# ON A CONJECTURE OF HAMMERSLEY

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Denote by $\Sigma_{n,s}$ the sum of the products of the first $n$ natural numbers taken $s$ at a time, *i.e.* the $s$-th elementary symmetric function of $1, 2, \ldots, n$. Hammersley† conjectured that the value of $s$ which maximises $\Sigma_{n,s}$ for a given $n$ is unique. In the present note I shall prove this conjecture and discuss some related problems.

We shall denote by $f(n)$ the largest value of $s$ for which $\Sigma_{n,s}$ assumes its maximum value. As Hammersley† remarks, it follows immediately from a theorem of Newton that

$$\Sigma_{n,1} < \Sigma_{n,2} < \ldots < \Sigma_{n,f(n)-1} \leqslant \Sigma_{n,f(n)} > \Sigma_{n,f(n)+1} > \ldots > \Sigma_{n,n} = n!. \quad (1)$$

Thus it follows from (1) that the uniqueness of the maximising $s$ will follow if we can prove that

$$\Sigma_{n,f(n)-1} < \Sigma_{n,f(n)}. \quad (2)$$

Hammersley proves (2) for $1 \leqslant n \leqslant 188$. He also proves that

$$f(n) = n - \left[\log(n+1) + \gamma - 1 + \frac{\zeta(2) - \zeta(3)}{\log(n+1) + \gamma - \frac{3}{2}} + \frac{h}{\left(\log(n+1) + \gamma - \frac{3}{2}\right)^2}\right], \quad (3)$$

* Received 27 February, 1952; read, 20 March, 1952.

† J. M. Hammersley, *Proc. London Math. Soc.* (3), 1 (1951), 435–452.

where $[x]$ denotes the integral part of $x$, $\gamma$ denotes Euler's constant, $\zeta(k)$ is the Riemann $\zeta$-function and $-1{\cdot}1 < h < 1{\cdot}5$. Thus for $n > 188 > e^5$ we obtain by a simple computation

$$[\log n - \tfrac{1}{2}] \leqslant n - f(n) \leqslant [\log n]. \tag{4}$$

First we prove

Theorem 1. *For sufficiently large $n$ all the integers $\Sigma_{n,s}$, $1 \leqslant s \leqslant n$, are different.*

We evidently have*

$$\Sigma_{n,n-k} < \frac{n!}{k!}\left(\sum_{l=1}^{n} \frac{1}{l}\right)^k < \frac{n!}{k!}(1+\log n)^k < n!\left\{\frac{e}{k}(1+\log n)\right\}^k \leqslant n! = \Sigma_{n,n} \tag{5}$$

for $k \geqslant e(\log n + 1)$. Thus from (1) and (5) it follows that to prove Theorem 1 we have only to consider the values

$$0 \leqslant k < e(\log n + 1). \tag{6}$$

The Prime Number Theorem in its slightly sharper form states that for every $l$

$$\pi(x) = \int_2^x \frac{dy}{\log y} + O\left(\frac{x}{(\log x)^l}\right). \tag{7}$$

From (7) we have that for sufficiently large $x$ there is a prime between $x$ and $x + x/(\log x)^2$. Thus we obtain that for $n > n_0$ and $k < e(\log n + 1)$ there always is a prime $p_k$ satisfying

$$\frac{n}{k+1} < p_k \leqslant \frac{n}{k}.$$

We have

$$\Sigma_{n,n-k} \not\equiv 0 \pmod{p_k}. \tag{8}$$

For $\Sigma_{n,n-k}$ is the sum of $\binom{n}{k}$ products each having $n-k$ factors. Clearly only one of these products is not a multiple of $p_k$ (viz., the one in which none of the $k$ multiples not exceeding $n$ of $p_k$ occur); thus (8) is proved. For $r < k$ all the $\binom{n}{r}$ summands of $\Sigma_{n,n-r}$ are multiples of $p_k$. Thus

$$\Sigma_{n,n-r} \equiv 0 \pmod{p_k}. \tag{9}$$

(8) and (9) complete the proof of Theorem 1.

We now give an elementary proof of Theorem 1 which will be needed in the proof of Hammersley's conjecture. Let

$$r < k < e(\log n + 1). \tag{10}$$

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* The proof is similar to the one in a joint paper with Niven, *Bull. Amer. Math. Soc.*, 52 (1946), 248–251. We prove there that for $n > n_0$, $\Sigma_{n,s} \not\equiv 0 \pmod{n!}$.

We shall prove that for $n > 10^8$

$$\Sigma_{n,\,n-r} \neq \Sigma_{n,\,n-k}. \tag{11}$$

Let $q$ be a prime satisfying $n/2k < q \leqslant n/k$. Assume that

$$\frac{n}{l+1} < q \leqslant \frac{n}{l}, \quad k \leqslant l \leqslant 2k-1.$$

Clearly
$$\Sigma_{n,\,n-r} \equiv 0 \pmod{q^{l-r}}. \tag{12}$$

Now we compute the residue of $\Sigma_{n,\,n-k}$ (mod $q^{l-k+1}$). Clearly $\Sigma_{n,\,n-k} \equiv 0$ (mod $q^{l-k}$). The only summands of $\Sigma_{n,\,n-k}$ which are not multiples of $q^{l-k+1}$ are those which contain $\Pi' t$ where the product is extended over the integers $1 \leqslant t \leqslant n$, $t \not\equiv 0$ (mod $q$). $\Pi' t$ contains $n-l$ factors, and the remaining $l-k$ factors of the summands in question of $\Sigma_{n,\,n-k}$ must be among the integers $q, 2q, \ldots, lq$. Thus clearly

$$\Sigma_{n,\,n-k} \equiv \Sigma_{l,\,l-k} \,.\, \Pi' t \,.\, q^{l-k} \pmod{q^{l-k+1}}. \tag{13}$$

Therefore if (11) does not hold we must have

$$\Sigma_{l,\,l-k} \equiv 0 \pmod{q} \; \left(i.e.\ \Sigma_{n,\,n-k} \equiv \Sigma_{n,\,n-r} \equiv 0 \pmod{q^{l-k+1}}\right).$$

Thus if (11) is false

$$\prod_{n/2k < q \leqslant n/k} q \,\Big|\, \prod_{l=k}^{2k-1} \Sigma_{l,\,l-k}. \tag{14}$$

Now evidently (we can of course assume that $k \geqslant 2$ for if $k = 1$ then (11) clearly holds)

$$\prod_{l=k}^{2k-1} \Sigma_{l,\,l-k} < \prod_{l=k}^{2k-1} \binom{l}{k} l^{l-k} < \prod_{l=k}^{2k-1} (2k)^l < (2k)^{\frac{3}{2}k^2} \leqslant k^{3k^2} < (3 \log n)^{27 (\log n)^2}, \tag{15}$$

since for $n > 10^8 > e^{10}$, $k < e(1+\log n) < 3 \log n$. Define

$$\vartheta(x) = \sum_{p \leqslant x} \log p.$$

By the well-known results of Tchebycheff* we have

$$\vartheta(2x) - \vartheta(x) > 0{\cdot}7 \,.\, x - 3{\cdot}4 \,.\, x^{\frac{1}{2}} - 4{\cdot}5 (\log x)^2 - 24 \log x - 32.$$

Thus for $n > 10^4$ we have by a simple computation

$$\vartheta(2x) - \vartheta(x) > \tfrac{1}{2} x. \tag{16}$$

For $n > 10^8$, we have $n/2k > n/(6 \log n) > 10^4$. Thus from (16) we have

$$\prod_{n/2k < q \leqslant n/k} q > e^{n/4k} > e^{n/(12 \log n)}. \tag{17}$$

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* E. Landau, *Verteilung der Primzahlen*, I, 91.

From (14), (15) and (17) we have

$$(3 \log n)^{27(\log n)^3} \geqslant e^{n/(12 \log n)}.$$

Thus on taking logarithms and using $\log(3 \log n) < \log n$ for $n > 10^8$,

$$27(\log n)^3 > n/(12 \log n) \quad \text{or} \quad 324(\log n)^4 > n,$$

which is false for $n > 10^8$. Thus the proof of Theorem 1 is complete.

THEOREM 2 (Hammersley's conjecture). *The value of s which maximises* $\Sigma_{n,s}$ *is unique; in other words*

$$\Sigma_{n, f(n)-1} \neq \Sigma_{n, f(n)}. \tag{18}$$

It follows from the second proof of Theorem 1 that Theorem 2 certainly holds if for $n > 10^8$. Thus since Hammersley proved Theorem 2 for $n \leqslant 188$ it suffices to consider the interval $188 < n \leqslant 10^8$.

Put $n - f(n) = t$. We have, from (4),

$$\log n - 2 \leqslant t \leqslant \log n. \tag{19}$$

As was shown in the first proof of Theorem 1, (18) certainly holds if there is a prime satisfying

$$n/(t+2) < p \leqslant n/(t+1). \tag{20}$$

It follows from (19) that if $1500 < n \leqslant 10^8$

$$150 < n/(t+2) < 10^7.$$

The tables of primes* show that for $150 < x < 10^7$ there always is a prime $q$ satisfying $x < q < x + x^{\frac{1}{2}}$. For $n > 1500$ we have

$$\frac{n}{t+2} + \left(\frac{n}{t+2}\right)^{\frac{1}{2}} < \frac{n}{t+1},$$

since

$$\frac{n}{(t+1)(t+2)} > \left(\frac{n}{t+2}\right)^{\frac{1}{2}},$$

or, by using (19), $\quad n > (1 + \log n)^2 (2 + \log n),$

which holds for $n > 1500$. Thus for $1500 < n < 10^7$ there always is a prime in the interval (20) and thus Theorem 2 is proved for $n > 1500$.

To complete our proof we only have to dispose of the $n$ satisfying $188 < n \leqslant 1500$. Hammersley† showed that for $n \leqslant 1500$ the only doubtful values of $n$ are: $189 \leqslant n \leqslant 216$, $539 \leqslant n \leqslant 580$. He also showed that if $189 \leqslant n \leqslant 216$ and (18) does not hold, then $t = 5$. But then $p = 31$ is in the interval (20), which shows that (18) holds in this case. If $539 \leqslant n \leqslant 590$ and (18) does not hold, he shows that $t = 6$. But then

* A. E. Western, *Journal London Math. Soc.*, 9 (1934), 276–278.

† See footnote †, p. 232.

either $p = 73$ or $p = 79$ lies in the interval (20). Thus (18) holds here too, and the proof of Theorem 2 is complete.

By slightly longer computations we could prove that for $n \geqslant 5000$ Theorem 1 holds. Theorem 1 is certainly not true for all values of $n$ since $\Sigma_{3,1} = \Sigma_{3,3}$. Hammersley proved that for $n \leqslant 12$ this is the only case for which Theorem 1 fails, and it is possible that Theorem 1 holds for all $n > 3$. The condition $n \geqslant 5000$ could be considerably relaxed, but to prove Theorem 1 for $n > 3$ would require much longer computations.

Let $u_1 < u_2 < \ldots$ be an infinite sequence of integers. Denote again by $\Sigma_{n,s}$ the sum of the products of the first $n$ of them taken $s$ at a time. It seems possible that for $n > n_0$ ($n_0$ depends on the sequence) the maximising $s$ is unique and even that for $n > n_1$ all the $n$ numbers $\Sigma_{n,s}$, $1 \leqslant s \leqslant n$ are distinct. If the $u$'s are the integers $\equiv a \pmod d$ it is not hard to prove this theorem.

Stone and I proved by elementary methods the following

**THEOREM.** *Let $u_1 < u_2 < \ldots$ be an infinite sequence of positive real numbers such that*

$$\Sigma \frac{1}{u_i} = \infty \quad \text{and} \quad \Sigma \frac{1}{u_i^2} < \infty.$$

*Denote by $\Sigma_{n,s}$ the sum of the product of the first n of them taken s at a time and denote by $f(n)$ the largest value of s for which $\Sigma_{n,s}$ assumes its maximum value. Then*

$$f(n) = n - \left[\sum_{i=1}^{n} \frac{1}{u_i} - \sum_{i=1}^{\infty} \frac{1}{u_i^2}\left(1+\frac{1}{u_i}\right)^{-1} + o(1)\right].$$

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