MATHEMATICS

SOME LINEAR AND SOME QUADRATIC RECURSION FORMULAS. I

BY

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§ 1. *Introduction*

We shall mainly deal with linear recursion formulas of the type

$$(1.1)\qquad f(1)=1;\qquad f(n)=\sum_{k=1}^{n-1} c_k f(n-k)\qquad (n=2,3,\ldots),$$

and with quadratic formulas of the type

$$(1.2)\qquad f(1)=1;\qquad f(n)=\sum_{k=1}^{n-1} d_k f(k)f(n-k)\qquad (n=2,3\ldots).$$

We assume that $c_k>0$, $d_k>0$ $(k=1,2,\ldots)$. In a previous paper [1] we discussed (1. 1) under the condition $\Sigma_1^\infty c_k=1$, and further special assumptions. Presently we deal with it more generally. We shall show that $\lim\,\{f(n)\}^{-1/n}$ always exists, and we shall give several sufficient conditions for the existence of $\lim f(n)/f(n+1)$. Some of the results can be applied to (1. 2) (see § 6), and some of the methods can be extended to recurrence relations with coefficients c depending on n also (see § 3 and § 7).

In [1] as well as in the earlier paper of ERDÖS, FELLER and POLLARD [3], referred to below, the condition on the c_k was $c_k\geqslant 0$ $(k=1,2,\ldots)$, whereas the g.c.d. of the k's with $c_k=0$ was assumed to be 1. For convenience we assume $c_k>0$ throughout. Consequently we have, both for (1. 1) and for (1. 2), $f(n)>0$ $(n=1,2,\ldots)$.

Dealing with the linear relation (1. 1) we put formally

$$(1.3)\qquad C(x)=\sum_1^\infty c_n x^n\quad,\quad F(x)=\sum_1^\infty f(n)\,x^n,$$

and we have formally

$$(1.4)\qquad F(x)=x+C(x)\,F(x).$$

Furthermore, if ϱ is a positive number, and if we put

$$(1.5)\qquad f(n)=\varrho^{-n+1}\,g(n),$$

then we have

$$(1.6)\qquad g(n)=\sum_{k=1}^{n-1} b_k\,g(n-k)\quad,\quad g(1)=1,$$

where $b_k=c_k\varrho^k$. Formula (1. 6) is again of the type (1. 1), and $b_k>0$ for all k.

§ 2. *Linear recursions, different cases*

We discern amongst 5 different cases with respect to the behaviour of the series $C(x)$ (see (1. 3)). Let R be the radius of convergence $(0 \leqslant R \leqslant \infty)$ and let γ be the l.u.b. of the numbers a with $C(a) \leqslant 1$.

Case 1. $\gamma = R = 0$.

Case 2. $0 < \gamma < R \leqslant \infty$, $C(\gamma) = 1$.

Case 3. $0 < \gamma = R < \infty$, $C(\gamma) = 1$, $0 < C'(\gamma) < \infty$.

Case 4. $0 < \gamma = R < \infty$, $C(\gamma) = 1$, $C'(\gamma) = \infty$.

Case 5. $0 < \gamma = R < \infty$, $0 < C(\gamma) < 1$.

Since the coefficients c_k are positive it is easily seen that all possibilities are listed here.

§ 5 will be specially devoted to case 1; nevertheless case 1 is not excluded in §§ 2, 3, 4 unless explicitly stated.

In all cases we can show (§ 3)

$$(f(n))^{-\frac{1}{n}} \to \gamma, \tag{2. 1}$$

In case 1 we infer that also $F(x)$ has 0 as its radius of convergence. In the other cases we can transform by (1. 5), taking $\varrho = \gamma$. Apart from case 5, this leads to (1. 6) with $\Sigma b_k = 1$. Therefore we can apply the results of Erdös, Feller and Pollard [3], and we obtain

$$\lim_{n\to\infty} f(n)\,\gamma^n \begin{cases} = \{C'(\gamma)\}^{-1} & \text{in cases 2 and 3,} \\ = \quad 0 & \text{in case 4.} \end{cases} \tag{2. 2}$$

If the limit is $= 0$, we have not yet an asymptotic formula for $f(n)$, and such a formula seems to be hard to obtain without introducing very special assumptions (see [1]).

In case 5 we have, just as in case 4, $f(n)\gamma^n \to 0$. For, it follows from (1. 4) that

$$\textstyle\sum_1^\infty f(n)\,\gamma^n = \gamma/(1-C(\gamma)); \tag{2. 3}$$

hence the series on the left is divergent in cases 2, 3, 4 but convergent in case 5.

In case 2 it can be shown that for some $C > 0$ and some $\delta > \gamma$ we have

$$f(n) = C\,\gamma^{-n} + O(\delta^{-n}). \tag{2. 4}$$

For, the coefficients of $C(x)$ being positive, we have $C(x) \neq 1$ $(|x| \leqslant \gamma,$ $x \neq \gamma)$ and $C'(\gamma) \neq 0$. Now (1. 4) shows that $F(x)$ is regular in $|x| \leqslant \gamma$ apart from a simple pole at $x = \gamma$. This proves (2. 4).

Apart from case 1 we have $\gamma > 0$, $C(\gamma) \leqslant 1$ and so, by induction

$$f(n) \leqslant \gamma^{1-n} \qquad (n = 1, 2, 3, \dots). \tag{2. 5}$$

In all cases we put

$$\liminf_{n\to\infty} \frac{f(n)}{f(n+1)} = \alpha \quad , \quad \limsup_{n\to\infty} \frac{f(n)}{f(n+1)} = \beta ,$$

and we have

$$(2.\ 6) \qquad 0 \leqslant \alpha \leqslant \gamma \leqslant \beta \leqslant c_1^{-1} < \infty .$$

For, (2. 1) shows that $\alpha \leqslant \gamma \leqslant \beta$, and $\beta \leqslant c_1^{-1}$ follows from the inequality $f(n+1) \geqslant c_1 f(n)$, which immediately follows from (1. 1).

§ 3. *Linear recursion; existence of lim* $\{f(n)\}^{-1/n}$

We shall show (theorem 2) that $\{f(n)\}^{-1/n}$ tends to a finite limit in all cases. Denoting the limit by L, it is easily proved afterwards that $L = \gamma$.

The existence of the limit will be shown for a slightly more general recursion formula.

Theorem 1. Let $0 < c_{k,k+1} \leqslant c_{k,k+2} \leqslant c_{k,k+3} \leqslant \dots \quad (k = 1, 2, 3, \dots)$.

$$(3.\ 1) \qquad f(1) = 1 \quad , \quad f(n) = \sum_{k=1}^{n-1} c_{k,n} f(n-k) \qquad (n = 2, 3, \dots).$$

Then we have

$$(3.\ 2) \qquad f(n+k-1) \geqslant f(n)\, f(k) \qquad (k, n = 1, 2, 3, \dots).$$

Proof. We apply induction with respect to n. If $n = 1$, (3. 2) is trivial. Now assume that (3. 2) holds for $n = 1, \dots, N$. Then we have

$$\begin{aligned} f(N+k) &= \sum_{l=1}^{N+k-1} c_{l,N+k} f(N+k-l) \geqslant \\ &\geqslant \textstyle\sum_1^N c_{l,N+k} f(N+k-l) \geqslant \sum_1^N c_{l,N+1} f(N+k-l) \geqslant \\ &\geqslant \textstyle\sum_1^N c_{l,N+1} f(N+1-l)\, f(k) = f(N+1)\, f(k), \end{aligned}$$

and the induction is complete.

Theorem 2. Under the assumptions of theorem 1 we have, putting

$$\inf \{ f(n+1)\}^{-1/n} = L \qquad (0 \leqslant L < \infty),$$

that

$$\lim_{n\to\infty} \{f(n+1)\}^{-1/n} = L.$$

Proof. Clearly we have $f(n) > 0$ $(n = 1, 2, \dots)$. Putting

$$g(n) = -\log f(n+1),$$

we infer from (3. 2) that $g(n)$ is sub-additive:

$$g(n+k) \leqslant g(n) + g(k) \qquad (n, k = 0, 1, 2, \dots).$$

It follows that

$$-\infty \leqslant \inf \frac{g(n)}{n} = \lim_{n\to\infty} \frac{g(n)}{n} < \infty .$$

(See [4], vol. 1, p. 17 and 171. An extension of this theorem will be given in § 7).

We next show for the equation (1. 1) that $L = \gamma$. We have $f(n) \geqslant c_{n-1}$ for all $n > 1$; therefore the radius of convergence of $F(x)$ is $\leqslant R$, and so $L \leqslant R$. In case 1 this means $L = 0 = \gamma$.

In case 2 we have $L = \gamma$ by (2. 4).

In the remaining cases we have $R = \gamma$, and so $L \leqslant \gamma$. On the other hand (2. 5) gives $L \geqslant \gamma$.

§ 4. *Linear recursion; existence of* $\lim f(n)/f(n+1)$

If $\lim f(n)/f(n+1)$ exists, it equals γ (see (2. 6)). In the cases 2 and 3 the limit exists (by (2. 2)). In the other cases $f(n)/f(n+1)$ can be oscillating, and we can even have (with the notations of (2. 6)) $\beta > \alpha = 0$.

In cases 4 and 5 we construct an example as follows. Let σ be a number, $0 < \sigma \leqslant 1$; and let $p_1 + p_2 + \ldots$, be a series of positive terms whose sum is $\frac{1}{2}\sigma$. We shall construct a series $c_1 + c_2 + \ldots$ with $c_k \geqslant p_k$, whose sum is σ, and such that $c_n/f(n)$ is not bounded.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence with $\varepsilon_k > 0$, $\varepsilon_k \to 0$. Take $c_k = p_k$ for $k = 1, 2, \ldots, K_1 - 1$, where K_1 is the first k with $f(k) < \frac{1}{4}\varepsilon_1\sigma$. The existence of this k follows from the inequality

$$(4.\ 1) \qquad f(1) + \ldots + f(m) < \{1 - \sum_1^{m-1} c_k\}^{-1},$$

which is obtained by addition of the formulas (1. 1) with $n = 1, 2, \ldots, m$, respectively.

Now take $c_k = \frac{1}{4}\sigma + p_k$ if $k = K_1$, which does not alter the values of $f(1), \ldots, f(K_1)$. If $k = K_1 + 1, \ldots, K_2 - 1$ we take $c_k = p_k$ again, where K_2 is the first $k > K_1$ with $f(k) < \frac{1}{8}\varepsilon_2\sigma$. For $k = K_2$ take $c_k = \frac{1}{8}\sigma + p_k$ etc. If $k = K_1, K_2, \ldots$ we have $c_k/f(k) > \varepsilon_1^{-1}, \varepsilon_2^{-1}, \ldots$, respectively. As $f(k+1) > c_k$ for all k, we also find that $f(k+1)/f(k)$ is not bounded. Therefore $\alpha = 0$. On the other hand we have $\beta > 0$ by (2. 6), since γ is positive. It can be shown that $\gamma = 1$, $C(\gamma) = \sigma$.

A sufficient condition for α to be positive is that $\Sigma c_k/f(k) < \infty$. For, writing down (1. 1) with $n = N + 1$ and $n = N$, respectively, we infer

$$\frac{f(N+1)}{f(N)} \leqslant \max_{1 \leqslant k < N} \frac{f(k+1)}{f(k)} + \frac{c_N}{f(N)},$$

whence $f(n+1) = O\{f(n)\}$.

In case 1 the series $\Sigma c_k/f(k)$ does not converge since it would lead to $\alpha > 0$. In cases 2 and 3 the series always converges (see (2. 2)). In case 4 the condition may be useful, and we can show that it implies $\alpha = \beta$ (theorem 11). In case 5 however the condition never applies:

Theorem 3. In case 5 we have $\Sigma c_k/f(k) = \infty$.

Proof. We have $\Sigma_1^\infty c_k\gamma^k < 1$. Assume $\Sigma c_k/f(k) < \infty$.

Put $1 - \Sigma_1^\infty c_k\gamma^k = 2\varepsilon$. Choose l such that $2\gamma \Sigma_{l+1}^\infty c_k/f(k) < \varepsilon$, and $\delta > 0$ such that $e^{\delta l}\Sigma_1^l c_k\gamma^k < 1 - \varepsilon$, $e^\delta < 2$. Then we can show by induction

$$f(k) \leqslant 2e^{-\delta k}\gamma^{1-k}. \tag{4.2}$$

If $k = 1$, (4. 2) is trivial. Next assume (4. 2) to be true for $k = 1, \ldots, n-1$. Then by (1. 1)

$$f(n) \leqslant \sum_1^s c_k f(n-k) + \sum_{s+1}^{n-1} \frac{c_k}{f(k)} f(k) f(n-k),$$

where $s = \min(n-1, l)$, and the second sum is empty if $n - 1 \leqslant l$. It follows that

$$f(n) \leqslant \sum_1^s c_k e^{\delta k}\gamma^k \cdot 2e^{-\delta n}\gamma^{1-n} + 4\sum_{s+1}^{n-1} \frac{c_k}{f(k)} e^{-\delta n}\gamma^{2-n} \leqslant$$
$$\leqslant 2e^{-\delta n}\gamma^{1-n}\{e^{\delta l}\textstyle\sum_1^l c_k\gamma^k + 2\gamma\sum_{l+1}^\infty c_k/f(k)\} < 2e^{-\delta n}\gamma^{1-n}.$$

This proves (4. 2). However, (4. 2) contradicts (2. 1). Therefore our assumption $\Sigma c_k/f(k) < \infty$ is false.

We next discuss the condition $c_k = o\{f(k)\}$. We do not know whether this guarantees the existence of $\lim f(n)/f(n+1)$. On the other hand it is a necessary condition in cases 2, 3 and 4 (theorem 4), but it is not necessary in case 5.

In case 5 we can give an example where

$$\frac{f(n+1)}{f(n)} \to 1, \quad \frac{c_{n+1}}{c_n} \to 1, \quad \frac{c_n}{f(n)} \to \tfrac{1}{4}. \tag{4.3}$$

In order to construct this example, require (1. 1) and $c_n = \frac{1}{4}f(n)$ for all n. Then we have $F(x) - x = \frac{1}{4}F^2(x)$, and so

$$F(x) = 2\{1 - (1-x)^{\frac{1}{2}}\}, \; f(n) = \frac{4^{-n}}{2n-1}\frac{(2n)!}{n!\,n!}.$$

We are in case 5 indeed, for the radius of convergence of $C(x) = \frac{1}{4}F(x)$ equals 1, and

$$\textstyle\sum_1^\infty c_k = M \cdot F(1) = \frac{1}{2}.$$

Theorem 4. If, in case 2, 3 or 4, $\lim f(n)/f(n+1)$ exists [1]), then we have $c_n = o\{f(n)\}$.

Proof. If the limit exists, we know that it equals γ. And, if $n > k + 1$, we have

$$f(n+1) \geqslant c_1 f(n) + \ldots + c_{k+1} f(n-k) + c_n. \tag{4.4}$$

Dividing by $f(n)$ and making $n \to \infty$, we infer

$$\gamma^{-1} \geqslant c_1 + c_2\gamma + \ldots + c_k\gamma^{k-1} + \limsup c_n/f(n),$$
$$\limsup c_n/f(n) \leqslant \gamma^{-1}\{1 - c_1\gamma - c_2\gamma^2 - \ldots - c_k\gamma^k\}.$$

This holds for every k. Since $\Sigma c_k\gamma^k = 1$ we infer $c_n = o\{f(n)\}$.

[1]) In case 2 or 3 the limit exists automatically.

Theorem 5. If, in case 2, 3, or 4, $\lim c_{n+1}/c_n$ exists, then we have $c_n = o\{f(n)\}$.

Proof. The limit of c_{n+1}/c_n equals γ^{-1}, of course. If $n > k$, we have

$$f(n) \geqslant c_n f(1) + c_{n-1} f(2) + \ldots + c_{n-k} f(k).$$

Dividing by c_n and making $n \to \infty$, we infer

$$\liminf f(n)/c_n \geqslant f(1) + f(2)\gamma + \ldots + f(k)\gamma^{k-1}.$$

The theorem follows from the fact that $\Sigma f(k)\gamma^{k-1} = \infty$ (see (2. 3)).

The following simple theorem applies to the cases 2, 3, 4, 5 (in case 1 the condition is never satisfied).

Theorem 6. If, for some fixed k, we have $c_n = O(c_{n-1} + c_{n-2} + \ldots + c_{n-k})$, then $f(n+1) = O\{f(n)\}$, that is $\alpha > 0$.

Proof. For $n > k$ we have

$$\frac{c_{n-k} f(k+1) + \ldots + c_n f(1)}{c_{n-k} f(k) + \ldots + c_{n-1} f(1)} \leqslant \max_{1 \leqslant i \leqslant k} \frac{f(j+1)}{f(j)} + \frac{C(c_{n-k} + \ldots + c_{n-1})}{c_{n-k} f(k) + \ldots c_{n-1} f(1)} < B,$$

B not depending on n. Furthermore, if $n > k$,

$$f(n+1) = \textstyle\sum_1^n c_j f(n+1-j) \leqslant$$

$$\leqslant \sum_1^{n-k-1} c_j f(n-j) \cdot \max_{1 \leqslant l < n} \frac{f(l+1)}{f(l)} + B \sum_{n-k}^{n-1} c_j f(n-j) \leqslant$$

$$\leqslant f(n) \max \left\{ B, \max_{1 \leqslant l < n} \frac{f(l+1)}{f(l)} \right\}.$$

It follows by induction that $f(n+1) \leqslant B f(n)$ for all n.

We shall give a necessary and sufficient condition for the existence of $\lim f(n)/f(n+1)$ in the cases 2, 3, 4, 5. That is, we assume

$$(4.\ 5) \qquad \gamma > 0,\ \textstyle\sum_1^\infty c_k \gamma^k \leqslant 1;\ 1 < \sum_1^\infty c_k x^k \leqslant \infty \text{ if } x > \gamma.$$

Put, if $1 \leqslant k < n$,

$$(4.\ 6) \quad \begin{cases} \dfrac{\gamma\{c_k f(n-k+1) + \ldots + c_n f(1)\} - \{c_k f(n-k) + \ldots + c_{n-1} f(1)\}}{f(n)} = S_{n,k}; \\ \limsup_{n\to\infty} |S_{n,k}| = \varphi(k) \leqslant \infty. \end{cases}$$

Theorem 7. In the cases 2, 3, 4, 5 a necessary and sufficient condition for the existence of $\lim f(n)/f(n+1)$ is that $\varphi(k) \to 0$ when $k \to \infty$.

Proof. We have, if $1 \leqslant k < n$,

$$(4.\ 7) \qquad \gamma f(n+1) - f(n) = \gamma \sum_1^{k-1} c_j f(n+1-j) - \sum_1^{k-1} c_j f(n-j) + f(n) S_{n,k}.$$

If $f(n)/f(n+1) \to \gamma$, it easily follows by making $n \to \infty$ that $\varphi(k) = 0$ for all k.

We next show that $\varphi(k) \to 0$ is also sufficient. We have (see (2. 6))

$0 \leqslant \alpha \leqslant \beta < \infty$. First we prove that $\alpha > 0$. We have $f(l+1) \geqslant c_1 f(l)$ for all l. Hence, dividing (4. 7) by $f(n)$ we obtain

$$\gamma \frac{f(n+1)}{f(n)} \leqslant 1 + \sum_1^{k-1} c_j c_1^{1-j} + |S_{n,k}|.$$

Choose k such that $\varphi(k) < \infty$, and make $n \to \infty$. It follows that $f(n+1) = O(f(n))$, that is $\alpha > 0$.

Let $\{n_i\}$ be a sequence for which

$$f(n_i)/f(n_i+1) \to \alpha \qquad (i \to \infty). \tag{4. 8}$$

Then we have, for any fixed $l \geqslant 0$, also

$$f(n_i - l)/f(n_i + 1 - l) \to \alpha \qquad (i \to \infty). \tag{4. 9}$$

The same holds if α is replaced by β both times. We only prove it for the lower limit; the other case can be proved analogously.

Assume (4. 9) false for some $l > 0$. Then there is a subsequence $\{m_i\}$ and a number δ $(\delta > \alpha)$ such that

$$f(m_i - l) > \delta f(m_i + 1 - l) \qquad (i = 1, 2, \ldots).$$

Further, if $\varepsilon > 0$ and $i > i_0(\varepsilon, k)$ then we have

$$f(m_i - j) > (\alpha - \varepsilon) f(m_i + 1 - j) \qquad (1 \leqslant j < k)$$

It follows, if $k > l$, $i > i_0(\varepsilon, k)$, that

$$\sum_{j=1}^{k-1} c_j \{\gamma f(m_i + 1 - j) - f(m_i - j)\} <$$
$$< \sum_{j=1}^{k-1} c_j (\gamma - \alpha + \varepsilon) f(m_i + 1 - j) - c_l (\delta - \alpha) f(m_i + 1 - l) <$$
$$< (\gamma - \alpha + \varepsilon) f(m_i + 1) - c_l (\delta - \alpha) f(m_i + 1 - l),$$

and so, by (4. 7),

$$(\alpha - \varepsilon) f(m_i + 1) + c_l (\delta - \alpha) f(m_i + 1 - l) \leqslant f(m_i) \{|S_{m_i,k}| + 1\}.$$

If $i \to \infty$, we have $f(m_i)/f(m_i + 1) \to \alpha$, $\liminf f(m_i + 1 - l)/f(m_i + 1) \geqslant \alpha^l$. Therefore

$$\alpha - \varepsilon + c_l (\delta - \alpha) \alpha^l \leqslant \alpha + \alpha\varphi(k),$$

which holds whenever $k > l$, $\varepsilon > 0$. Making $k \to \infty$, $\varepsilon \to 0$ we obtain $\delta = \alpha$, and a contradiction has been found. This proves (4. 9).

We can now show that $\alpha = \gamma$. Assume $\alpha < \gamma$, and let the sequence $\{n_i\}$ satisfy (4. 8). Now write down (4. 7) with $n = n_i$, divide by $f(n_i + 1)$ and make $i \to \infty$ (k is fixed). We obtain

$$\left|\gamma - \alpha - \sum_1^{k-1} c_j (\gamma \alpha^j - \alpha^{j+1})\right| \leqslant \alpha \varphi(k),$$

which leads to

$$\left|1 - \sum_1^{k-1} c_j \alpha^j\right| \leqslant \frac{\alpha \varphi(k)}{\gamma - \alpha}.$$

Making $k \to \infty$ we infer $C(\alpha) = 1$, which is impossible since $\alpha < \gamma$.

In the same way the assumption $\beta > \gamma$ leads to $C(\beta) = 1$. Thus the proof of theorem 7 is completed.

For some applications we can better deal with $T_{n,k}$, where, if $n > k \geqslant 1$,

(4.10) $$T_{n,k} = S_{n,k} - \gamma \frac{c_k f(n-k+1)}{f(n)} = \frac{1}{f(n)} \sum_{j=k}^{n-1} f(n-j)\{\gamma c_{j+1} - c_j\},$$

and put $\limsup_{n\to\infty} |T_{n,k}| = \psi(k) \leqslant \infty$.

Theorem 8. In the cases 2, 3, 4, 5 a necessary and sufficient condition for the existence of $\lim f(n)/f(n+1)$ is that $\psi(k) \to 0$ as $k \to \infty$.

Proof. In the first place, if $f(n)/f(n+1) \to \gamma$ is given, then we deduce

$$\lim_{n\to\infty} |T_{n,k} - S_{n,k}| = c_k \gamma^k,$$

and $c_k \gamma^k \to 0$ since $\Sigma c_k \gamma^k$ converges. Hence $\psi(k) \to 0$.

Next assume $\psi(k) \to 0$. As in the beginning of the proof of theorem 7 we deduce $f(n+1) < Cf(n)$ for some C and all n. Therefore we have, if $n > 2K$

$$\min_{K\leqslant k\leqslant 2K} \frac{\gamma c_k f(n-k+1)}{f(n)} \leqslant \frac{\gamma}{K f(n)} \sum_{K}^{2K} c_k f(n-k+1) \leqslant \frac{\gamma C}{K},$$

and hence

(4.11) $$\lim_{K\to\infty} \limsup_{n\to\infty} \min_{K\leqslant k\leqslant 2K} |S_{n,k}| = 0.$$

It is easily seen that with this condition, instead of $\varphi(k) \to 0$, we are also able to give the remaining part of the proof of theorem 7.

Theorem 9. In all cases the condition $c_n/c_{n+1} \to \gamma$ implies

$$f(n)/f(n+1) \to \gamma.$$

Proof. We exclude case 1 here; the proof for case 1 will be given in § 5.

If $\varepsilon > 0$, then for $j > A(\varepsilon)$ we have

$$|\gamma c_{j+1} - c_j| < \varepsilon c_j.$$

Hence, for $k > A(\varepsilon)$, $n > k$, we have by (4. 10),

$$f(n)\,|T_{n,k}| < \sum_{k}^{n-1} \varepsilon c_j f(n-j) < \varepsilon f(n).$$

Therefore $\psi(k) \to 0$ as $k \to \infty$, and theorem 8 can be applied.

Theorem 10. In the cases 2, 3, 4, 5, the condition

$$\sum_{2}^{\infty} \frac{|\gamma c_n - c_{n-1}|}{f(n)} < \infty$$

implies $f(n)/f(n+1) \to \gamma$.

Proof. By (4. 10) and by theorem 1 we have, if $n > k > 1$,

$$f(n)\,|T_{n,k}| < \sum_{k}^{n-1} \frac{f(n)}{f(j+1)}\,|\gamma\, c_{j+1} - c_j| < f(n) \sum_{k-1}^{\infty} \frac{|\gamma\, c_j - c_{j-1}|}{f(j)}.$$

Consequently $\psi(k) \to 0$ as $k \to \infty$, and theorem 8 can be applied.

Theorem 11. If $\Sigma c_n/f(n) < \infty$, then $f(n)/f(n+1) \to \gamma$.

Proof. As was remarked before, the convergence of the series implies $f(n+1) = O\{f(n)\}$, and it excludes case 1. Thus we may apply theorem 10, since

$$\sum_{2}^{\infty} \frac{c_{n-1}}{f(n)} = \sum_{1}^{\infty} \frac{c_n}{f(n+1)} < \sum_{1}^{\infty} \frac{c_n}{c_1\, f(n)} < \infty.$$

Possibly the condition

$$\sum_{1}^{\infty} \left| \frac{c_{n+1}}{f(n+1)} - \frac{c_n}{f(n)} \right| < \infty \tag{4. 15}$$

is also sufficient for $f(n)/f(n+1) \to \gamma$, but we could not decide this.

A sufficient condition which applies to all cases, is

Theorem 12. If $c_{n+1}\, c_{n-1} \geqslant c_n^2$ $(n > 1)$, then $f(n)/f(n+1) \to \gamma$.

Proof. It was proved in [**1**] that $c_{n+1}\, c_{n-1} \geqslant c_n^2$ $(n > 1)$ implies $f(n+1) \cdot f(n-1) \geqslant f^2(n)$ $(n > 1)$. (The proof did not depend on the assumption $\Sigma c_k = 1$ which was made throughout that paper). Consequently $f(n)/f(n+1)$ is non-increasing, and its limit exists.

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(To be continued)