P. ERDÖS and J. F. KOKSMA: *On the uniform distribution modulo 1 of lacunary sequences.*

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§ 1. As is well known one calls the sequence of real numbers $u_1, u_2, \ldots$ uniformly distributed modulo 1, if the number N' of those among the numbers

$$u_1-[u_1],\ u_2-[u_2],\ldots,u_N-[u_N],$$

which fall into an arbitrarily given part $\alpha \leqq u < \beta$ of the unit interval $0 \leqq u < 1$ satisfies the condition

$$\frac{N'}{N} \to \beta-\alpha,\ \text{if}\ N \to \infty.$$

The difference $\left|\frac{N'}{N}-(\beta-\alpha)\right|$ is always $\leqq 1$ and for fixed $N \geqq 1$ one calls its upper bound, (if (α, β) is supposed to run through all couples with $0 \leqq \alpha < \beta \leqq 1$), the discrepancy $D(N)$ of the sequence. If

$$ND(N)=o(N), \quad \ldots\ldots \quad (1)$$

it is trivial that the sequence is uniformly distributed modulo 1 and as was proved by WEYL [1]), inversively (1) is a consequence of the distribution modulo 1, defined above.

One gets an interesting special case when putting

$$u_n=\theta\lambda_n \qquad (n=1,2,\ldots) \quad \ldots\ldots \quad (2)$$

where

$$\lambda_1<\lambda_2<\ldots \quad \ldots\ldots \quad (3)$$

denotes an increasing sequence of integers. FATOU [1]) already proved that such a sequence is everywhere dense modulo 1 in the unit interval for almost all values of θ, provided that the sequence (3) is lacunary, i.e. that for some positive constant δ

$$\lambda_{n+1} \geqslant (1+\delta)\lambda_n \qquad (n=1,2,\ldots). \quad \ldots\ldots \quad (4)$$

HARDY–LITTLEWOOD [1]) and WEYL [1]) proved that for each sequence of integers (3) the sequence of numbers (2) is uniformly distributed modulo 1 for almost all θ. Hence for such sequences (1) holds. FOWLER [1]), KOKSMA [1]) and DREWES [2]) deduced improvements of (1). In the special case

$$\lambda_n=2^n,$$

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[1]) References in "Diophantische Approximationen", Erg. d. Math. IV, 4 (1936) by J. F. KOKSMA (Kap. VIII and IX).

[2]) A. DREWES, Diophantische Benaderingsproblemen, Thesis Free University, Amsterdam (1945).

the problem is equivalent to the question how the digits 0, 1 are distributed in the dyadic expansion of θ. Here KHINTCHINE [1]) proved very sharp results.

Generally speaking, the problem is somewhat easier to handle for lacunary sequences (4) than in the general case (3). In this paper we consider the case of *lacunary* sequences of numbers

$$u_n = f(n, \theta),$$

which form a generalisation of the sequences defined by (2). The method used in this paper leads to great difficulties, if one tries to apply it in the general case. In a following paper we treat the *general* case with an other method, which in the specialised cases which are considered in the present paper would give a slightly less sharp result than we deduce here.

§ 2. In this paper we prove a general theorem in which as a special case is contained the following

Theorem 1. *Let δ denote an arbitrary positive constant and $\omega(n)$ a positive increasing function of $n = 1, 2, \ldots$ with $\omega(n) \to \infty$, if $n \to \infty$. Then for any sequence of positive numbers $\lambda_1, \lambda_2, \ldots$, which satisfy (4), the discrepancy $D(N)$ of the sequence (2) satisfies the inequality*

$$ND(N) = o\left(N^{\frac{1}{2}} \log^{\frac{5}{2}} N (\log \log N)^{\frac{1}{2}} \omega(N)\right) \quad . \quad . \quad . \quad . \quad (5)$$

for almost all θ.

This estimate is sharper than all known results. The exponent $\frac{1}{2}$ in the factor $N^{\frac{1}{2}}$ cannot be improved, as KHINTCHINE proved that in the special case $\lambda_n = 2^n$, we have

$$ND(N) = \Omega\left(N^{\frac{1}{2}} \sqrt{\log \log N}\right).$$

Another application of our theorem is the following

Theorem 2. *For almost all values of $\theta \geqq 1$ the discrepancy of the sequence*

$$\theta, \theta^2, \theta^3, \ldots$$

satisfies the inequality (5), if $\omega(n)$ denotes a positive increasing function such that $\omega(n) \to \infty$ as $n \to \infty$.

That the sequence $\theta, \theta^2, \ldots$ for almost all θ is uniformly distributed (modulo 1) had already been proved by KOKSMA [1]), whereas the sharpest estimate for the discrepancy of this sequence known till now was given by DREWES [2]).

§ 3. The theorems quoted above are contained in the following Theorem 3, which itself is a special case of the main Theorem 5.

Theorem 3. *Let $a < b$, $\delta > 0$ be given real numbers. Let $f(1, \theta)$, $f(2, \theta)$, ..., denote a sequence of real functions which are defined on $a \leqq \theta \leqq b$, such that*

$$f'_\theta(n+1, \theta) \geqq (1+\delta) f'_\theta(n, \theta) > 0; \; f''_\theta(n+1, \theta) \geqq (1+\delta) f''_\theta(n, \theta) \geqq 0 \qquad (n = 1, 2, \ldots)$$

for all values of θ on $a \leqq \theta \leqq b$. Let $\omega(n)$ denote an increasing function of $n = 1, 2, \ldots$, such that $\omega(n) \to \infty$ as $n \to \infty$.

Then for almost all θ on $a \leqq \theta \leqq b$ the discrepancy in the uniform distribution of the sequence

$$f(1, \theta), \; f(2, \theta), \ldots$$

satisfies the relation (5).

Remark. It is clear that the sequences of the theorems 1 and 2 satisfy the conditions of Theorem 3. In the first case we put without loss of generality $a = 0$, $b = 1$ and in the second case we put $a = 1 + \delta$, $b > a$ and after application of Theorem 3, we let

$$\delta \to 0, \quad b \to \infty.$$

The reader will find the deduction of Theorem 3 from Theorem 4 in § 8.

§ 4. For the proof of our main theorem we deduce a lemma (Lemma 2), which has some interest in itself. For the special case, considered in Theorem 3, it runs as follows:

Theorem 4. *Suppose that the conditions of Theorem 3 are satisfied. Let K denote a positive constant. Then for almost all θ the following statement is true: If N and k are integer such that $1 \leqq k \leqq N^K$, then*

$$\left| \sum_{n=1}^{N} e^{2\pi i k f(n, \theta)} \right| \leqq C(\theta) N^{\frac{1}{2}} \log^{\frac{3}{2}} N (\log \log N)^{\frac{1}{2}} \omega(N),$$

where $C(\theta)$ does not depend on N or k.

The reader finds its deduction in § 9.

§ 5. Before we state the main theorem, we make some

Preliminary Remarks. Let N and r denote positive integers. Out of the N integers $n = 1, 2, \ldots, N$, we can form N^r different r-tuples; such an r-tuple we shall denote by $(n_1, \ldots, n_r)$. There are $C^r_{\overline{N}}$ different r-tuples among them for which $n_1 \leqq n_2 \leqq \ldots \leqq n_r$. Such a special r-tuple we shall denote also by $\{n_1, \ldots, n_r\}$. The elements $n_1, \ldots, n_r$ of the r-tuple $\{n_1, \ldots, n_r\}$ have a number of different permutations, which we shall denote by $A\{n_1, \ldots, n_r\}$. Then we obviously have

$$A\{n_1, \ldots, n_r\} \leqq r! \qquad (6)$$

and

$$\sum_{\{n_1, \ldots, n_r\}} A\{n_1, \ldots, n_r\} = N^r. \qquad (7)$$

For later purposes we put

$$A_N^r = \sum_{\{n_1,\dots,n_r\}} A^2\{n_1,\dots,n_r\}. \quad (8)$$

Definition. If $\{n_1, \dots, n_r\}$ and $\{m_1, \dots, m_r\}$ are two different r-tuples of the above kind, we say that the first is greater than the second:

$$\{n_1,\dots,n_r\} > \{m_1,\dots,m_r\},$$

if and only if for some τ $(1 \leqq \tau \leqq r)$

$$n_\tau > m_\tau \quad , \quad n_\varrho = m_\varrho \quad (\varrho = \tau+1, \tau+2, \dots, r).$$

Condition A. Let $g(x, \theta)$ for $x = 1, 2, \dots, N$ denote a function of θ on the segment $a \leqq \theta \leqq b$ such that for each couple of r-tuples $\{n_1, \dots, n_r\} > \{m_1, \dots, m_r\}$ the function

$$\Phi(\theta) = \Phi(n_1,\dots,n_r; m_1,\dots,m_r; \theta) = \sum_{\varrho=1}^{r} g(n_\varrho, \theta) - \sum_{\varrho=1}^{r} g(m_\varrho, \theta) \quad . \quad (9)$$

has a derivative for $a \leqq \theta \leqq b$, which is continuous, $\neq 0$, and either non-decreasing or non-increasing in the segment $a \leqq \theta \leqq b$. We then put

$$\left.\begin{array}{l}\Psi = \Psi(n_1,\dots,n_r; m_1,\dots,m_r) = \\ \operatorname{Min}\{\Phi'_\theta(n_1,\dots,n_r; m_1,\dots,m_r; a), \Phi'_\theta(n_1,\dots,n_r; m_1,\dots,m_r; b)\},\end{array}\right\} \quad . \quad (10)$$

$$\left.\begin{array}{r}B_N = N^{-r} \sum\limits_{\{n_1,\dots,n_r\} > \{m_1,\dots,m_r\}} \sum A\{n_1,\dots,n_r\}\, A\,\{m_1,\dots,m_r\} \\ \Psi^{-1}(n_1,\dots,n_r; m_1,\dots,m_r).\end{array}\right\} \quad (11)$$

Now we state our main theorem:

Theorem 5. I. *Let a and b denote real constants with* $a < b$. *Let* $f(n, \theta)$ *for* $n = 1, 2 \dots$ *denote a real function of* θ *on the segment* $a \leqq \theta \leqq b$. *Let* N_0 *be a positive integer. Let* $r = r(N)$ *and* $s = s(N)$ *be positive integers which are defined for each integer* $N \geqq N_0$, *such that*

$$S(N) \leqq N.$$

Let for each integer $N \geqq N_0$ *and each integer* $\sigma = 1, \dots, s(N)$ *the* N_σ *functions*

$$g_\sigma(x, \theta) = f(\sigma + (x-1)s, \theta) \qquad \left(x = 1, 2, \dots, N_\sigma = \left[\frac{N-\sigma}{s}\right] + 1\right)$$

be considered and let the condition A be satisfied with g_σ *instead of* g *and with* N_σ *instead of* N.

II. *Putting*

$$B_N^* = \operatorname*{Max}_{1 \leqq \sigma \leqq s} B_{N_\sigma}, \quad (12)$$

we assume that a non-decreasing sequence $\psi(1), \psi(2), \dots$ *of positive numbers exists such that the series*

$$\sum_{n=N_0}^{\infty} s \cdot \{(b-a)\, r!\, N^{\frac{1}{2}} + B_N^* \log N\} \left\{ \psi\left(\left[\frac{N-s}{s}\right] + 1\right) \right\}^{-2r} \quad . \quad (13)$$

converges. Then almost all numbers θ of $a \leqq \theta \leqq b$, have the property that the discrepancy $D(N)$ of the sequence $f(1, \theta), f(2, \theta), \ldots$ satisfies the inequality

$$ND(N) \leqq K_1 \, s^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \cdot \psi\left(\left[\frac{N-1}{s}\right]+1\right) \log N \text{ for } N \geqq N_0^*, . \quad (14)$$

where K_1 denotes a numerical constant, whereas N_0 denotes an index depending on θ.

§ 6. We now prove

Lemma 1. *Let N and r denote positive integers and let $g(x, \theta)$ satisfy the condition A. Then for each fixed integer $h \neq 0$ we have*

$$\int_a^b \left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} d\theta = (b-a) A_N^r + \frac{2\vartheta}{\pi h} B_N N^r \, (|\vartheta| \leqq 1). \quad . \quad (15)$$

Proof.

$$\left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} = \left\{ \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right\}^r \left\{ \sum_{x=1}^{N} e^{-2\pi i h g(x,\theta)} \right\}^r$$

$$= \sum_{(n_1, \ldots, n_r)} e^{2\pi i h (g(n_1,\theta) + \ldots + g(n_r,\theta))} \cdot \sum_{(n_1, \ldots, n_r)} e^{-2\pi i h (g(n_1,\theta) + \ldots + g(n_r,\theta))},$$

where both sums are to be expanded over all N^r r-tuples of integers

$$n_\varrho = 1, 2, \ldots, N.$$

Applying the preliminary remark of § 5, we write

$$\left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} = \sum_{\{n_1, \ldots, n_r\}} A\{n_1, \ldots, n_r\} \, e^{2\pi i h (g(n_1,\theta) + \ldots + g(n_r,\theta))}$$

$$\cdot \sum_{\{n_1, \ldots, n_r\}} A\{n_1, \ldots, n_r\} \, e^{-2\pi i h (g(n_1,\theta) + \ldots + g(n_r,\theta))}$$

$$= \sum_{\{n_1, \ldots, n_r\}} \sum_{\{m_1, \ldots, m_r\}} A\{n_1, \ldots, n_r\} A\{m_1, \ldots, m_r\} \, e^{2\pi i h \left\{ \sum_{\varrho=1}^{r} g(n_\varrho,\theta) - \sum_{\varrho=1}^{r} g(m_\varrho,\theta) \right\}}$$

$$= \sum_{\{n_1, \ldots, n_r\}} A^2\{n_1, \ldots, n_r\} + 2 \sum_{\{n_1, \ldots n_r\} > \{m_1, \ldots, m_r\}} \sum$$

$$A\{n_1, \ldots, n_r\} A\{m_1, \ldots, m_r\} \cos 2\pi h \Phi(\theta)$$

by (9). Hence by (8)

$$\left. \begin{aligned} &\int_a^b \left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} d\theta = (b-a) A_N^r + \\ &+ 2 \sum_{\{n_1, \ldots, n_r\} > \{m_1, \ldots, m_r\}} \sum A\{n_1, \ldots, n_r\} A\{m_1, \ldots, m_r\} \int_a^b \cos 2\pi h \Phi(\theta) \, d\theta. \end{aligned} \right\} . \quad (16)$$

Now choosing the new variable of integration u by the substitution

$$u = \Phi(\theta),$$

we find

$$du = \Phi'(\theta)\, d\theta$$

and

$$\int_a^b \cos 2\pi h\, \Phi(\theta) \cdot d\theta = \int_{\Phi(a)}^{\Phi(b)} \cos 2\pi h u\, \frac{du}{\Phi'(\theta)}$$

and therefore using BONNET's theorem and (10)

$$\left| \int_a^b \cos 2\pi h\, \Phi(\theta) \cdot d\theta \right| \leqq \frac{1}{\pi h\, \Psi}.$$

Hence by (16) and (11)

$$\int_a^b \left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} d\theta = (b-a)\, A_N^r + \frac{2\vartheta N^r}{\pi h} B_N \cdot (|\vartheta| \leqq 1).$$

Q.e.d.

Lemma 2. *Let the conditions* I *of the main theorem* 5 *be satisfied.* II. *Defining* B_N^* *by* (12), *we suppose that a non-decreasing sequence* $\psi(1), \psi(2), \ldots$ *and a non-decreasing sequence of integers* $\Lambda(1), \Lambda((2), \ldots$ $(\Lambda(N) \geqq 3$, *if* $N \geqq N_0)$ *exist, such that the series*

$$\sum_{n=N_0}^{\infty} s\{(b-a)\, r!\, \Lambda(N) + 2B_N^* \log \Lambda(N)\} \left\{ \psi\left(\left[\frac{N-s}{s} \right] + 1 \right) \right\}^{-2r}. \quad (13a)$$

converges. Then almost all numbers θ *of* $a \leqq \theta \leqq b$ *have the property that for all integers* $h = 1, 2, \ldots, \Lambda(N)$

$$\left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right| \leqq 2 s^{\frac{1}{2}} N^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s} \right] + 1 \right), \text{ if } N \geqq N_0^*(\theta) \quad . \quad (17)$$

Proof. Let N be a fixed integer $\geqq N_0$. Let (h, σ) denote a couple of integers which satisfy the inequalities

$$1 \leqq h \leqq \Lambda(N), \qquad 1 \leqq \sigma \leqq s.$$

Then the Lemma 1 with

$$g(x, \theta) = g_\sigma(x, \theta) = f(\sigma + (x-1)\, s, \theta), \quad N = N_\sigma$$

learns

$$\int_a^b \left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x,\theta)} \right|^{2r} d\theta \leqq b-a)\, r!\, N_\sigma^r + \frac{1}{h} B_{N_\sigma} N_\sigma^r \quad . \quad . \quad (18)$$

because of (6) and (7).

Now let $S(N, h, \sigma)$ denote the set of all numbers θ on $a \leqq \theta \leqq b$ for which

$$\left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x, \theta)} \right| \geqq N_\sigma^{\frac{1}{2}} \psi(N_\sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Then by (18) we obviously find for its measure $mS(N, h, \sigma)$ the inequality

$$m\, S(N, h, \sigma)\, N_\sigma^r \{\psi(N_\sigma)\}^{2r} \leqq (b-a)\, r!\, N_\sigma^r + \frac{1}{h} B_{N_\sigma} N_\sigma^r,$$

hence

$$m\, S(N, h, \sigma) \leqq \left\{ (b-a)\, r! + \frac{1}{h} B_{N_\sigma} \right\} \{\psi(N_\sigma)\}^{-2r}$$

and therefore by

$$N_\sigma \geqq \left[\frac{N-s}{s}\right] + 1 \text{ and } (12):$$

$$m\, S(N, h, \sigma) \leqq \left\{ (b-a)\, r! + \frac{1}{h} B_N^* \right\} \left\{ \psi\left(\left[\frac{N-s}{s}\right] + 1 \right) \right\}^{-2r}.$$

Now let $S(N)$ denote the set

$$S(N) = \sum_{(h, \sigma)} S(N, h, \sigma),$$

where the summation is to be extended over all couples (h, σ) which satisfy $1 \leqq h \leqq \Lambda(N)$, $1 \leqq \sigma \leqq s$. Then we have

$$\left. \begin{aligned} m\, S(N) &\leqq \left\{ (b-a)\, r!\, s\, \Lambda(N) + s B_N^* \sum_{h=1}^{\Lambda(N)} \frac{1}{h} \right\} \left\{ \psi\left(\left[\frac{N-s}{s}\right] + 1 \right\}^{-2r} \\ &< s\, \{(b-a)\, r!\, \Lambda(N) + 2 B_N^* \log \Lambda(N)\} \left\{ \psi\left(\left[\frac{N-s}{s}\right] + 1 \right) \right\}^{-2r}. \end{aligned} \right\} \quad (20)$$

Each θ of $a \leqq \theta \leqq b$, which does not belong to $S(N)$ $(N \geqq N_0)$ has the property that the inequality

$$\left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x, \theta)} \right| \leqq N_\sigma^{\frac{1}{2}} \psi(N_\sigma) < \left(\frac{N-1}{s} + 1 \right)^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s}\right] + 1 \right)$$

is valid for all couples (h, σ) which satisfy the inequalities

$$1 \leqq h \leqq \Lambda(N), \quad 1 \leqq \sigma \leqq s.$$

Therefore we have for such a θ

$$\left| \sum_{n=1}^{N} e^{2\pi i h f(n, \theta)} \right| \leqq \left| \sum_{\sigma=1}^{s} \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x, \theta)} \right| \leqq 2 s^{\frac{1}{2}} N^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s}\right] + 1 \right),$$

for all integers $h = 1, 2, \ldots, \Lambda(N)$.

Now as $mS(N)$ satisfies (20) and as the series (13a) converges, almost all numbers θ of $a \leqq \theta \leqq b$ belong to at most a finite number of the sets

$S(N)$ $(N = N_0, N_0+1, \ldots)$. Therefore for almost all θ of $a \leq \theta \leq b$ an index N_0^* can be found, such that

$$\left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right| \leqq 2 s^{\frac{1}{2}} N^{\frac{1}{2}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right),$$

whatever the value of the integer $h = 1, 2, \ldots A(N)$ may be. Q.e.d.

§ 7. In order to prove the main theorem, we quote the following theorem, which is an improvement proved by ERDÖS–TURÁN [3]) of the one dimensional case of a theorem of VAN DER CORPUT–KOKSMA [1]).

Lemma 3. *If $u_1, u_2, \ldots$ is a real sequence and if $D(N)$ denotes its discrepancy, then for each integer $m \geq 1$, we have*

$$ND(N) \leqq K \left\{ \frac{N}{m+1} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h u_n} \right| \right\}, \quad . \quad . \quad . \quad (21)$$

where K denotes a numerical constant.

Proof of the Theorem 5. Put

$$u_n = f(n, \theta), \; m = [\sqrt{N}], \; A(N) = [\sqrt{N}].$$

Then (using Lemma 2) for almost all θ we have by (17) and (21), if $N \geq N_0^*(\theta)$

$$ND(N) \leqq K \sqrt{N} + K \sum_{h=1}^{[\sqrt{N}]} \frac{2}{h} s^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right)$$
$$\leqq K_1 s^{\frac{1}{2}} N^{\frac{1}{2}} \psi \left(\left[\frac{N-1}{s} \right] + 1 \right) \log N.$$

Q.e.d.

§ 8. **Proof of Theorem 3.** Be $\omega(N)$ the function of Theorem 3 and let N_0 be a sufficiently large integer. We shall prove that the functions $f(n, \theta)$ of Theorem 3 satisfy the conditions of Theorem 5, if we put for $N \geq N_0$

$$\left. \begin{aligned} s(N) &= \left[\frac{2}{\log(1+\delta)} \log\log N \right]; \\ r(N) &= \left[\frac{\log N}{\log \sqrt{\omega([\sqrt{N}])}} \right] + 1; \; \psi(N) = \sqrt{\log N^2} \cdot \sqrt{\omega(N)} \end{aligned} \right\} \quad . \quad (22)$$

where δ denotes the constant of Theorem 3. Now for $N \geqq N_0$ we consider the $s = s(N)$ sequences

$$g_\sigma(x, \theta) = f(\sigma + (x-1)s, \theta) \left(1 \leqq \sigma \leqq s; \; x = 1, 2, \ldots, N_\sigma = \left[\frac{N-\sigma}{s} \right] + 1 \right).$$

[3]) P. ERDÖS and P. TURÁN, On a problem in the theory of uniform distribution. Proc Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 1146—1154, 1262—1269 (1948); Ind. Math. **10**, 370—378, 406—413 (1948).

By hypothesis we have (the prime meaning differentiation by θ):

$$\frac{f'(n+1,\theta)}{f'(n,\theta)} \geqq 1+\delta,$$

hence for $N \geqq N_0$

$$\frac{g'_\sigma(x+1,\theta)}{g'_\sigma(x,\theta)} \geqq (1+\delta)^s > (1+\delta)^{3/2 \frac{\log\log N}{\log(1+\delta)}} = (\log N)^{3/2} > r+1.$$

Therefore if we take two r-tuples

$$\{n_1, \ldots, n_r\} > \{m_1, \ldots, m_r\},$$

we have, using the notation of the Preliminary Remarks:

$$\begin{aligned} \Phi'(\theta) &= g'_\sigma(n_1,\theta) + \ldots + g'_\sigma(n_r,\theta) - g'_\sigma(m_1,\theta) - \ldots - g'_\sigma(m_r,\theta) \\ &\geqq g'_\sigma(n_\tau,\theta) - r g'_\sigma(n_\tau - 1,\theta) > g'_\sigma(n_\tau - 1,\theta) \geqq f'(1,a) = c_0 \quad (23) \end{aligned}$$

and we conclude that if we range the r-tuples $\{n_1, \ldots, n_r\}$ in order of increasing magnitude, the corresponding sums

$$g'_\sigma(n_1,\theta) + \ldots + g'_\sigma(n_2,\theta)$$

with each step will increase by at least the amount c_0, whatever the value of θ $(a \leqq \theta \leqq b)$ may be. Hence, if the r-tuple $\{n_1, \ldots, n_r\}$ is fixed, we have by (10) and (9)

$$\sum_{\{n_1,\ldots,n_r\} > \{m_1,\ldots,m_r\}} \; \sum \{\Psi(n_1,\ldots n_r; m_1,\ldots,m_r\}^{-1} \leqq \frac{1}{c_0} \sum_{k=1}^{N_\sigma^r} \frac{1}{k} \leqq \frac{1+r\log N}{c_0},$$

as there are at most $N_\sigma^r \leqq N^r$ r-tuples $\{m_1, \ldots, m_r\}$.

Therefore we find by (11) and (6)

$$B_{N_\sigma} \leqq N_\sigma^{-r}\, r!\, \frac{1+r\log N}{c_0} \sum_{\{n_1,\ldots,n_r\}} A\{n_1,\ldots,n_r\} = \frac{r!}{c_0}(1+r\log N)$$

by (7). Hence we find by (12) a fortiori

$$B_N^* \leqq \frac{2r}{c_0}\, r! \log N < r^r \log N \text{ for } N \geqq N_0.$$

Thus we find that the general term of our series (13) is at most

$$\begin{aligned} &\{(b-a)\,s\,r^r N^{\frac{1}{2}} + s\,r^r \log^2 N\}\,\{\psi([\sqrt{N}]+1)\}^{-2r} \leqq \\ &\qquad \leqq c_1\, s\,(\log N)^r N^{\frac{1}{2}} \cdot \sqrt{\log N}^{-2r} \{\sqrt{\omega([\sqrt{N}])}\}^{-2r} \end{aligned}$$

by (22) and therefore by (22)

$$\leqq c_1\, s\, N^{\frac{1}{2}} \{\sqrt{\omega([\sqrt{N}])}\}^{-\frac{2\log N}{\log\sqrt{\omega([\sqrt{N}])}}} < N^{-\frac{5}{4}}, \text{ if } N \geqq N_0.$$

Hence, our series (13) converges.

By hypothesis we further have $f''_{(n+1,\theta)} \geqq (1+\delta)\, f''_{(n,\theta)} \geqq 0$.

Repeating the proof of (23) with f'' instead of f' and g''_σ instead of g'_σ, we find $\Phi''(\theta) \geqq 0$. Hence $\Phi'(\theta)$ is non-decreasing.

From our result we conclude that for almost all θ the inequality (14) holds, i.e. because of (22):

$$ND(N) \leqq K_2 N^{\frac{1}{2}} (\log N)^{\frac{3}{2}} (\log\log N)^{\frac{1}{2}} \sqrt{\omega(N)}, \text{ if } N \geqq N_0^*(\theta).$$

Hence (5) follows immediately.

§ 9. **Proof of Theorem 4.** We shall use Lemma 2 and we put

$$\left.\begin{aligned} A(N) = [N^K],\ s(N) = \left[\frac{2}{\log(1+\delta)} \log\log N\right],\ r(N) = \\ = \left[\frac{(K+2)\log N}{\log \omega([\sqrt{N}])}\right] + 1,\ \psi(N) = \sqrt{\log N^2} \quad \omega(N), \end{aligned}\right\}. \tag{24}$$

where K, δ and $\omega(N)$ are defined in Theorem 4. Then in exactly the same way as in § 8 it follows that

$$B_N^* \leqq r^r \log N \text{ for } N \geqq N_0$$

and thus the general term of the series (13a) is at most

$$\{(b-a)\, s r^r N^K + s r^r \cdot 2K \log^2 N\} \{\psi([\sqrt{N}]+1)\}^{-2r} \leqq$$
$$\leqq c_2\, s\, (\log N)^r N^K (\sqrt{\log N})^{-2r} \{\omega([\sqrt{N}])\}^{-2r}$$

by (24) and therefore by (24)

$$< c_2 s \cdot N^K \{\omega([\sqrt{N}])\}^{\frac{-(2+K)\log N}{\log \omega([\sqrt{N}])}} < N^{-\frac{3}{2}}, \text{ if } N \geqq N_0.$$

Hence, the series (13a) converges. From our result we conclude that (17) holds for almost all θ on $a \leqq \theta \leqq b$, i.e.

$$\left|\sum_{n=1}^{N} e^{2\pi i h f(n,\theta)}\right| \leqq K_3 N^{\frac{1}{2}} \log^{\frac{1}{2}} N (\log\log N)^{\frac{1}{2}} \omega(N), \text{ if } N \geqq N_0^*(\theta).$$

From this Theorem 4 follows immediately.