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THE NUMBER OF MULTINOMIAL COEFFICIENTS

PAUL ERDÖS, National Bureau of Standards, and IVAN NIVEN, University of Oregon

The problem is to find the number of multinomial coefficients

$$(1) \qquad \frac{n!}{i_1!i_2! \cdots i_r!(n-k)!}, \qquad \sum_{j=1}^{r} i_j = k,$$

which are less than x, excluding the cases $r=1=i_1$ and $r=1=k$ for which (1) assumes the value n. The values of (1) are thus restricted by

(2) $$i_1 \leqq i_2 \leqq \cdots \leqq i_r \leqq n - k \leqq n - 2.$$

One of the writers [1] proved earlier that these multinomial coefficients have density zero; we now prove the following stronger result.

THEOREM. *The number of multinomial coefficients* (1) *which satisfy* (2) *and which are less than a fixed* $x>0$ *is* $(1+\sqrt{2})x^{1/2}+o(x^{1/2})$.

To prove this we divide the values (1) into 3 classes and treat each class separately. The first class, $f_1(x)$ in number, contains those having $k=2$; the second class, $f_2(x)$ in number, those with $3\leqq k\leqq n/2$; the third class, $f_3(x)$ in number, those with $k>n/2$. We prove that

(3) $$f_1(x) = (1+\sqrt{2})x^{1/2} + o(x^{1/2}), \qquad f_2(x) = o(x^{1/2}), \qquad f_3(x) = o(x^{1/2}),$$

which will establish the theorem.

Class 1. For $k=2$ the values (1) are the two types, $n(n-1)/2$ and $n(n-1)$. Now $n(n-1)<x$ for $x^{1/2}+o(x^{1/2})$ values of n, and $n(n-1)/2<x$ for $(2x)^{1/2}+o(x^{1/2})$ values of n. We must eliminate duplicates, that is cases where

(4) $$n(n-1)=m(m-1)/2.$$

We show that (4) has at most $c \log x$ solutions $<x$, and this will establish the first equation (3). Solving (4) for m we find that solutions exist if and only if $8n^2-8n+1$ is a perfect square, say u^2, and replacing $2n-1$ by z, we have (4) reduced to $u^2-2z^2=1$. The positive integral solutions of this equation are given (*cf.* [3]) by $u+z\sqrt{2}=(3+2\sqrt{2})^r$ for $r=1, 2, \cdots$, and the number of these less than x is of the order of $c \log x$.

Class 2. For any fixed k and n, the equation $k=\sum i_j$ indicates that the number of values of (1) is $p(k)$, the number of partitions of k into positive integers. The smallest of these $p(k)$ values is $\binom{n}{k}$, and so the admissible values of n and k will satisfy $\binom{n}{k}<x$. Now

(5) $$\binom{n}{k} = \prod_{j=0}^{k-1} \frac{n-j}{k-j} > \left(\frac{n}{k}\right)^k.$$

Hence the admissible values of n and k satisfy $\left(\frac{n}{k}\right)^k<x$ or $n<kx^{1/k}$. Thus for each k the maximum number of values of n is $kx^{1/k}$ and so

(6) $$f_2(x) < \sum_k (kx^{1/k})p(k),$$

the sum ranging over the admissible values of k. By definition of $f_2(x)$ the smallest k is $k=3$. To get an upper bound of k in terms of x we observe that the largest k corresponds to $n=2k$. Using $\binom{n}{k}<x$ again, we have that admissible values of k satisfy $\binom{2k}{k}<x$ and so satisfy $2^k<x$ since $2^k<\binom{2k}{k}$ by (5). Thus the range in the sum (6) can be taken as $k=3$ to $k=c \log x$.

Now [2] $p(k)<e^{c_1\sqrt{k}}$, so $p(c \log x)<e^{c_1(c \log x)^{1/2}}<x^\epsilon$ for arbitrary $\epsilon>0$ with x

sufficiently large. Maximizing each part of (6) gives

$$f_2(x) < (c \log x)(c \log x)x^{1/3}x^{\epsilon} = o(x^{1/2}).$$

Class 3. Every value (1) in this class is clearly

$$\geqq \frac{n!}{\left[\frac{n}{2}\right]!\left\{n - \left[\frac{n}{2}\right]\right\}!}.$$

Thus each admissible value of n satisfies $n \leqq 2h+1$ where h is chosen so that $\binom{2h}{h}$ exceeds x. Replacing $\binom{2h}{h}$ by 2^h as previously we see that $n < c \log x$. For any fixed n the number of values of (1) is maximized by $p(n)$. Thus $f_3(x) < c \log x \cdot p(c \log x) = o(x^{1/2})$, and the proof of (3) is complete.

A more careful analysis would improve the theorem to yield the estimate

$$(1 + \sqrt{2})x^{1/2} + c_3x^{1/3} + \cdots + c_mx^{1/m} + o(x^{1/m})$$

for every m. This could be proved by isolation of the cases $k=2, 3, \cdots, m$ for special treatment, where here we stopped at $k=2$.

References

1. Ivan Niven, The asymptotic density of sequences, Bull. Amer. Math. Soc., vol. 57, 1951, pp. 420–434, Theorem 2.

2. G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc. (2), vol. 17, 1918, pp. 75–115; or, Paul Erdös, On an elementary proof of some asymptotic formulas in the theory of partitions, Annals of Math. (2), vol. 43, 1942, pp. 437–450.

3. Nagell, T., Introduction to Number Theory, John Wiley (1952), Theorem 104, p. 197.