CONVERGENCE FIELDS OF ROW-FINITE AND ROW-INFINITE TOEPLITZ TRANSFORMATIONS

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During a recent conversation, R. P. Agnew suggested a determination of the validity of the proposition that row-infinite Toeplitz transformations are more powerful than row-finite transformations. Before this proposition is examined, it is necessary to assign a precise meaning to it. Corresponding to every sequence s a regular row-finite Toeplitz transformation A can be constructed such that the transform As converges to a finite limit. In order to be of interest, the proposition must therefore be interpreted in terms of convergence fields of individual transformations. Also, because every row-infinite regular Toeplitz matrix is the sum of two Toeplitz matrices A and B, where A is regular and row-finite and B has the norm zero, the convergence field in the space of bounded sequences of every regular Toeplitz transformation coincides with that of a row-finite regular transformation; in other words, the problem is trivial except in its reference to unbounded sequences.

THEOREM 1. *There exists a regular Toeplitz transformation whose convergence field is not contained in the convergence field of any regular row-finite Toeplitz matrix.*

Let A be the matrix

$$\left\| \begin{array}{ccccccc} 1/2 & 0 & 1/4 & 0 & 1/8 & 0 & 1/16 & \cdots \\ 0 & 1/2 & 0 & 0 & 0 & 1/4 & 0 & \cdots \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & \cdots \\ \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \end{array} \right\|$$

whose elements a_{nk} $(n=1, 2, \cdots ; k=1, 2, \cdots)$ are given by the rule

$$a_{nk} = 1/2^p \quad (n = 1, 2, \cdots ; k = 2^{n-1}(2p - 1); p = 1, 2, \cdots),$$
$$a_{nk} = 0 \quad \text{otherwise.}$$

Corresponding to every regular row-finite matrix B a sequence s will be constructed such that the sequence As converges to zero while the sequence Bs diverges. The construction hinges on the fact, readily

shown, that each column of A contains precisely one positive element; and it will be carried out in two ways, according as the matrix B falls into one or the other of two classes.

In Case I there exists, for every positive integer m, a nonzero element b_{nk}, well to the right in its row, such that the positive element in the kth column of A occurs below the first m rows. To be precise: to each pair of positive integers m and h corresponds a pair of integers n and k $(k > h)$ such that

(i) $$b_{nk} \neq 0,$$

(ii) $$a_{rk} = 0 \qquad\qquad (r = 1, 2, \cdots, m).$$

Let $b(n_1, k_1)$ be a nonzero element in B; let

$$s_k = 0 \qquad\qquad (k = 1, 2, \cdots, k_1 - 1),$$
$$s_k = 1/b(n_1, k_1) \qquad\qquad (k = k_1),$$
$$s_k = 0 \qquad\qquad (k = k_1 + 1, k_1 + 2, \cdots, k_1'),$$

where the index k_1' is large enough so that $b(n_1, k) = 0$ whenever $k > k_1'$. It is then certain that the n_1st element of the sequence Bs has the value one. On the other hand, there exist integers k_1'' $(k_1'' > k_1')$ and n_1' such that $a(n_1', k_1'') \neq 0$. If

$$s_k = 0 \qquad\qquad (k = k_1' + 1, k_1' + 2, \cdots, k_1'' - 1),$$
$$s_k \text{ has an appropriate value for } k = k_1'',$$
$$s_k = 0 \qquad \text{for all } k \, (k > k_1'') \text{ for which } a(n_1', k) > 0,$$

then all except finitely many of the partial sums of the series $\sum a(n_1', k)s_k$ are zero.

Suppose that the element s_k of the sequence s has been defined for $k = 1, 2, \cdots, k_{r-1}''$. Let $b(n_r, k_r)$ be a nonzero element in B, subject to the consistent conditions on the indices that $k_r > k_{r-1}''$, $n_r > n_{r-1}$, and $a(n, k_r) = 0$ when $n \leq n_{r-1}$; and let $a(n_r', k_r)$ be the positive element in the k_rth column of A. Let k_r' be an integer large enough so that $b(n_r, k) = 0$ when $k > k_r'$, and k_r'' an integer greater than k_r' and such that the element $a(n_r', k_r'')$ is positive. The elements s_k are chosen to be zero for $k_{r-1}'' < k < k_r$ and for $k_r < k < k_r''$; the elements s_k $(k = k_r$ and $k = k_r'')$ are chosen so that the n_rth element of the sequence Bs has the value $(-1)^{r+1}$ and the k_r''th partial sum of the series $\sum a(n_r', k)s_k$ is zero; and the elements s_k $(k > k_r'')$ for which $a(n_r', k) > 0$ are chosen to be zero.

Let this construction of the sequence s be continued indefinitely. Because each column of A contains only one positive element, the multiple definition that is accorded to some of the elements s_k is only

apparent: all definitions assign the value zero to such elements. The transform As is the sequence $0, 0, \cdots$, while the transform Bs contains infinitely many elements of value 1 and -1, respectively.

Case II is that in which there exists a pair of integers N and K such that b_{nk} is zero whenever the three conditions $a_{nk} > 0$, $n > N$, $k > K$ are satisfied. In this case, the sequence s to be constructed is subjected to the condition that $s_k = 0$ whenever $k \leq K$ or one of the elements a_{nk} $(n > N)$ is positive. This condition implies that all partial sums of the series $\sum a_{nk} s_k$ $(n = N+1, N+2, \cdots)$ are zero. The task at hand will be accomplished when the remaining elements s_k are chosen so that the transform Bs diverges and each of the N series $\sum a_{nk} s_k$ $(n = 1, 2, \cdots, N)$ converges to some finite value a_n.

Let $(n_1, k_1), (n_2, k_2), \cdots$ be a sequence of pairs of integers such that

(i) $$n_1 > N,$$

(ii) $$b(n_r, k) = 0 \qquad (k = k_r + 1, k_r + 2, \cdots),$$

(iii) $$\sum_{k=1}^{k_{r-1}} |b(n_r, k)| < 1/2^r \qquad (r = 2, 3, \cdots).$$

The elements s_k of the sequence s which have not yet been chosen are now defined to have the value $(-1)^r$ when the index k is subject to the inequality $k_{r-1} < k \leq k_r$ $(r = 2, 3, \cdots)$. Then, with the notation $Bs = t$, the equation $\lim [t(n_r) - (-1)^r] = 0$ is satisfied. On the other hand, the N series $\sum a_{nk} s_k$ $(n = 1, 2, \cdots, N)$ converge absolutely, and the proof of Theorem 1 is complete.

The proof that has just been given needs only minor modifications to lead to the following result.

THEOREM 2. *If A is a regular Toeplitz matrix with the property that each row of A contains infinitely many nonzero elements and each column of A contains precisely one nonzero element, the convergence field of A is not contained in that of any row-finite matrix.*

Theorems 1 and 2 assert the existence of row-infinite Toeplitz matrices that are in some sense more powerful than any row-finite matrices. But this superiority is not a uniform property of row-infinite matrices.

THEOREM 3. *There exists a regular Toeplitz matrix whose elements are all positive and whose convergence field is identical with that of the Hölder transformation.*

Let A be a the matrix

$$
\begin{Vmatrix}
1 & 1/2! & 1/3! & 1/4! & \cdots \\
1 & 1/2^2 & 1/2^3 & 1/2^4 & \cdots \\
1/2 & 1/2 & 1/3! & 1/4! & \cdots \\
1/2 & 1/2 & 1/2^3 & 1/2^4 & \cdots \\
1/3 & 1/3 & 1/3 & 1/4! & \cdots \\
& & & &
\end{Vmatrix}.
$$

As is seen by examining separately the elements below and above the broken line, this matrix is the sum of the Hölder matrix (with each row in duplicate) and another matrix whose elements b_{nk} are zero when $2k \leqq n$ and whose remaining elements tend to zero uniformly with respect to n as k becomes large; the rapidity with which the elements b_{nk} tend to zero depends only on the parity of n. If the sequence s is evaluated by the Hölder transformation, $s_n = o(n)$, and therefore the series

$$
\sum_{k=1}^{\infty} \frac{s_k}{k!} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{s_k}{2^k}
$$

converge. This implies that the sums

$$
\sum_{k=r}^{\infty} \frac{s_k}{k!} \quad \text{and} \quad \sum_{k=r}^{\infty} \frac{s_k}{2^k}
$$

tend to zero as the lower limit r becomes large, and it follows that the sequence As converges to the same limit as the sequence Hs.

On the other hand, suppose that the sequence s is not evaluated by the Hölder transformation, but is evaluated to zero by the matrix A. Then

$$
\limsup_{r \to \infty} \left| \sum_{k=r}^{\infty} \frac{s_k}{k!} \right| > 0,
$$

and therefore

$$
\limsup_{h \to \infty} \left| s_k \right| / (k-2)! > 0.
$$

The last inequality implies that

$$
\limsup_{k \to \infty} \left| s_k \right| / 2^k > 0,
$$

that is, that the series $\sum a_{nk} s_k$ does not converge when n is even. In

other words, the transform As cannot exist, contrary to the supposition that it converges.

THEOREM 4. *There exists a regular row-finite Toeplitz matrix whose convergence field is not contained in the convergence field of any row-infinite matrix.*

Let A be the matrix

(1)
$$\left\|\begin{array}{cccccc} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/2 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1/2 & \cdots \\ \multicolumn{6}{c}{\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot} \end{array}\right\|.$$

That its existence proves Theorem 4 is a consequence of the following result.

THEOREM 5. *For the convergence field of a regular Toeplitz matrix B to contain the convergence field of the matrix (1) it is necessary and sufficient that B be row-finite and that there exist a constant k_0 such that the relation $b_{n,2k+1} = b_{n,2k+2}$ holds for all n whenever $k > k_0$.*

If B is not row-finite, there exists a sequence s whose transform by A is the sequence $0, 0, 0, \cdots$ and whose transform by B does not exist. If B is row-finite and there exists an infinite increasing sequence of integers k_r such that for some integer n $(n = n_r)$ the relation

$$b(n_r, 2k_r + 1) - b(n_r, 2k_r + 2) \neq 0$$

holds, it is possible to construct a sequence s which satisfies the conditions

$$s_{2k+1} = - s_{2k+2} \qquad\qquad (k = 1, 2, \cdots),$$
$$t(n_r) = (-1)^r \qquad\qquad (t = Bs).$$

Because B is row-finite, the sequence $\{n_r\}$ tends to infinity, and the necessity of the conditions follows.

Suppose, on the other hand, that B satisfies the condition of the theorem and that A evaluates the sequence s to zero. Then

$$s_{2k+1} = - s_{2k+2} + \epsilon_k$$

where ϵ is a null sequence. That B evaluates s to zero follows from the fact that the Toeplitz transformations are linear.

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