\sin^{-1}\tau+O(\sigma/\tau)+O(\epsilon^2).$$

Again

$$\int_\eta^\epsilon \int_0^\epsilon e^{-\sigma(A^2x^2+B^2y^2)} \frac{\phi(x,-y)-\phi(x,y)}{xy}\,dxdy$$

$$\geqslant (1-\epsilon^2) \int_\eta^\epsilon \int_0^\epsilon \frac{g(x,-y)-g(x,y)}{xy} e^{-(\sigma+8\epsilon^2)A^2x^2-(\sigma+9\epsilon^2)B^2y^2}\,dxdy$$

$$\geqslant \iint_0^\infty [e^{-\frac{1}{2}\{(1+2\sigma+16\epsilon^2)A^2x^2-2Pxy+(1+2\sigma+18\epsilon^2)B^2y^2\}}-$$

$$-e^{-\frac{1}{2}\{(1+2\sigma+16\epsilon^2)A^2x^2+2Pxy+(1+2\sigma+18\epsilon^2)B^2y^2\}}]x^{-1}y^{-1}\,dxdy+$$

$$+O(\epsilon^2)+O\left(\frac{\sqrt{\log(\epsilon A)}}{\epsilon A}\right)+O(\eta A).$$

Because the contribution from the range $\int\limits_0^\eta \int\limits_0^\epsilon$ cannot exceed

$$\int\limits_0^\eta \int\limits_0^\epsilon \frac{g(x,-y)-g(x,y)}{xy}\,dxdy$$

it is easily verified that this does not exceed $\sqrt{(2\pi)}A\eta$. And by Lemma 9 the contribution from $\int\limits_\epsilon^\infty \int\limits_0^\infty$ is $O\left(\frac{\sqrt{\log(\epsilon A)}}{\epsilon A}\right)$. It follows from Lemma 9 that this integral exceeds

$$\tfrac{1}{2}\pi^2-\pi\cos^{-1}\gamma_2+O(\epsilon^2)+O\left(\frac{\sqrt{\log(\epsilon A)}}{\epsilon A}\right)+O(\eta A),$$

where

$$\gamma_2 = 1-P^2[A^2B^2(1+2\sigma+16\epsilon^2)(1+2\sigma+18\epsilon^2)]^{-\frac{1}{2}} = \tau^2+O(\sigma+\epsilon^2),$$

so
$$\cos^{-1}\gamma_2 = \sin^{-1}\tau+O\left(\frac{\sigma+\epsilon^2}{\tau}\right).$$

LEMMA 12. *If* $0 \leqslant a_\nu \leqslant 1$, $0 \leqslant b_\nu \leqslant 1$, *and* $1 < A \leqslant B$, *then*

$$\int\limits_0^1 \mu(t)\,dt = \frac{1}{\pi}\sin^{-1}\tau+O\left(\frac{\sqrt{\log A}}{\tau^{\frac{1}{2}}A^{\frac{1}{2}}}\right).$$

Proof. Without loss of generality we may suppose $\tau > 2/\sqrt{A}$, because in any case the first member cannot exceed unity. We now put $\eta = 0$, $\sigma = 0$, and $\epsilon = \tau^{\frac{1}{2}}A^{-\frac{1}{2}}$ in Lemma 11. This is permissible since with $A^{-\frac{1}{2}} < \frac{1}{2}\tau$ we have $\epsilon^2 < \frac{1}{2}\tau^2$ as desired. We then obtain on combining Lemmas 8 and 11

$$\int\limits_0^\infty\int \frac{\phi(x,-y)-\phi(x,y)}{xy}\,dxdy = \tfrac{1}{2}\pi^2-\pi\sin^{-1}\tau+O\left(\frac{\sqrt{\log^+(\tau^{\frac{1}{2}}A^{\frac{1}{2}})}}{\tau^{\frac{1}{2}}A^{\frac{1}{2}}}\right).$$

The desired result now follows from Lemma 6.

4. In this section we shall extend the analysis of the preceding section to the case in which there are four sums of the form $\sum a_\nu r_\nu(t)$. This time we must define two functions $\mu(t)$; $\mu_1(t)$ which is the same as $\mu(t)$ of § 3 and $\mu_2(t)$ which is defined in the same way but for the sums $\sum c_\nu r_\nu(t)$ and $\sum d_\nu r_\nu(t)$. The object of this section is then to obtain an estimate for

$$\int\limits_0^1 \mu_1(t)\mu_2(t)\,dt.$$

Now it is intuitive that if c_ν and d_ν differ substantially from a_ν and b_ν then the above mean value should approximate closely to

$$\int_0^1 \mu_1(t)\,dt \int_0^1 \mu_2(t)\,dt,$$

and we shall in fact show that this is the case. This result is given in Lemma 17. Some further specification of the parameters enable us to simplify this result to the form in which it is applied in the sequel. This is given in Lemma 18.

LEMMA 13. *If $\mu_1(t)$ is the function $\mu(t)$ of* (3.1) *and if $\mu_2(t)$ is defined by replacing a_ν by c_ν and b_ν by d_ν in* (3.1), *then*

$$\int_0^1 \mu_1(t)\mu_2(t)\,dt = -\tfrac{1}{4}+\tfrac{1}{2}\int_0^1 \mu_1(t)\,dt+\tfrac{1}{2}\int_0^1 \mu_2(t)\,dt+ \\ +\frac{1}{2\pi^4}\iiiint_0^\infty \frac{\Delta^{(3)}\phi(x,y,z,t)}{xyzt}\,dxdydzdt,$$

where $$\phi(x,y,z,t) = \prod_0^\infty \cos(a_\nu x+b_\nu y+c_\nu z+d_\nu t), \tag{4.1}$$

and

$$\Delta^{(3)}\phi = \phi(x,y,z,t)-\phi(-x,y,z,t)-\phi(x,-y,z,t)-\phi(x,y,-z,t)- \\ -\phi(x,y,z,-t)+\phi(x,y,-z,-t)+\phi(x,-y,-z,t)+\phi(x,-y,z,-t).$$

Proof. The proof is similar to that of Lemma 6 and so we omit it.

LEMMA 14. *If* $0 \leqslant a_\nu \leqslant 1$, $0 \leqslant b_\nu \leqslant 1$, $0 \leqslant c_\nu \leqslant 1$, $0 \leqslant d_\nu \leqslant 1$, *if $\phi(x,y,z,t)$ is defined by* (4.1),

$$\phi_1(x,y) = \phi(x,y,0,0), \qquad \phi_2(x,y) = \phi(0,0,x,y),$$

and if

$$\sum_0^n a_\nu c_\nu \leqslant \sigma AC, \qquad \sum_0^n a_\nu d_\nu \leqslant \sigma AD, \qquad \sum_0^n b_\nu c_\nu \leqslant \sigma BC,$$

and $$\sum_0^n b_\nu d_\nu \leqslant \sigma BD,$$

then

$$\frac{\Delta^{(3)}\phi(x,y,z,t)}{\Delta\phi_1(x,y)\Delta\phi_2(z,t)} = 2(1+\eta_3)\exp\{\eta_4(A^2x^2+B^2y^2+C^2z^2+D^2t^2)\},$$

for $\eta_1 \leqslant x \leqslant \epsilon_1$, $0 \leqslant y \leqslant \epsilon_1$, $\eta_2 \leqslant z \leqslant \epsilon_2$, $0 \leqslant t \leqslant \epsilon_2$, *where, writing* $\epsilon = \max(\epsilon_1, \epsilon_2)$,

$$\eta_3 = \eta_3(x,y,z,t) = O(\epsilon^2)+O\left\{\sigma\left(\frac{(C+D)\epsilon_2}{\eta_1 AP}+\frac{(A+B)\epsilon_1}{\eta_2 CQ}\right)\right\}$$

and $\eta_4 = \eta_4(x,y,z,t)$ *satisfies* $|\eta_4| \leqslant 2\sigma$.

Proof. Consider first the ratio

$$\frac{\Delta_{yt}^{(2)}\phi(x,y,z,t)}{\phi_1(y,0)\phi_2(0,t)}\bigg/\frac{\Delta\phi_1(x,y)}{\phi_1(0,y)}\cdot\frac{\Delta\phi_2(z,t)}{\phi_2(0,t)}.$$

We calculate this by Cauchy's mean value theorem, and, on differentiating numerator and denominator partially with respect to y and t, we obtain for $|y'|<y$, $|t'|<t$,

$$\frac{\phi(x,y',z,t')}{\phi_1(x,y')\phi_2(z,t')}\times\frac{\sum\limits_0^n b_\nu\{\tan(a_\nu x+b_\nu y'+c_\nu z+d_\nu t')-\tan b_\nu y'\}}{\sum\limits_0^n b_\nu\{\tan(a_\nu x+b_\nu y')-\tan b_\nu y'\}}\times$$

$$\times\frac{\sum\limits_0^n d_\nu\{\tan(a_\nu x+b_\nu y'+c_\nu z+d_\nu t')-\tan d_\nu t'\}}{\sum\limits_0^n d_\nu\{\tan(c_\nu z+d_\nu t')-\tan d_\nu t'\}}$$

$$=\Pi_1\times\Pi_2\times\Pi_3.$$

We consider first Π_2. Now,

$$\begin{aligned}\tan(a_\nu x+b_\nu y'+c_\nu z+d_\nu t')-\tan b_\nu y'&\\ &=\frac{\sin(a_\nu x+c_\nu z+d_\nu t')}{\cos(a_\nu x+b_\nu y'+c_\nu z+d_\nu t')\cos b_\nu y'}\\ &=\{1+O(\epsilon^2)\}\frac{\sin a_\nu x}{\cos(a_\nu x+b_\nu y')\cos b_\nu y'}\\ &\quad+\{1+O(\epsilon^2)\}\frac{c_\nu z+d_\nu t'}{\cos(a_\nu x+b_\nu y')\cos b_\nu y'}.\end{aligned}$$

Hence

$$\begin{aligned}\Pi_2&=1+\frac{\sum\limits_0^n b_\nu(c_\nu z+d_\nu t')}{\sum\limits_0^n b_\nu\sin a_\nu x}+O(\epsilon^2)\\ &=1+O\left\{\frac{\epsilon_2\sigma(C+D)}{xAP}\right\}+O(\epsilon^2).\end{aligned}$$

Similarly $$\Pi_3=1+O\left\{\frac{\epsilon_1\sigma(A+B)}{zCQ}\right\}+O(\epsilon^2).$$

Finally $$\Pi_1=\prod_0^n\{1-\tan(a_\nu x+b_\nu y')\tan(c_\nu z+d_\nu t')\},$$

and under our hypotheses

$$\begin{aligned}\exp\{-2(a_\nu x+b_\nu|y'|)(c_\nu z+d_\nu|t'|)\}&\\ &\leqslant 1-\tan(a_\nu x+b_\nu y')\tan(c_\nu z+d_\nu t')\\ &\leqslant\exp\{2b_\nu|y'|d_\nu|t'|\},\end{aligned}$$

and hence

$$\exp\{-2\sigma(A^2x^2+B^2y^2+C^2z^2+D^2t^2)\} \leqslant \Pi_1 \leqslant \exp\{2\sigma(B^2y^2+D^2t^2)\},$$

using the fact that

$$ACxz \leqslant \tfrac{1}{2}(A^2x^2+C^2z^2), \text{ etc.}$$

We have therefore proved that

$$\frac{\Delta_{yt}^{(2)}\phi(x,y,z,t)}{\Delta\phi_1(x,y)\Delta\phi_2(z,t)} = (1+\eta_3)\exp\{\eta_4(A^2x^2+B^2y^2+C^2z^2+D^2t^2)\}, \quad (4.2)$$

where

$$\eta_3 = \eta_3(x,y,z,t) = O(\epsilon^2)+O\left\{\frac{\epsilon_2\,\sigma(C+D)}{\eta_1 AP}+\frac{\epsilon_1\,\sigma(A+B)}{\eta_2 CQ}\right\},$$

and $\eta_4 = \eta_4(x,y,z,t)$ satisfies $|\eta_4| \leqslant 2\sigma$.

Now $$\Delta^{(3)}\phi = \Delta_{y,t}^{(2)}\phi(x,y,z,t)-\Delta_{y,t}^{(2)}\phi(x,y,-z,t).$$

But $$\frac{\Delta_{yt}^{(2)}\phi(x,y,-z,t)}{\Delta\phi_1(x,y)\Delta\phi_2(-z,t)} = -\frac{\Delta_{yt}^{(2)}\phi(x,y,-z,t)}{\Delta\phi_1(x,y)\Delta\phi_2(z,t)}$$

satisfies a similar inequality to (4.2) and combining these two results we get the desired inequality.

LEMMA 15. *We have*

$$\int\!\!\int_0^\infty\!\!\int\!\!\int \frac{\Delta^{(3)}\phi}{xyzt}\,dxdydzdt$$
$$= \int_{\eta_1}^{\epsilon_1} dx \int_0^{\epsilon_1} dy \int_{\eta_2}^{\epsilon_2} dz \int_0^{\epsilon_2} \frac{\Delta^{(3)}\phi}{xyzt}\,dxdydzdt + O(\eta_1 A+\eta_2 C)+$$
$$+O\left\{\frac{\log\epsilon_1 A}{\epsilon_1 A}+\frac{\log\epsilon_1 B}{\epsilon_1 B}+\frac{\log\epsilon_2 C}{\epsilon_2 C}+\frac{\log\epsilon_2 D}{\epsilon_2 D}\right\},$$

and

$$\int\!\!\int_0^\infty \frac{\Delta\phi_1}{xy}\,dxdy = \int_{\eta_1}^{\epsilon_1}\int_0^{\epsilon_1} \frac{\Delta\phi_1}{xy}\,dxdy + O(\eta_1, A)+O\left(\frac{\log\epsilon_1 A}{\epsilon_1 A}+\frac{\log\epsilon_1 B}{\epsilon_1 B}\right).$$

Proof. We have

$$\int_0^{\eta_1}\int\!\!\int_0^\infty\!\!\int \frac{\phi^3}{xyzt}\,dxdydzdt$$
$$= -\frac{8}{2^n}\sum_{k=1}^{2^{n+1}} \int_0^{\eta_1} \frac{\sin A_k x}{x}\,dx \int\!\!\int_0^\infty\!\!\int \frac{\sin B_k y\sin C_k z\sin D_k t}{yzt}\,dydzdt$$
$$= -\frac{8}{2^n}\left(\frac{\pi}{2}\right)^3 \sum_{k=1}^{2^{n+1}} \operatorname{sgn}(A_k B_k C_k D_k) \int_0^{\eta|A_k|} \left(\frac{\sin\theta}{\theta}\right)d\theta.$$

Hence the modulus of the first member does not exceed

$$\frac{\pi^3}{2^n}\eta_1 \sum_{k=1}^{2^n+1} |A_k| = 2\pi^3\eta_1 \operatorname*{av}_{\epsilon_\nu=\pm 1}\left(\left|\sum_0^n \epsilon_\nu a_\nu\right|\right) = O(\eta_1, A).$$

The proof that

$$\int_{\epsilon_1}^{\infty} dx \iiint_0^{\infty} \frac{\Delta^{(3)}\phi}{xyzt}\, dydzdt = O\left(\frac{\log \epsilon_1 A}{\epsilon_1 A}\right)$$

is similar to that of Lemma 7. It is now evident that the first inequality of the theorem follows from these two inequalities and two similar ones where the variable z plays the role of x in the above inequalities.

The proof of the second inequality of the lemma is on the same lines.

Lemma 16. *We have*

$$\iiiint_0^{\infty} \frac{\Delta^{(3)}\phi}{xyzt}\, dxdydzdt$$

$$= \tfrac{1}{2}\pi^4 - \pi^3(\sin^{-1}\tau_1 + \sin^{-1}\tau_2) + 2\pi^2 \sin^{-1}\tau_1 \sin^{-1}\tau_2 +$$

$$+O\left(\frac{\sigma+\epsilon_1^2}{\tau_1}+\frac{\sigma+\epsilon_2^2}{\tau_2}\right)+O\left\{\frac{\sqrt{\log(\epsilon_1 A)}}{\epsilon_1 A}+\frac{\sqrt{\log(\epsilon_2 C)}}{\epsilon_2 C}\right\}+$$

$$+O(\eta_1 A+\eta_2 C)+O(\epsilon^2)+$$

$$+O\left[\sigma\left\{\frac{\epsilon_2(C+D)}{\eta_1 AP}+\frac{\epsilon_1(A+B)}{\eta_2 CQ}\right\}\right],$$

where $\sum a_\nu c_\nu \leqslant \sigma AC$, *etc., and* $P = \dfrac{\sum a_\nu b_\nu}{AB}$, $Q = \dfrac{\sum c_\nu d_\nu}{CD}$.

Proof. By Lemma 14

$$\int_{\eta_1}^{\epsilon_1} dx \int_0^{\epsilon_1} dy \int_{\eta_2}^{\epsilon_2} dz \int_0^{\epsilon_2} \frac{\Delta^{(3)}\phi}{xyzt}\, dt$$

$$= 2\int_{\eta_1}^{\epsilon_1}\int_0^{\epsilon_1} \frac{\Delta\phi_1}{xy}\, dxdy \int_{\eta_2}^{\epsilon_2}\int_0^{\epsilon_2} (1+\eta_3)\frac{\Delta\phi_2}{t} e^{\eta_4(A^2x^2+B^2y^2+C^2z^2)}\, dzdt$$

$$= 2\int_{\eta_1}^{\epsilon_1}\int_0^{\epsilon_1} \frac{\Delta\phi_1}{xy} e^{\eta_4(A^2x^2+B^2y^2)}\, dxdy \int_{\eta_2}^{\epsilon_2}\int_0^{\epsilon_2} \frac{\Delta\phi_2}{zt} e^{\eta_4(C^2z^2+D^2t^2)}\, dzdt +$$

$$+O(\epsilon^2)+O\left\{\sigma\left(\frac{\epsilon_2(C+D)}{\eta_1 AP}+\frac{\epsilon_1(A+B)}{\eta_2 CQ}\right)\right\}.$$

And by Lemma 11 this is

$$2\left\{\frac{\pi^2}{2}-\pi\sin^{-1}\tau_1+O\left(\frac{\sigma+\epsilon_1^2}{\tau_1}\right)+O\left(\frac{\sqrt{\log(\epsilon_1 A)}}{\epsilon_1 A}\right)+O(\eta_1 A)\right\}\times$$
$$\times\left\{\frac{\pi^2}{2}-\pi\sin^{-1}\tau_2+O\left(\frac{\sigma+\epsilon_2^2}{\tau_2}\right)+O\left(\frac{\sqrt{\log(\epsilon_2 C)}}{\epsilon_2 C}\right)+O(\eta_2 C)\right\}+$$
$$+O(\epsilon^2)+O\left\{\sigma\left(\frac{\epsilon_2(C+D)}{\eta_1 AP}+\frac{\epsilon_1(A+B)}{\eta_2 CQ}\right)\right\},$$

which gives the desired result.

Lemma 17.

$$\int_0^1 \mu_1(t)\mu_2(t)\,dt=\int_0^1 \mu_1(t)\,dt\int_0^1 \mu_2(t)\,dt+$$
$$+O\left(\frac{\sqrt{\log A}}{\tau_1^{\frac12}A^{\frac12}}\right)+O\left(\frac{\sqrt{\log C}}{\tau_2^{\frac12}C^{\frac12}}\right)+$$
$$+O\left(\frac{\sigma+\epsilon_1^2}{\tau_1}+\frac{\sigma+\epsilon_2^2}{\tau_2}\right)+O\left\{\frac{\sqrt{\log(\epsilon_1 A)}}{\epsilon_1 A}+\frac{\sqrt{\log(\epsilon_2 C)}}{\epsilon_2 C}\right\}+$$
$$+O\left\{\left(\frac{\epsilon_2\,\sigma(C+D)}{P}\right)^{\frac12}\right\}+O\left\{\left(\frac{\epsilon_1\,\sigma(A+B)}{Q}\right)^{\frac12}\right\}.$$

Proof. We have merely to combine Lemmas 12, 13, and 16, and at the same time choose η_1 and η_2 so that

$$\eta_1 A=\left\{\frac{\epsilon_2\,\sigma(C+D)}{P}\right\}^{\frac12},\qquad \eta_2 C=\left\{\frac{\epsilon_1\,\sigma(A+B)}{Q}\right\}^{\frac12},$$

and observe that $\epsilon^2<\epsilon^2/\tau$.

The following lemma is obtained from Lemma 17 by further specification of the conditions.

Lemma 18. *If, in addition to the hypotheses of Lemma* 17, *we suppose that*

(i) $A\leqslant B\leqslant 2A;\qquad C\leqslant D\leqslant 2C;\qquad 1<A\leqslant C;$

(ii) $P\geqslant\frac14,\qquad Q\geqslant\frac14,$

and write
$$\tau=\min(\tau_1,\tau_2),$$
then

$$\int_0^1 \mu_1(t)\mu_2(t)\,dt=\int_0^1 \mu_1(t)\,dt\int_0^1 \mu_2(t)\,dt+O\left(\frac{\sqrt{\log A}}{\tau^{\frac12}A^{\frac12}}\right)+$$
$$+O(\sigma^{\frac12}\log 1/\sigma)+O(\sigma/\tau).$$

Proof. Without loss of generality we may suppose $\tau>2/\sqrt{A}$, $\sigma<\tau$, since neither the first member nor the first term of the second member can exceed unity. After making the obvious simplifications in Lemma 17 we obtain three terms involving ϵ_1. They are

$$O\left(\frac{\epsilon_1^2}{\tau}\right), \qquad O\left(\frac{\sqrt{\log^+(\epsilon_1 A)}}{\epsilon_1 A}\right), \qquad O\{\sqrt{(\epsilon_1 \sigma A)}\},$$

and we proceed to choose ϵ_1 so as to make these three of the same order of magnitude. We distinguish two cases, (i) $\sigma \leqslant 1/\tau A^2$, and (ii) $\sigma > 1/\tau A^2$. In case (i) choose $\epsilon_1 = \tau^{\frac{1}{3}} A^{-\frac{1}{3}}$, and the three terms become

$$O\left(\frac{\sqrt{\log A}}{\tau^{\frac{1}{3}} A^{\frac{2}{3}}}\right).$$

In case (ii) choose $\epsilon_1 = 1/\sigma^{\frac{1}{3}} A$, and then under our conditions the three terms are

$$O(\sigma^{\frac{1}{3}} \log 1/\sigma).$$

The terms involving ϵ_2 can be treated in the same way and in view of conditions (i) and (ii) we obtain the desired result.

5. We turn now to the special case of our theorem. We write $\delta = (\log n)^{\frac{1}{3}}$ and define p_0 and p_1 so that

$$(1+\delta)^{-p_0-1} < \tfrac{1}{2} \leqslant (1+\delta)^{-p_0},$$

$$(1+\delta)^{-p_1} \leqslant \frac{1}{2n} < (1+\delta)^{-p_1+1}.$$

We define
$$x_p = 1-(1+\delta)^{-p}, \qquad p_0 < p \leqslant p_1,$$
$$x_{p_0} = \tfrac{1}{2}, \qquad x_{p_1+1} = 1.$$

We further define p_2 and p_3 in the following way:

$$(1+\delta)^{-p_2} \leqslant \exp\{-(\log n)^{\frac{1}{3}}\} < (1+\delta)^{-p_2+1}, \tag{5.1}$$

and
$$p_3 = p_1 - p_2.$$

Then it is clear that
$$p_2 \sim (\log n)^{\frac{2}{3}}. \tag{5.2}$$

Then, by Lemma 5,

$$\int_0^1 \sum_{p=p_0}^{p_1} \{N_p(t) - N_p^*(t)\}\, dt = O\{\log^{\frac{2}{3}} n (\log\log n)^{\frac{1}{2}}\},$$

and so
$$\sum_{p=p_0}^{p_1} \{N(t) - N_p^*(t)\} = o\{\log^{\frac{2}{3}} n (\log\log n)\},$$

except for a set of t of measure at most $o\{(\log\log n)^{\frac{1}{2}}\}$. Using (5.1) and the fact that $0 \leqslant N_p^*(t) \leqslant 1$, we see that outside this exceptional set

$$\sum_{p=p_0}^{p_1} N_p(t) = \sum_{p=p_2}^{p_3} N_p^*(t) + o\{\log^{\frac{2}{3}} n (\log\log n)\}. \tag{5.3}$$

We shall now apply the results of the two preceding paragraphs to obtain an estimate for the first sum in the second member. Write, for

$$p_2 \leqslant p < q \leqslant p_3,$$
$$a_\nu = a_\nu^{(p)} = x_p^\nu, \qquad b_\nu = b_\nu^{(p)} = x_{p+1}^\nu,$$
$$c_\nu = c_\nu^{(p)} = x_q^\nu, \qquad d_\nu = d_\nu^{(q)} = x_{q+1}^\nu.$$

Then, for $p_2 \leqslant p \leqslant p_3$,

$$x_p^n < \{1-(1+\delta)^{-p_3}\}^n \sim \{1-n^{-1}e^{(\log n)^{\frac{1}{2}}}\}^n$$
$$< \exp\{-e^{(\log n)^{\frac{1}{2}}}\}.$$

Hence $$A = A^{(p)} = \frac{1-x_p^{2n+2}}{1-x_p^2} \sim \frac{1}{(1-x_p^2)^{\frac{1}{2}}} \sim \frac{1}{\sqrt{2}}(1+\delta)^{p/2}. \tag{5.4}$$

Further we may take $$\sigma = (1+\delta)^{-(q-p)/2}, \tag{5.5}$$

while

$$\tau^2 = \tau_p^2 = 1 - \frac{\left(\sum_0^n x_p^\nu x_{p+1}^\nu\right)^2}{\sum_0^n x_p^{2\nu} \sum_0^n x_{p+1}^{2\nu}} \sim 1 - \frac{(1-x_p^2)(1-x_{p+1}^2)}{(1-x_p\, x_{p+1})^2} = \frac{(x_{p+1}-x_p)^2}{(1-x_p\, x_{p+1})^2},$$

so that

$$\tau_p \sim \frac{x_{p+1}-x_p}{1-x_p\, x_{p+1}} = \frac{\delta}{(1+\delta)^{p+1}} \frac{1}{1-x_p\, x_{p+1}}$$
$$\sim \delta\{2+\delta-(1+\delta)^{-p}\}^{-1}.$$

That is $$\tau_p = \tfrac{1}{2}\delta + O(\delta^2), \tag{5.6}$$

and also $$\sin^{-1}\tau_p = \tfrac{1}{2}\delta + O(\delta^2).$$

By Lemma 12 and (5.4) we have

$$\int_0^1 \mu_p(t)\, dt = \frac{1}{\pi}\sin^{-1}\tau_p + O\{\sqrt{p}\,\delta^{\frac{1}{2}}(1+\delta)^{-p/3}\}.$$

But the function $\mu_p(t)$ of § 3 is now identical with the function $N_p^*(t)$, so using (5.1) and (5.6) we have

$$\int_0^1 N_p^*(t)\, dt = \frac{\delta}{2\pi} + O(\delta^2), \tag{5.7}$$

for $p \geqslant p_2$. Writing $$m_p = \int_0^1 N_p^*(t)\, dt,$$

we have from Lemma 18, (5.4), (5.5), and (5.6),

$$\int_0^1 N_p^*(t) N_q^*(t)\, dt = m_p\, m_q + O\{\sqrt{p}\delta^{\frac{1}{2}}(1+\delta)^{-p/3}\} +$$
$$+ O\{\delta(q-p)(1+\delta)^{-(q-p)/6}\} +$$
$$+ O\{\delta^{-1}(1+\delta)^{-(q-p)/2}\} \tag{5.8}$$

for $p_2 \leqslant p < q \leqslant p_3$. Consider

$$\int_0^1 \Big\{\sum_{p=p_2}^{p_3} (N_p^*(t)-m_p)\Big\}^2\,dt = \int_0^1 \Big\{\sum_{p=p_2}^{p_3} N_p^*(t)\Big\}^2\,dt - \Big(\sum_{p=p_2}^{p_3} m_p\Big)^2$$

$$= \sum_{p=p_2}^{p_3}\sum_{q=p_2}^{p_3}\Big\{\int_0^1 N_p^*(t)N_q^*(t)\,dt - m_p m_q\Big\}$$

$$\leqslant 2\sum_{p=p_2}^{p_3-k}\sum_{q=p_2+k}^{p_3}\Big\{\int_0^1 N_p^*(t)N_q^*(t)\,dt - m_p m_q\Big\}+$$

$$+\sum_{p=p_2}^{p_3}\sum_{|p-q|<k}\int_0^1 N_p^*(t)N_q^*(t)\,dt$$

$$= \Sigma_1 + \Sigma_2,$$

where $k = \Big[\dfrac{4}{\delta}\log\dfrac{1}{\delta}\Big]$. But by (5.8)

$$\Sigma_1 = O(p_3\,\delta^{1/6})\sum_{p=p_2}^{p_3-k}\sqrt{p}(1+\delta)^{-p/3}+$$

$$+O(\delta)\sum_{p=p_2}^{p_3-k}\sum_{q=p+k}^{p_3}(q-p)(1+\delta)^{-(q-p)/6}+$$

$$+O(\delta^{-1})\sum_{p=p_2}^{p_3-k}\sum_{q=p+k}^{p_3}(1+\delta)^{-(q-p)/2}$$

$$= O\{p_2^{3/2}\,\delta^{-5/6}(1+\delta)^{-p_2/3}\}+$$

$$+O\{p_2(k+\delta^{-1})(1+\delta)^{-k/6}\}+O\{p_2\,\delta^{-2}(1+\delta)^{-k/2}\}.$$

Using (5.1) and inserting the values of k and δ, we get

$$\Sigma_1 = O(p_1).$$

On the other hand, since $0 \leqslant N_p^*(t) \leqslant 1$,

$$\int_0^1 N_p^*(t)N_q^*(t)\,dt \leqslant \int_0^1 N_p^*(t)\,dt = O(\delta)$$

by (5.6) and (5.7). Therefore

$$\Sigma_2 = O(p_1\,k\delta) = O(p_1 \log 1/\delta).$$

We deduce that

$$\int_0^1 \Big\{\sum_{p_2}^{p_3}(N_p^*(t)-m_p)\Big\}^2\,dt = O(p_1 \log 1/\delta) = O(\log^{2/3} n \operatorname{loglog} n),$$

and hence that

$$\Big|\sum_{p_2}^{p_3}\{N_p^*(t)-m_p\}\Big| = o\,(\log^{2/3} n \operatorname{loglog} n),$$

except for a set of t of measure at most $o\{(\log\log n)^{-\frac{1}{2}}\}$. Combining this with (5.3), we see that, outside the exceptional set,

$$\sum_{p=p_0}^{p_1} N_p(t) = \sum_{p_2}^{p_3} m_p + o\{\log^{\frac{2}{3}} n \log\log n\},$$

and using (5.5), (5.6), and (5.7) that this expression is

$$\frac{1}{2\pi}(p_3-p_2)\delta + O(p_3\,\delta^2) + o\{\log^{\frac{2}{3}} n \log\log n\}.$$

But $$p_3-p_2 = p_1-2p_2 = \delta^{-1}\log n + O(\log^{\frac{2}{3}} n).$$

Hence, outside a set of measure at most $o\{(\log\log n)^{-\frac{1}{2}}\}$

$$\sum_{p=p_0}^{p_1} N_p(t) = \frac{1}{2\pi}\log n + o\{\log^{\frac{2}{3}} n \log\log n\}.$$

But the first member denotes the number of zeros of the equation parameter t in the range $(\frac{1}{2}, 1)$, and as explained in § 1 this completes the proof of our theorem.

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