Remarks on the size of $L(1, \chi)$.

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§ 1. Introduction.

In this paper we consider the value of the Dirichlet $L(s, \chi)$ functions at $s = 1$, χ being a non-principal residue-character and $L(s, \chi)$ being defined for $\Re(s) > 0$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

(For the basic properties of residue-charakters and L-functions see LANDAU [1, 4]). It is known[1]) that if k is the modulus of χ, then

$$C_\varepsilon k^{-\varepsilon} < |L(1, \chi)| < \log k, \tag{1}$$

where ε is any positive number and C_ε is a positive number depending only upon ε.

It is obviously of interest on the other hand to obtain results showing that $|L(1, \chi)|$ actually can be small or large relative to k. It is known (cf. CHOWLA [3, 4]) that for any positive ε there are infinitely many real primitive χ satisfying any one of the folloving four pairs of conditions[2]) (γ is EULER's constant):

$$L(1, \chi) > (1 - \varepsilon)\, e^{\gamma} \log\log k, \quad \chi(-1) = 1; \tag{2}$$

$$L(1, \chi) < \frac{1 + \varepsilon}{6\, \pi^{-2} e^{\gamma} \log\log k}, \quad \chi(-1) = 1; \tag{3}$$

$$L(1, \chi) > (1 - \varepsilon)\, e^{\gamma} \log\log k, \quad \chi(-1) = -1; \tag{4}$$

$$L(1, \chi) < \frac{1 + \varepsilon}{6\, \pi^{-2} e^{\gamma} \log\log k}, \quad \chi(-1) = -1. \tag{5}$$

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[1]) For the proof of the left-hand inequality of (1) for real primitive χ see SIEGEL [1], LANDAU [5], HEILBRONN [1], CHOWLA [5], and ESTERMANN [1]. The extension to any real non-principal χ is immediate. For complex χ see LANDAU [2]. The right-hand side of (1) is proved trivially by partial summation; cf. § 9 below. Of course the extended RIEMANN hypothesis gives much stronger results than (1); cf. LITTLEWOOD [1].

[2]) If χ is a real non principal character, $L(1, \chi)$ is positive. Actually CHOWLA proved only (2) and (3) explicitly, but (4) and (5) can be obtained merely by replacing $\left(\frac{8an+b}{m}\right)$ wherever it occurs in either of CHOWLA's two papers by $\left(\frac{-8an+b}{m}\right)$, and $\left(\frac{m}{8an+b}\right)$ wherever it occurs by $\left(\frac{m}{8an-b}\right)$.

Now the only real primitive (non-principal) characters $\chi(n)$ are given for positive n by the KRONECKER symbol $\left(\frac{d}{n}\right)$, where d is a fundamental discriminant (cf. WALFISZ [1]). Thus if we put

$$L_d(s)=\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n^s} \qquad \left(\Re(s)>0, \left(\frac{d}{n}\right) \text{ the Kronecker symbol}\right),$$

statements (2)—(5) may be written as follows:

If d runs through positive fundamental discriminants,

$$\overline{\lim_{d\to\infty}}\frac{L_d(1)}{\log\log d}\geqq e^{\gamma}, \tag{2'}$$

$$\underline{\lim_{d\to\infty}}\,(\log\log d)\,L_d(1)\leqq\frac{1}{6\pi^{-2}e^{\gamma}}; \tag{3'}$$

if d runs through negative fundamental discriminants,

$$\overline{\lim_{d\to-\infty}}\frac{L_d(1)}{\log\log|d|}\geqq e^{\gamma}, \tag{4'}$$

$$\underline{\lim_{d\to-\infty}}\,(\log\log|d|)\,L_d(1)\leqq\frac{1}{6\pi^{-2}e^{\gamma}}. \tag{5'}$$

The statements (2)—(5) or (2')—(5') say nothing about the nature of the modulus k or $|d|$. We shall prove in this paper that statements similar to these can still be made if we restrict the modulus to be prime. (The results are poorer by the numerical factor 18). If q is a prime congruent to 1 modulo 4, then q is a fundamental discriminant and the Kronecker symbol $\left(\frac{q}{n}\right)$ is the same as the Legendre symbol $\left(\frac{n}{q}\right)$, so that

$$L_q(s)=\sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{1}{n^s}.$$

If q is a prime congruent to 3 modulo 4, then $-q$ is a fundamental discriminant and the Kronecker symbol $\left(\frac{-q}{n}\right)$ is the same as the Legendre symbol $\left(\frac{n}{q}\right)$, so that

$$L_{-q}(s)=\sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{1}{n^s}.$$

Our results are as follows:

Theorem 1. *If q runs through the primes congruent to 1 modulo 4 then*

$$\overline{\lim_{q\to\infty}}\frac{L_q(1)}{\log\log q}=\overline{\lim_{q\to\infty}}\frac{\sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{1}{n}}{\log\log q}\geqq\frac{e^{\gamma}}{18}, \tag{A}$$

$$\text{(B)}\qquad \overline{\lim_{q\to\infty}}\ (\log\log q)\, L_q(1) = \overline{\lim_{q\to\infty}}\ (\log\log q) \sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{1}{n} \leqq \frac{18}{6\pi^{-2}e^{\gamma}}.$$

If q runs throught the primes congruent to 3 modulo 4, then

$$\text{(C)}\qquad \overline{\lim_{q\to\infty}}\ \frac{L_{-q}(1)}{\log\log q} = \overline{\lim_{q\to\infty}}\ \frac{\sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{1}{n}}{\log\log q} \geqq \frac{e^{\gamma}}{18},$$

$$\text{(D)}\qquad \overline{\lim_{q\to\infty}}\ (\log\log q)\, L_{-q}(1) = \overline{\lim_{q\to\infty}}\ (\log\log q) \sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{1}{n} \leqq \frac{18}{6\pi^{-2}e^{\gamma}}.$$

On the other side we give the following upper estimate for $L(1,\chi)$ for any non-principal χ, which is an improvement on the right-hand inequality in (1) for those k which have many distinct small prime factors (here $\varphi(k)$ denotes Euler's function):

Theorem 2. *If χ is any non-principal character, modulo k, then*

$$|L(1,\chi)| < \left(1+\frac{\log 64}{\log 6}\right)\frac{\varphi(k)}{k}\log k + 1 < \frac{10}{3}\,\frac{\varphi(k)}{k}\log k + 1.$$

Further, if ε is a small positive number, then for k sufficiently large

$$|L(1,\chi)| < \left(\frac{1}{2} + e^{\gamma}\log 2 + \varepsilon\right)\frac{\varphi(k)}{k}\log k < \frac{5}{4}\,\frac{\varphi(k)}{k}\log k.$$

The proof of Theorem 2 is rather simple. Theorem 1 requires a generalization by RÉNYI [2, 3] of the large sieve of LINNIK and the work of PAGE [1] on primes in arithmetic progressions. The factor 18 in Theorem 1 could be improved to 4 by using stronger results (see the remarks at the end of § 7), but a factor greater than 1 definitely enters because of the limitations of the sieving method.

BATEMAN and CHOWLA [1] have remarked that (4) or (4′) implies the following Ω-result for the summatory function of a real primitive character (a slightly stronger form of the Ω -result of PALEY [1]): If

$$S_d(m) = \sum_{n=1}^{m}\left(\frac{d}{n}\right), \quad A_d = \frac{S_d(1)+\ldots+S_d(|d|)}{|d|},$$

then if d runs through negative fundamental discriminants

$$\overline{\lim_{d\to-\infty}}\ |d|^{1/2}\frac{A_d}{\log\log|d|} \geqq \frac{e^{\gamma}}{\pi}.$$

(This is an immediate consequence of the formula

$$A_d = \frac{1}{|d|}\sum_{m=1}^{|d|}\sum_{n=1}^{m}\left(\frac{d}{n}\right) = \frac{|d|^{1/2}}{\pi}L_d(1),$$

which holds for d a negative fundamental discriminant. Cf. LANDAU [4, Satz

217]). We now remark that similarly part (C) of Theorem 1 implies that if q runs through the primes congruent to 3 modulo 4 then

$$\overline{\lim_{q\to\infty}} \frac{A_{-q}}{q^{1/2}\log\log q} \geqq \frac{e^{\gamma}}{18\pi}.$$

Consequently for q a prime (and $\left(\frac{n}{q}\right)$ the Legendre symbol)

$$\max_{m} \sum_{n=1}^{m} \left(\frac{n}{q}\right) = \Omega_R\left(q^{1/2}\log\log q\right),$$

a result previously proved by CHOWLA [1] only under the assumption of the extended Riemann hypothesis.

PROOF OF THEOREM 1.

§ 2. **Necessary lemmas.**

We shall need the following lemmas. The letter p always runs over the prime numbers with limitations as specified.

Lemma 1. (RÉNYI [2, Theorem 3] and RÉNYI [3, Theorem 3]). *Suppose we have a sequence of Z integers $n_1 < n_2 < \ldots < n_Z \leqq N$. Let $f(p)$ and $Q(p)$ be two arbitrary arithmetical functions with $0 < f(p) \leqq p$ and $1 < Q(p)$. Put*

$$\min_{p<\frac{1}{2}N^{1/3}} \frac{f(p)}{p} = \tau, \quad \max_{p<\frac{1}{2}N^{1/3}} Q(p) = Q.$$

If $Z(p,h)$ denotes the number of integers of the sequence n_j $(j=1,2,\ldots,Z)$ which are congruent to h modulo p, then we have for every prime number $p<\frac{1}{2}N^{1/3}$, except possibly for at most $9NQ^2/(Z\tau)$ abnormal primes, and for every residue h modulo p, except possibly for at most $f(p)$ irregular residues, the relation

$$\left| Z(p,h) - \frac{Z}{p} \right| < \frac{Z}{pQ(p)}.$$

In the application of Lemma 1 we shall refer to the primes p as the "sieving primes".

Lemma 2. (PAGE [1, pp. 128 and 135]). *There exist absolute positive constants a and b with the following property. If u is a positive integer there is at most one real primitive character with modulus not exceeding u such that the associated L-function has a real zero greater than $1 - a/(\log u)$. If k_1 is the modulus of this character (if it exists) and if k does not exceed u and is not a multiple of k_1, then for $m \geqq \exp(\log u)^2$ and $(l,k)=1$ we have*

$$\sum_{p\leqq m,\, p\equiv l \pmod k} 1 = \frac{1}{\varphi(k)} \sum_{n=2}^{m} \frac{1}{\log n} + O\left(\frac{m}{e^{b\sqrt{\log m}}}\right)$$

where the constant implied by the O-symbol is an absolute one.

Lemma 3. *In Lemma 2 a given real primitive character can be exceptional with respect to at most finitely many positive integers u. Also, if (as in Lemma 2) $k \leqq u$, $k_1 \nmid k$, $m \geqq \exp(\log u)^2$, and $(l, k) = 1$, then*

$$\sum_{p \leqq m,\, p \equiv l \pmod k} \frac{1}{p} = \frac{1}{\varphi(k)} \sum_{n=2}^{m} \frac{1}{n \log n} + C_{l,k} + O(e^{-\frac{1}{2} b \sqrt{\log m}}),$$

where $C_{l,k}$ is a number depending only upon l and k and where the constant implied by the O-symbol is an absolute one.

The first part of Lemma 3 is clear. To prove the second part we assume that m is integral and put

$$g(m) = \sum_{p \leqq m,\, p \equiv l \pmod k} 1 - \frac{1}{\varphi(k)} \sum_{2 \leqq n \leqq m} \frac{1}{\log n},$$

$$h(m) = \sum_{p \leqq m,\, p \equiv l \pmod k} \frac{1}{p} - \frac{1}{\varphi(k)} \sum_{2 \leqq n \leqq m} \frac{1}{n \log n}.$$

Then

$$h(m) = \sum_{n=2}^{m} \frac{g(n) - g(n-1)}{n} = \sum_{n=2}^{m} \frac{g(n)}{n(n+1)} + \frac{g(m)}{m+1}.$$

Since $g(n) = O(n e^{-b\sqrt{\log n}})$ for $n \geqq m$, we may write

$$\begin{aligned} h(m) &= \sum_{n=2}^{\infty} \frac{g(n)}{n(n+1)} - \sum_{n=m+1}^{\infty} \frac{g(n)}{n(n+1)} + \frac{g(m)}{m+1} \\ &= \sum_{n=2}^{\infty} \frac{g(n)}{n(n+1)} + O\left(\sqrt{\log m}\, e^{-b\sqrt{\log m}}\right) \\ &= \sum_{n=2}^{\infty} \frac{g(n)}{n(n+1)} + O\left(e^{-\frac{1}{2} b \sqrt{\log m}}\right). \end{aligned}$$

This proves Lemma 3.

§ 3. Outline of the proof.

We shall give in detail the proof of (A) and shall indicate what changes are necessary in order to prove the other parts of Theorem 1.

It suffices to show that for every large positive integer x there exists a prime q not exceeding x and congruent to 1 modulo 4 such that

$$\log L_q(1) \geqq \log\log\log x + \gamma - \log 18 + o(1). \tag{6}$$

(In the proof of Theorem 1 the notation "o" is with respect to x tending to infinity, statements made shall be understood to be accompanied by the phrase „for large x", and the constants implied by the O-symbol are absolute ones). To prove (6) we shall define a certain set $\mathfrak{S} = \mathfrak{S}(x)$ of primes q (congruent to 1 modulo 4) not exceeding x and shall prove that

$$\sum_{q \in \mathfrak{S}} \log L_q(1) \geqq S \log\log\log x + S(\gamma - \log 18) + o(S), \tag{7}$$

where $S=S(x)$ is the number of primes in $\mathfrak{S}$. (In general the quantities introduced in the course of the proof of Theorem 1 will depend upon x, unless otherwise specified).

The set $\mathfrak{S}$ is defined as follows. Consider the odd primes $p_1, p_2, \ldots, p_m$ not exceeding

$$y=\frac{\sqrt{\log x}}{(\log\log x)^2}. \tag{8}$$

Put $M=8p_1p_2\ldots p_m$ and consider the moduli

$$\frac{M}{p_m}, \frac{M}{p_{m-1}}, \ldots, \frac{M}{p_2}, \frac{M}{p_1}. \tag{9}$$

Each of the moduli (9) is not greater than

$$\frac{1}{3}M=\frac{4}{3}e^{\theta(y)}=e^{y+o(y)}<e^{\sqrt{\log x}},$$

and their greatest common divisor is 8. (Here $\theta(y)$ denotes the sum of the logarithms of the primes not exceeding y). Now we apply Lemma 2 with $u=[e^{\sqrt{\log x}}]$ and $m=x$. By Lemma 3 the corresponding exceptional modulus k_1 of Lemma 2 tends to infinity with x (if it exists) and so (if it exists) is greater[3]) than 8 for large x. Hence at least one of the moduli (9) is not a multiple of k_1. Suppose $k=M/p_r$ is the smallest such modulus, that is, the first in the order in which the moduli are written in (9). (If k_1 does not exist, $p_r=p_m$). Since if k_1 exists,

$$k_1\Big|\frac{M}{p_m}, \ldots, k_1\Big|\frac{M}{p_{r+1}},$$

we see that $k_1|(p_1\ldots p_r)$ and so p_r tends infinity with x (It can be proved that $1/p_r=o(1/\log\log x)$, but this is not needed). We have

$$k=\frac{M}{p_r}=8p_1\ldots p_{r-1}p_{r+1}\ldots p_m, \quad k=e^{y+o(y)}. \tag{10}$$

If $(l,k)=1$ we know by Lemma 2 that

$$\sum_{p\leqq x,\, p\equiv l \pmod k} 1=\frac{1}{\varphi(k)}\sum_{n=2}^{x}\frac{1}{\log n}+O\Big(\frac{x}{e^{b\sqrt{\log x}}}\Big). \tag{11}$$

We define a residue l modulo k in the following way. Suppose g_i is a certain quadratic residue modulo p_i $(1\leqq i\leqq m,\ i\neq r)$. We define l by

$$l\equiv 1 \pmod 8,\ l\equiv g_i \pmod{p_i}\ (1\leqq i\leqq m,\ i\neq r). \tag{12}$$

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[3]) As a matter of fact it is easy to see that the L-functions corresponding to real primitive characters with modulus not exceeding 8 have no positive real zero, so that k_1 is always greater than 8. Cf. CHOWLA [2] and ROSSER [3].

Now of the primes congruent to l modulo k and not exceeding x there is by Lemma 2 (with $u=x$) at most one prime q_0 such that the corresponding L-function

$$\sum_{n=1}^{\infty}\left(\frac{n}{q_0}\right)\frac{1}{n^s}$$

has a real zero greater than $1-a/(\log x)$. For our set $\mathfrak{S}$ of primes we now take those primes q such that

$$q\equiv l \pmod{k},\quad \sqrt{x}\leqq q\leqq x,\quad q\neq q_0. \tag{13}$$

By (11) the number S of primes in $\mathfrak{S}$ satisfies

$$S=\frac{1}{\varphi(k)}\sum_{n=2}^{x}\frac{1}{\log n}+O\left(\frac{x}{e^{b\sqrt{\log x}}}\right). \tag{14}$$

Since by (8) and (10) $e^{b\sqrt{\log x}}$ is of larger order of magnitude than $k\log x$, (14) implies in particular

$$S=\{1+o(1)\}\frac{x}{\varphi(k)\log x}. \tag{15}$$

Now, since the product formula for the $L(s,\chi)$ functions holds for $s=1$ for non-principal χ (cf. LANDAU [1, § 109]), we have

$$\log L_q(1)=-\sum_{p}\log\left\{1-\left(\frac{p}{q}\right)\frac{1}{p}\right\}.$$

Hence for q in $\mathfrak{S}$ we have (using (12), (13), and the quadratic reciprocity law)

$$\begin{aligned}\log L_q(1)&=-\sum_{p\leqq y,\,p\neq p_r}\log\left\{1-\frac{1}{p}\right\}-\log\left\{1-\left(\frac{p_r}{q}\right)\frac{1}{p_r}\right\}-\sum_{p>y}\log\left\{1-\left(\frac{p}{q}\right)\frac{1}{p}\right\}\\&=-\sum_{p\leqq y}\log\left(1-\frac{1}{p}\right)+O\left(\frac{1}{p_r}\right)+\sum_{p>y}\left\{\left(\frac{p}{q}\right)\frac{1}{q}+O\left(\frac{1}{p^2}\right)\right\}.\end{aligned} \tag{16}$$

Now by a theorem of MERTENS (cf. LANDAU [1, § 36])

$$\begin{aligned}-\sum_{p\leqq y}\log\left(1-\frac{1}{p}\right)&=\log\log y+\gamma+O\left(\frac{1}{\log y}\right)\\&=\log\left(\frac{1}{2}\log\log x-\log\log\log x\right)+\gamma+O\left(\frac{1}{\log\log x}\right)\\&=\log\log\log x-\log 2+\gamma+O\left(\frac{\log\log\log x}{\log\log x}\right).\end{aligned} \tag{17}$$

Combining (16) and (17) gives

$$\begin{aligned}\log L_q(1)=\log\log\log x+\gamma-\log 2+\sum_{p>y}\left(\frac{p}{q}\right)\frac{1}{p}\\+O\left(\frac{1}{p_r}\right)+O\left(\frac{\log\log\log x}{\log\log x}\right).\end{aligned} \tag{18}$$

Thus

$$\sum_{q \in \mathfrak{S}} \log L_q(1) = S \log\log\log x + S(\gamma - \log 2) + R + o(S), \tag{19}$$

where

$$R = \sum_{q \in \mathfrak{S}} \sum_{p > y} \left(\frac{p}{q}\right) \frac{1}{p}.$$

To prove (7) from (19) we must consider how the double sum R behaves for large x. We split the sum into four parts R_1, R_2, R_3, R_4, according as the summation over p is extended over the following intervals respectively:

$$I_1: \ y < p \leqq \frac{1}{2} x^{1/3},$$

$$I_2: \ \frac{1}{2} x^{1/3} < p \leqq (2x)^3,$$

$$I_3: \ (2x)^3 < p \leqq \exp(\log x)^{2+2\delta}$$

$$I_4: \ \exp(\log x)^{2+2\delta} < p.$$

Here δ can be any positive number, which for convenience we take less than $\frac{1}{2}$. In the four subsequent sections we shall show that

$$R_1 = o(S), \tag{20}$$

$$|R_2| < S \log 9 + o(S), \tag{21}$$

$$R_3 = o(S), \tag{22}$$

$$R_4 = o(S). \tag{23}$$

These estimates give

$$|R| = |R_1 + R_2 + R_3 + R_4| < S \log 9 + o(S).$$

Thus, in view of (19), we get (7) and thus part (A) of Theorem 1.

The estimation of R_2 is trivial, while R_4 is estimated rather simply by means of PAGE's theorem (Lemma 2). For R_1 we use RÉNYI's theorem (Lemma 1) with the primes p in I_1 as the sieving primes, while for R_3 we use RÉNYI's theorem with the primes q in $\mathfrak{S}$ as the sieving primes.

For the proof of (B) we must replace the condition $l \equiv 1 \pmod 8$ in (12) by $l \equiv 5 \pmod 8$ and the condition that g_i be a quadratic residue modulo p_i by the condition that it be a quadratic non-residue. For (C) we would need $l \equiv 7 \pmod 8$ and $-g_i$ a quadratic residue modulo p_i, while for (D) we would have $l \equiv 3 \pmod 8$ and $-g_i$ a quadratic non-residue modulo p_i. Also for (C) and (D) the quadratic reciprocity law $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$ which we used in (16) above and which we shall use in (24) below must of course be replaced by $\left(\frac{p}{q}\right) = \left(\frac{-q}{p}\right)$.

Further for (B) and (D) we must replace (17) by

$$-\sum_{p\leqq y}\log\left(1+\frac{1}{p}\right)=\sum_{p\leqq y}\log\left(1-\frac{1}{p}\right)-\sum_{p\leqq y}\log\left(1-\frac{1}{p^2}\right)=$$

$$=\sum_{p\leqq y}\log\left(1-\frac{1}{p}\right)+\log\frac{\pi^2}{6}+\sum_{p>y}\log\left(1-\frac{1}{p^2}\right)=$$

$$=-\log\log\log x+\log 2-\gamma+\log\frac{\pi^2}{6}+O\left(\frac{\log\log\log x}{\log\log x}\right).$$

§ 4. **Estimation of R_1.**

We use Lemma 1 with the primes q of $\mathfrak{S}$ as the integers $n_1,\ldots,n_Z$ and the primes p in I_1 as the sieving primes. Then $Z=S$ and $N=x$. We take $f(p)=p\,(\log p)^{-5}$ and $Q(p)=(\log p)^5$. Now by the quadratic reciprocity law

$$R_1=\sum_{q\in\mathfrak{S}}\sum_{p\in I_1}\left(\frac{p}{q}\right)\frac{1}{p}=\sum_{p\in I_1}\frac{1}{p}\sum_{q\in\mathfrak{S}}\left(\frac{q}{p}\right). \tag{24}$$

We use Σ' to denote summation over the normal primes p and Σ^* to denote summation over the abnormal primes p. Thus

$$R_1=\sum_{p\in I_1}{}'\frac{1}{p}\sum_{q\in\mathfrak{S}}\left(\frac{q}{p}\right)+\sum_{p\in I_1}{}^*\frac{1}{p}\sum_{q\in\mathfrak{S}}\left(\frac{q}{p}\right). \tag{25}$$

First we consider the normal primes p. If we denote by $S(p,h)$ the number of elements of $\mathfrak{S}$ which are congruent to h modulo p, then

$$\sum_{q\in\mathfrak{S}}\left(\frac{q}{p}\right)=\sum_{h=1}^{p-1}\left(\frac{h}{p}\right)S(p,h).$$

By Lemma 1 if p is normal we have

$$\left|S(p,h)-\frac{S}{p}\right|<\frac{S}{p\,Q(p)} \tag{26}$$

except for at most $f(p)$ irregular residues h modulo p. For p normal we use Σ' to denote summation over the regular residues h modulo p (that is, those for which (26) holds) and Σ^* to denote summation over the irregular residues h modulo p (other than the residue zero). Now for the total number of q in $\mathfrak{S}$ which fall into regular residue classes modulo the normal prime p we have by (26)

$$\sum_h{}'S(p,h)\geqq\sum_h{}'\left\{\frac{S}{p}-\frac{S}{p\,Q(p)}\right\}\geqq\left\{p-f(p)\right\}\left\{\frac{S}{p}-\frac{S}{p\,Q(p)}\right\}\geqq$$

$$\geqq S-S\frac{f(p)}{p}-\frac{S}{Q(p)}.$$

Hence for the total number of q which fall into irregular residue classes modulo the normal prime p we have

$$\sum_h^* S(p,h) \leqq S\frac{f(p)}{p} + \frac{S}{Q(p)}. \tag{27}$$

Thus for p normal we have by (26) and (27)

$$\left|\sum_{h=1}^{p-1}\left(\frac{h}{p}\right)S(p,h)\right| = \left|\sum_{h=1}^{p-1}\left(\frac{h}{p}\right)\left\{S(p,h)-\frac{S}{p}\right\}\right| \leqq$$

$$\leqq \sum_h{}' \frac{S}{pQ(p)} + \sum_h^*\left\{S(p,h)+\frac{S}{p}\right\} \leqq$$

$$\leqq \frac{S}{Q(p)} + S\frac{f(p)}{p} + \frac{S}{Q(p)} + f(p)\frac{S}{p} = \frac{4S}{(\log p)^5}.$$

Thus

$$\left|\sum_{p\in I_1}{}'\frac{1}{p}\sum_{p\in\mathfrak{S}}\left(\frac{q}{p}\right)\right| = \left|\sum_{p\in I_1}{}'\frac{1}{p}\sum_{h=1}^{p-1}\left(\frac{h}{p}\right)S(p,h)\right| \leqq \tag{28}$$

$$\leqq \sum_{p\in I_1}{}'\frac{4S}{p(\log p)^5} \leqq \sum_{p>y}\frac{4S}{p(\log p)^5} = O\left(\frac{S}{(\log y)^5}\right) = O\left(\frac{S}{(\log\log x)^5}\right).$$

Now we consider the abnormal p. If we put

$$z = \frac{\sqrt{\log x}}{\log\log x} = y\log\log x,$$

we claim first that there can be at most one abnormal prime not exceeding e^z. For (with k as in §3) consider the moduli pk for the primes p in the interval $y<p\leqq e^z$. By (10) we have

$$pk \leqq e^{z+o(z)} < e^{\sqrt{\log x}}$$

for large x. Since k is not a multiple of the basic exceptional modulus k_1 (with respect to $u=[e^{\sqrt{\log x}}]$), it follows that pk can be a multiple of k_1 for at most one p in the interval $y<p\leqq e^z$. Hence for any p in this interval except possibly one, and for any residue h modulo p prime to p, we have (by (13), Lemma 2, (14), (15), (10), and (8))

$$S(p,h) = \sum_{q\in\mathfrak{S},\, q\equiv h(\mathrm{mod}\, p)} 1 = \sum_{q\leqq x,\, q\equiv l(\mathrm{mod}\, k),\, q\equiv h(\mathrm{mod}\, p)} 1 + O(\sqrt{x}) =$$

$$= \frac{1}{(p-1)\varphi(k)}\sum_{n=2}^{x}\frac{1}{\log n} + O\left(\frac{x}{e^{b\sqrt{\log x}}}\right) = \frac{S}{p-1} + O\left(\frac{x}{e^{b\sqrt{\log x}}}\right) =$$

$$= \frac{S}{p} + O\left(\frac{S}{p^2}\right) + O\left(\frac{S\varphi(k)\log x}{e^{b\sqrt{\log x}}}\right) = \frac{S}{p} + O\left(\frac{S}{p^2}\right) + O\left(\frac{S}{e^{\frac{1}{2}b\sqrt{\log x}}}\right) = \frac{S}{p} + O\left(\frac{S}{p^2}\right).$$

Thus for any p but one in the interval $y<p\leqq e^z$ and for any residue h prime to p we have (for x large)

$$\left|S(p,h) - \frac{S}{p}\right| < \frac{S}{p(\log p)^5}.$$

Thus except for one prime p_a all primes p in the interval $y<p\leqq e^z$ are normal. (Actually the primes p in this interval other than p_a have no non-zero irregular residues).

With this information the abnormal primes are easily settled. By Lemma 1 the total number of abnormal primes p does not exceed

$$\frac{9x(\log x)^{10}}{S(\log x)^{-5}} = \left\{1+o(1)\right\} 9\varphi(k)(\log x)^{15}.$$

Since, aside from p_a, all the abnormal primes are greater than e^z, we have for the sum over the abnormal primes

$$\left|\sum_{p\in I_1}^{*}\frac{1}{p}\sum_{q\in\mathfrak{S}}{}'\left(\frac{q}{p}\right)\right| \leqq S\sum_{p\in I_1}^{*}\frac{1}{p} \leqq \frac{S}{p_a}+O\left(\varphi(k)(\log x)^{15}\frac{S}{e^z}\right)\leqq \tag{29}$$

$$\leqq \frac{S}{y}+O\left(\frac{S}{e^{\frac{1}{2}z}}\right)=O\left(\frac{S}{y}\right).$$

Finally (25), (28), and (29) give (20).

§ 5. Estimation of R_2.

Actually the estimation of R_2 is effected by estimating for each q in $\mathfrak{S}$ separatety and then multiplying by S. Thus we could have made the appropriate *estimate already in (18) before summing over q. A similar remark is true* for R_4.

By another asymptotic formula of MERTENS (cf. LANDAU [1, §36]) we have

$$\left|\sum_{p\in I_2}{}'\left(\frac{p}{q}\right)\frac{1}{p}\right| \leqq \sum_{\frac{1}{2}x^{1/3}<p\leqq(2x)^3}\frac{1}{p} = \log\log(2x)^3-\log\log\left(\frac{1}{8}x\right)^{1/3}+O\left(\frac{1}{\log x}\right)=$$

$$=\log 9+\log\left(\frac{\log x+\log 2}{\log x-\log 8}\right)+O\left(\frac{1}{\log x}\right)=\log 9+O\left(\frac{1}{\log x}\right).$$

Thus

$$|R_2|=\left|\sum_{q\in\mathfrak{S}}\sum_{p\in I_1}\left(\frac{p}{q}\right)\frac{1}{p}\right|<S\log 9+O\left(\frac{S}{\log x}\right),$$

which proves (21).

§ 6. Estimation of R_3.

We divide the interval I_3 into intervals J_t of the form

$$J_t:\quad t<p\leqq te^{(\log x)^{-5}}.$$

Clearly the number of such intervals needed to fill out I_3 is less than $(\log x)^{7+2\delta}\leqq(\log x)^8$. In each interval J_t we use Lemma 1 with the primes p in J_t as the integers $n_1,\ldots,n_Z$ and the primes q of $\mathfrak{S}$ as the sieving primes.

This is possible, since $\delta \leqq \frac{1}{2} t^{1/3}$ by the definition of I_3. We note in advance that if Z_t is the number of primes p in J_t, then

$$(30)\quad Z_t = \sum_{t < n \leqq te^{(\log x)^{-5}}} \frac{1}{\log n} + O\left(\frac{t}{e^{b\sqrt{\log t}}}\right) = \frac{te^{(\log x)^{-5}} - t}{\log t} + O\left(\frac{t}{(\log t)^2 (\log x)^{10}}\right) =$$

$$= \frac{t}{(\log t)(\log x)^5} + O\left(\frac{t}{(\log t)(\log x)^{10}}\right).$$

To estimate

$$\sum_{b \in \mathfrak{S}} \sum_{p \in J_t} \left(\frac{p}{q}\right) \frac{1}{p}$$

we first replace $1/p$ by $1/t$ before applying the large sieve. By (30) the error made in doing this is

$$(31)\quad \sum_{q \in \mathfrak{S}} \sum_{p \in J_t} \left(\frac{p}{q}\right) \frac{1}{p} - \sum_{q \in \mathfrak{S}} \sum_{t \in J_t} \left(\frac{p}{q}\right) \frac{1}{t} = O\left(\sum_{q \in \mathfrak{S}} \sum_{p \in J_t} \left\{ \frac{1}{t} - \frac{1}{te^{(\log x)^{-5}}} \right\}\right) =$$

$$= \sum_{q \in \mathfrak{S}} \sum_{p \in J_t} O\left(\frac{(\log x)^{-5}}{t}\right) = O\left(\frac{SZ_t}{t(\log x)^5}\right) = O\left(\frac{S}{(\log x)^{11}}\right).$$

We now apply the large sieve to estimate

$$\frac{1}{t} \sum_{q \in \mathfrak{S}} \sum_{p \in J_t} \left(\frac{p}{q}\right)$$

in the manner outlined above. Thus, in Lemma 1, $Z = Z_t$ and $N = te^{(\log x)^{-5}}$. We take $f(q) = q(\log q)^{-5}$ and $Q(q) = (\log q)^5$. As in § 4 we use Σ' to denote summation over the normal q and Σ^* to denote summation over the abnormal q, and for a normal q we use Σ' to denote summation over the regular residues j modulo q and Σ^* to denote summation over the irregular residues j modulo q (other than the residue zero).

The abnormal primes q are easily disposed of. In fact since $\log t \leqq (\log x)^{2+2\delta} \leqq (\log x)^3$, the number of abnormal q does not exceed

$$O\left(\frac{t(\log t)^{10}}{Z_t (\log t)^{-5}}\right) = O\left(\frac{t(\log x)^{45}}{Z_t}\right).$$

Hence for the sum over the abnormal q we have

$$(32)\qquad \left| \frac{1}{t} \sum_{q \in \mathfrak{S}}^{*} \sum_{p \in J_t} \left(\frac{p}{q}\right) \right| = O((\log x)^{45}) = O\left(\frac{S}{\sqrt{x}}\right).$$

Suppose we denote by $Z_t(q, j)$ the number of p in J_t which are congruent to j modulo q. Then if q is normal we have for the total number of p in J_t which fall into regular residue classes modulo q

$$\sum_{j}' Z_t(q, j) \geqq \sum_{j}' \left\{ \frac{Z_t}{q} - \frac{Z_t}{qQ(q)} \right\}$$

$$\geqq \{q - f(q)\} \left\{ \frac{Z_t}{q} - \frac{Z_t}{qQ(q)} \right\} \geqq Z_t - Z_t \frac{f(q)}{q} - \frac{Z_t}{Q(q)}.$$

Hence for the total number of p in J_t which fall into irregular residue classes modulo a normal q we have

$$\sum_j^* Z_t(q,j) \leqq Z_t \frac{f(q)}{q} + \frac{Z_t}{Q(q)}.$$

Thus if q is normal

$$\left|\sum_{p\in J_t}\left(\frac{p}{q}\right)\right| = \left|\sum_{j=1}^{q-1}\left(\frac{j}{q}\right) Z_t(q,j)\right| = \left|\sum_{j=1}^{q-1}\left(\frac{j}{q}\right)\left\{ Z_t(q,j) - \frac{Z_t}{q}\right\}\right| \leqq$$

$$\leqq \sum_j' \frac{Z_t}{qQ(q)} + \sum_j^*\left\{ Z_t(q,j) + \frac{Z_t}{q}\right\} \leqq \frac{Z_t}{Q(q)} + Z_t\frac{f(q)}{q} + \frac{Z_t}{Q(q)} + f(q)\frac{Z_t}{q} =$$

$$= \frac{4Z_t}{(\log q)^5} \leqq \frac{4Z_t}{(\log \sqrt{x})^5} \leqq \frac{128 Z_t}{(\log x)^5}.$$

Hence by (30)

$$\left|\frac{1}{t}\sum_{q\in\mathfrak{S}}'\sum_{p\in J_t}\left(\frac{q}{p}\right)\right| \leqq \frac{128 S Z_t}{t(\log x)^5} = O\left(\frac{S}{(\log x)^{11}}\right). \tag{33}$$

Thus by (31), (32), and (33) we have

$$\left|\sum_{q\in\mathfrak{S}}\sum_{p\in J_t}\left(\frac{p}{q}\right)\frac{1}{p}\right| = O\left(\frac{S}{(\log x)^{11}}\right).$$

Since there are less than $(\log x)^8$ intervals J_t in I_3, we have finally

$$R_3 = \sum_{q\in\mathfrak{S}}\sum_{p\in I_3}\left(\frac{p}{q}\right)\frac{1}{p} = O\left(\frac{S}{(\log x)^3}\right),$$

which proves (22).

§ 7. Estimation of R_4 and concluding remarks.

Suppose v and w are integers such that

$$\exp(\log x)^{2+2\delta} \leqq v \leqq w$$

and suppose $q\in\mathfrak{S}$. Now by (13) $q\leqq x$ and $q \neq q_0$; therefore by Lemma 3 with $u=x$ and $m=v,w$ we have

$$\sum_{v<p\leqq w}\left(\frac{p}{q}\right)\frac{1}{p} = \sum_{j=1}^{q-1}\left(\frac{j}{q}\right)\sum_{p\equiv j \pmod q,\ v<p\leqq w}\frac{1}{p} =$$

$$= \sum_{j=1}^{q-1}\left(\frac{j}{q}\right)\left\{\frac{1}{q-1}\sum_{n=v+1}^{w}\frac{1}{n\log n} + O\left(e^{-\frac{1}{2}b\sqrt{\log v}}\right)\right\} = O\left(q\,e^{-\frac{1}{2}b\sqrt{\log v}}\right).$$

Hence

$$\sum_{p>\exp(\log x)^{2+2\delta}}\left(\frac{p}{q}\right)\frac{1}{p} = O\left(x\,e^{-\frac{1}{2}b(\log x)^{1+\delta}}\right) = O\left(e^{-\log x}\right) = O\left(x^{-1}\right).$$

Thus

$$R_4 = \sum_{q \in \mathfrak{S}} \sum_{p > \exp(\log x)^{2+2\delta}} \left(\frac{p}{q}\right) \frac{1}{p} = O\left(\frac{S}{x}\right),$$

which proves (23). This completes the proof of Theorem 1.

The reduction of the numerical factor 18 in Theorem 1 to the factor 4 (the possibility of which we mentioned in the introduction) would be effected as follows. Firstly, by using the results of RODOSSKIĬ [1] we could replace the definition of y given in (8) by $y=(\log x)^{1-\epsilon}$, where ϵ is an arbitrarily small positive number. This procedure would replace the term $(-\log 2)$ in (17) by $\log(1-\epsilon)$ and thus improve our final results by a factor 2. Secondly we could replace Lemma 1 by another form of RÉNYI's generalized large sieve in which the range of the sieving primes extends to $\sqrt{N}$ at the expense of a relatively harmless increase in the upper bound for the number of abnormal primes to $3\pi N^2 Q^3/(2Z^2\tau^{3/2})$. (This is stated in RÉNYI [2] to be a consequence of the method used in Chapter 2 of RÉNYI [1]. We have used the form of the large sieve given in Lemma 1 because it is proved explicitly in both RÉNYI [2] and RÉNYI [3]). This alternative version of the large sieve would enable us to change the limits of I_2 to $\sqrt{x}$ and x^2 respectively and thus (21) would be replaced by $|R_2| < S\log 4 + o(S)$; this procedure would improve our final results by a factor of 9/4.

PROOF OF THEOREM 2.

§ 8. **Necessary Lemmas.**

Lemma 4. *For k a positive integer greater than 1 let $\varphi(x,k)$ denote the number of positive integers not exceeding x and relatively prime to k. Then*

$$\left|\varphi(x,k) - x\frac{\varphi(k)}{k}\right| < 2^{\omega(k)-1},$$

where $\omega(k)$ is the number of distinct prime divisors of k.

Proof. By a familiar theorem of Legendre

$$x\frac{\varphi(k)}{k} - \varphi(x,k) = \sum_{d \mid k} \mu(d)\left(\frac{x}{d} - \left[\frac{x}{d}\right]\right),$$

where μ is the MÖBIUS function and the sum extends over all the divisors of k. Hence

$$-\sum_{d \mid k,\, \mu(d)=-1} 1 < x\frac{\varphi(k)}{k} - \varphi(x,k) < \sum_{d \mid k,\, \mu(d)=1} 1.$$

Now since

$$\sum_{d \mid k} \mu(d) = 0, \quad \sum_{d \mid k} |\mu(d)| = 2^{\omega(k)},$$

we have

$$\sum_{d|k,\,\mu(d)=1} 1 = \sum_{d|k,\,\mu(d)=-1} 1 = 2^{\omega(k)-1},$$

This proves the lemma.

Lemma 5. *Let p_n denote the n^{th} prime and $\theta(x)$ the sum of the logarithms of the primes not exceeding x. Then as r tends to infinity*

$$\frac{r}{\theta(p_r)\prod_{n=1}^{r}\left(1-\frac{1}{p_n}\right)} = \{1+o(1)\}\,e^{\gamma},$$

while for r any positive integer

$$\frac{r}{\theta(p_r)\prod_{n=1}^{r}\left(1-\frac{1}{p_n}\right)} \leqq \frac{6}{\log 6},$$

equality occurring only for $r=2$.

Proof. The first part of the lemma follows immediately from the theorem of MERTENS referred to earlier (cf. LANDAU [1, § 36) and the prime number theorem:

$$\frac{r}{\theta(p_r)\prod_{n=1}^{r}\left(1-\frac{1}{p_n}\right)} = \{1+o(1)\}\frac{r e^{\gamma}\log p_r}{p_r} = \{1+o(1)\}\,e^{\gamma}.$$

The second part of the lemma may easily be checked for $r<15$. For $r\geqq 15$ we proceed as follows. ROSSER [1] has proved that

$$\log r \prod_{n=1}^{r}\left(1-\frac{1}{p_n}\right)$$

increases with r. Hence for $r\geqq 15$ we have

$$\log r \prod_{n=1}^{r}\left(1-\frac{1}{p_n}\right) \geqq \log 15 \prod_{n=1}^{15}\left(1-\frac{1}{p_n}\right) > 0.3755.$$

Also by Theorems 5, 7, 23, 29 of ROSSER [2] $\theta(x) > 0.8\,x$ for $x\geqq 100$. Thus with the aid of a little computation for $15\leqq r\leqq 25$ we see that $\theta(p_r) > 0.8\,p_r$ for $r\geqq 15$. By ROSSER [1] $p_r > r\log r$, and so we have

$$\theta(p_r) > 0.8\,r\log r \qquad (r\geqq 15).$$

Hence for $r\geqq 15$

$$\frac{r}{\theta(p_r)\prod_{n=1}^{r}\left(1-\frac{1}{p_n}\right)} < \frac{1}{(0.8)(0.3755)} < \frac{6}{\log 6}.$$

Thus the second part of the lemma is proved for all r.

Lemma 6. *If k is any positive integer*

$$\omega(k) \leqq \frac{6}{\log 6} \frac{\varphi(k)}{k} \log k .$$

As k tends to infinity

$$\omega(k) \leqq \{1 + o(1)\} e^{\gamma} \frac{\varphi(k)}{k} \log k .$$

Proof. For those k for which $\omega(k)$ has a given value r, the function $k^{-1}\varphi(k) \log k$ is smallest for the product of the first r primes. Hence

$$\frac{\omega(k)}{\frac{\varphi(k)}{k} \log k} \leqq \frac{\omega(k)}{\theta(p_{\omega(k)}) \prod_{n=1}^{\omega(k)} \left(1 - \frac{1}{p_n}\right)} .$$

Thus the first part of the lemma follows from Lemma 5. For those large k such that $\omega(k) > \log\log k$ the second part of the lemma also follows from Lemma 5, since for such k certainly $\omega(k)$ tends to infinity with k. On the other hand if $\omega(k) \leqq \log\log k$, the second part of the lemma follows from the fact that

$$\log\log k = o(k^{-1}\varphi(k) \log k).$$

This completes the proof of the lemma.

§ 9. The estimation proper.

For comparison we recall how the right-hand inequality in (1) is proved. Suppose χ is a non-principal residue-character modulo k and let us put

$$S(n) = \sum_{m=1}^{n} \chi(m) .$$

Let d be the smallest integer such that $|S(n)| \leqq d$ for all n. Clearly $d \leqq \frac{1}{2}(k-1)$. Now

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{n=1}^{\infty} \frac{S(n)}{n(n+1)} .$$

For $n < d$ we use the estimate $|S(n)| \leqq n$, while for $n \geqq d$ we use the estimate $|S(n)| \leqq d$. This gives

$$|L(1, \chi)| < \sum_{n=1}^{d-1} \frac{n}{n(n+1)} + \sum_{n=d}^{\infty} \frac{d}{n(n+1)} = \frac{1}{2} + \ldots + \frac{1}{d} + 1 <$$

$$< \int_{\frac{1}{2}}^{d+\frac{1}{2}} \frac{dx}{x} = \log(2d+1) \leqq \log k .$$

The change we make in the above proof in order to get Theorem 2 is as follows. In addition to the above two estimates for $S(n)$ we use a third

estimate which is valuable in a certain intermediate range of summation. This estimate is an immediate consequence of Lemma 4; it is

$$|S(n)| \leq \varphi(n,k) \leq n\frac{\varphi(k)}{k} + 2^{\omega(k)-1}.$$

Let b and c be positive integers to be chosen later, $b<c$. Then, using the three estimates for $S(n)$ in the intervals $1 \leq n < b$, $b \leq n < c$, $c \leq n < \infty$ respectively, we have

$$|L(1,\chi)| \leq \sum_{n=1}^{b-1} \frac{1}{n+1} + \sum_{n=b}^{c-1} \frac{n\varphi(k)k^{-1} + 2^{\omega(k)-1}}{n(n+1)} + \sum_{n=c}^{\infty} \frac{d}{n(n+1)} <$$

$$< \log b + \frac{\varphi(k)}{k} \log \frac{c}{b} + 2^{\omega(k)-1}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{d}{c}.$$

(fI we choose $b=2^{\omega(k)}$, $c=d\cdot 2^{\omega(k)}$, we get

$$|L(1,\chi)| < (\log 2)\,\omega(k) + \frac{\varphi(k)}{k} \log d + 1. \tag{34}$$

To get the first part of Theorem 2 we now use the first part of Lemma 6 in (34) and apply the trivial estimate $d<k$. To get the second part of Theorem 2 we apply the second part of Lemma 6 and use the deeper estimate

$$d = O(\sqrt{k} \log k).$$

Cf. LANDAU 3, pp. 85-86]).

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References.

P. T. BATEMAN and S. CHOWLA: [1] Averages of character sums, *Proc. Amer. Math. Soc.* vol. **1** (1950).

S. CHOWLA: [1] A theorem on characters, *J. Indian Math. Soc.* vol. **19** (1932) pp. 279—284; *Tôhoku Math. J.* vol. **39** (1934), pp. 248—252.

[2] Note on Dirichlet's L-functions, *Acta Arithmetica* vol. **1** (1936), pp. 113 - 114.

[3] On the class-number of the corpus P $(\sqrt{-k})$, *Proc. Nat. Acad. Sci. India* vol. **13** (1947), pp. 197—200.

[4] Improvement of a theorem of Linnik and Walfisz, *Proc. London Math. Soc.* (2) vol. **50** (1949), pp. 423—429.

[5] A new proof of a theorem of Siegel, *Ann. of Math.* (2) vol. **51** (1950), pp. 120—122.

T. ESTERMANN: [1] On Dirichlet's L-functions, *J. London Math. Soc.* vol. **23** (1948), pp. 275—279.

H. HEILBRONN: [1] On Dirichlet series which satisfy a certain functional equation, *Quart. J. Math.* (Oxford Ser.) vol. **9** (1938), pp. 194—195.

E. LANDAU: [1] Handbuch der Lehre von der Verteilung der Primzahlen, Band 1, (Leipzig, 1909).

[2] Über das Nichtverschwinden der Dirichletschen Reihen, welche komplexen Charakteren entsprechen, *Math. Ann.* vol. **70** (1911), pp. 69—78.

[3] Abschätzungen von Charaktersummen, Einheiten und Klassenzahlen, *Nachr. Göttingen* (1918), pp. 79–97.
[4] Vorlesungen über Zahlentheorie, Band 1 (Leipzig, 1927).
[5] Über einige neuere Fortschritte der additiven Zahlentheorie (Cambridge University Press, 1937).

J. E. Littlewood: [1] On the class-number of the corpus $P(\sqrt{-k})$, *Proc. London Math. Soc.* (2) vol. **27** (1928), pp. 358–372.

R. E. A. C. Paley: [1] A theorem on characters, *J. London Math. Soc.* vol. **7** (1932), pp. 28–32.

A. Page: [1] On the number of primes in an arithmetic progression, *Proc. London Math. Soc.* (2) vol. **39** (1935), pp. 116–141.

A. Rényi: [1] On the representation of even numbers as the sum of a prime number and an almost prime number, *Izvestiya Akad. Nauk SSSR.* Ser. Mat. vol. **12** (1948), pp. 57–78.
[2] On the large sieve of U. V. Linnik, *Compositio Math.* vol. 8 (1950) pp. 68–75.
[3] Un nouveau theoreme concernant les fonctions indépendantes et ses applications à la théorie des nombres, *J. Math. Pures Appl.* (N. S.) vol. **28** (1949), pp. 137–149.

K. A. Rodosskiĭ: [1] On the distribution of prime numbers in short arithmetic progressions, *Izvestiya Akad. Nauk SSSR.* Ser. Mat. vol. **12** (1948), pp. 123–128.

J. B. Rosser: [1] The n-th prime is greater than $n \log n$, *Proc. London Math. Soc.* (2) vol. **45** (1938), pp. 21–44.
[2] Explicit bounds for some functions of prime numbers, *Amer. J. Math,* vol. **63** (1941), pp. 211–232.
[3] Real roots of Dirichlet L-series, *Bull. Amer. Math. Soc.* vol. **55** (1949), pp. 906–913.

C. L. Siegel: [1] Über die Classenzahl quadratischer Zahlkörper, *Acta Arithmetica* vol. **1** (1936), pp. 83–86.

A. Walfisz: [1] Zur additiven Zahlentheorie II., *Math. Z.* vol. 40 (1936), pp. 592–607.