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On the representation of $1, 2, \ldots, N$ by differences

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P. ERDÖS and I. S. GÁL: *On the representation of 1, 2, ..., N by differences.*

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L. RÉDEI and A. RÉNYI called the set of integers $a_1, a_2, \ldots, a_{k(n)}$ in their paper [1]) a difference-basis with respect to n if every positive integer ν; $0 < \nu \leq n$ can be represented in the form $\nu = a_i - a_j$. Let $n^* = \min k(n)$ denote the minimal value of $k(n)$ for a given n. L. RÉDEI and A. RÉNYI proved, that

$$1^*) \qquad \lim_{n \to \infty} \frac{n^*}{\sqrt{n}} \text{ exists,}$$

$$2^*) \qquad \lim \frac{n^*}{\sqrt{n}} = \inf \frac{n^*}{\sqrt{n}} \text{ (inf denotes the greatest lower bound)}$$

$$3^*) \qquad \sqrt{2 + \frac{4}{3\pi}} \leqslant \lim_{n \to \infty} \frac{n^*}{\sqrt{n}} \leqslant \sqrt{\frac{8}{3}} \text{ holds.}$$

Somewhat earlier A. BRAUER [2]) considered the similar problem of a difference-basis $a_1 < a_2 < \ldots < a_{l(n)}$ with respect to n, the elements of which satisfy the inequality $0 \leq a_i \leq n$; $i = 1, 2, \ldots, n$. In what follows difference-bases of A. BRAUER's type shall be called "restricted difference-basis with respect to n".

L. RÉDEI proposed the following question: Let $n_0 = \min l(n)$ denote the minimal number of terms of a restricted difference-basis with respect to n, minimum being meant for fixed n. Does the set of numbers $\frac{n_0}{\sqrt{n}}$ converge to a limit? Further if the limit exists, how can it be estimated from above? In this note we prove the following results:

Theorem: *If $n_0 = \min l(n)$ for fixed n, where $l(n)$ denotes the number of terms of a restricted difference-basis with respect to n, then*

$$1^0) \qquad \lim_{n \to \infty} \frac{n_0}{\sqrt{n}} \text{ exists,}$$

$$2^0) \qquad \lim \frac{n_0}{\sqrt{n}} = \inf \frac{n_0}{\sqrt{n}},$$

$$3^0) \qquad \sqrt{2 + \frac{4}{3\pi}} \leqslant \lim \frac{n_0}{\sqrt{n}} \leqslant \sqrt{\frac{8}{3}} \text{ holds.}$$

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1) L. RÉDEI and A. RÉNYI, On the representation of 1, 2, ..., N by differences. Recueil Mathématique, T. 61, 1948.

2) A. BRAUER, A problem of additive number-theory and its application in electrical engineering, Journ. of the Elisha Mitchell Scientific Society, Vol. 61, pp. 55—66.

Proof: Obviously if we can prove 1⁰), then the inequality

$$\sqrt{2+\frac{4}{3\pi}} \leqslant \lim \frac{n_0}{\sqrt{n}}$$

follows at once from 3*). Similarly it can be seen from 2⁰) that

$$\lim \frac{n^*}{\sqrt{n}} \leqslant \lim \frac{n_0}{\sqrt{n}} \leqslant \sqrt{\frac{8}{3}}.$$

Namely the numbers 0, 1, 4, 6 form a restricted difference-basis with respect to $n = 6$, therefore

$$\inf \frac{n^*}{\sqrt{n}} \leqslant \inf \frac{n_0}{\sqrt{n}} \leqslant \frac{4}{\sqrt{6}} = \sqrt{\frac{8}{3}}.$$

Consequently it is sufficient to prove the statements 1⁰) and 2⁰). The following proof of these results contains a new proof of 1*) and 2*) too, only the restriction $0 \leq a_i \leq n$; $i = 1, 2, \ldots, n$ must be omitted [3]).

I. Consider a fixed value of n and denote

$$a_1 < a_2 < \ldots < n_0 \ ; \ (0 \leqslant a_i \leqslant n \ ; \ i = 1, 2, \ldots n_0) \quad . \quad . \quad . \quad (1)$$

the (restricted) difference-basis with respect to n, having a minimal number of terms. Further let us have $N \geq 7\,(n+1)$ and choose the prime p such that

$$M = N - (n+1)(p^2 + p + 1) \geqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Later we shall determine the exact value of the prime p.

J. Singer [4]) has proved that there exist $p+1$ integers b_k; $k = 1, 2, \ldots, p+1$ such that the differences $b_k - b_l$ represent a complete system of residues modulo $p^2 + p + 1$. We can choose these residues $b_1, b_2, \ldots, b_{p+1}$ in such a manner that

$$0 \leqslant b_1 < b_2 < \ldots < b_{p+1} < p^2 + p + 1 = m \quad . \quad . \quad . \quad . \quad (3)$$

Hence if $0 \leq \nu \leq m - 1$ (ν integer) there exist two residues b_k and b_l such that either $\nu = b_k - b_l$ or $\nu - m = b_k - b_l$.

Now let us consider the integers

$$a_i m + b_k \quad ; \quad i = 1, 2, \ldots, \bar{n} \quad ; \quad k = 1, 2, \ldots, p+1 \quad . \quad . \quad (4)$$

(If $0 \leq a_i \leq n$; $i = 1, 2, \ldots, n$ then according to (2) we have $0 \leq a_i m + b_k < m\,n + m \leq N$). Every ν; $0 \leq \nu \leq m\,n$ is the difference of two numbers $a_i m + b_k$ and $a_j m + b_l$. In fact put $\nu = \nu_1 m + \nu_2$, $0 \leq \nu_1 \leq n - 1$, $0 \leq \nu_2 \leq m - 1$. If ν_2 has a representation $\nu_2 = b_k - b_l$ then a_i and a_j shall be choosen so that $\nu_1 = a_i - a_j$. Consequently we obtain a

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3) Our proof is similar to that of Rédei and Rényi.

4) J. Singer, Trans. Amer. Math. Soc. 1938, T. 43, pp. 377—385, and Vijayaraghavan–S. Chowla, Proc. Nat. Acad. Sci. India, Sect. A. T. 15, 1945, p. 194.

representation $\nu = (a_i m + b_k) - (a_j m + b_l)$. If however $\nu_2 - m$ can be represented in the form $b_k - b_l$ then $\nu = (\nu_1 + 1)m + (\nu_2 - m)$ where $\nu_1 + 1 \leq n$. Consequently there exists a pair a_i, a_j with the property $\nu_1 + 1 = a_i - a_j$. Thus $\nu = (a_i m + b_k) - (a_j m + b_l)$. Taking all these facts into account, it follows that the set of the integers $a_i m + b_k$ in (4) is a restricted difference-basis with respect to mn.

Finally we consider the integers

$$0, 1, 2, \dots, [\sqrt{M}], N, N-[\sqrt{M}], N-2[\sqrt{M}], \dots, N-([\sqrt{M}]+1)[\sqrt{M}]. \quad (5)$$

(Every one of these numbers satisfies the condition $0 \leq \nu \leq N$.) Obviously we can represent every satisfying $N - [\sqrt{M}]([\sqrt{M}] + 2) \leq \nu \leq N$ as the difference of two members of the set (5). Taking into account the inequality $[\sqrt{M}] > \sqrt{M} - 1$ we obtain from (2) that

$$N-[\sqrt{M}]([\sqrt{M}]+2) < N-(\sqrt{M}-1)(\sqrt{M}+1) = N-M+1 = nm+1$$

and thus $N - [\sqrt{M}]([\sqrt{M}] + 2) \leq mn$. Consequently every ν satisfying $mn \leq \nu \leq N$ is the difference of two members of the set (5).

Therefore every ν; $0 \leq \nu \leq N$ is the difference of two integers of the sets (4) and (5) respectively. That is to say, the union of the sets (4) and (5) gives a restricted difference-basis of N. The sets (4) and (5) having $\bar{n}(p + 1)$ and $2[\sqrt{M}] + 2$ terms respectively, we obtain

$$N_0 \leqslant n_0(p+1) + 2[\sqrt{M}] + 2. \quad \dots\dots \quad (6)$$

II. Hitherto we have for p and N only the restrictions $N \geq 7(n + 1)$ and the inequality (2). Now we shall determine the exact value of the prime p. An immediate consequence of the prime number theorem is the following fact: If $\delta > 0$ and $x \geq x(\delta)$, there exists a prime such that $x \leq p < (1 + \delta)x$. Therefore $x^2 \leq p^2 < (1 + \delta)^2 x^2$ and thus

$$x^2 + x + 1 \leq p^2 + p + 1 < (1 + \delta)^2 (x^2 + x + 1).$$

Let us denote

$$(1+\delta)^2(x^2+x+1) = \frac{N}{n+1} \text{ and } (1+\delta)^{-2} = 1 - \frac{\varepsilon^2}{36}.$$

Consequently if $\varepsilon > 0$ is an arbitrary small fixed number, there exists a p such that

$$\left(1-\frac{\varepsilon^2}{36}\right)\frac{N}{n+1} \leqslant p^2+p+1 < \frac{N}{n+1}$$

if only $N \geq N_1(\varepsilon, n)$. Thus $O < M = N - (n + 1)(p^2 + p + 1) \leq \frac{\varepsilon^2}{36} N$ that is to say we can choose p such a manner that

$$2[\sqrt{M}] \leqslant 2\sqrt{M} < \frac{\varepsilon}{3}\sqrt{N}, \quad \dots\dots \quad (7)$$

if only $N \geq N_1(\varepsilon, n)$. According to (2) we have $N \geq mn = n(p^2 + p + 1) > np^2$ i.e.

$$p < \frac{\sqrt{N}}{\sqrt{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

if $N \geq 7\,(n+1)$. Taking into account that $\bar{n} \leq n$ and the fact that n and $\varepsilon > 0$ are fixed, we have $n_0 < \frac{\varepsilon}{3}\sqrt{N}$ if only $N \geq N_2(\varepsilon, n)$.

Consequently according to (6), (7) and (8) it follows

$$N_0 < \frac{n_0}{\sqrt{n}}\sqrt{N} + \frac{\varepsilon}{3}\sqrt{N} + 2 + n_0 < \sqrt{N}\left(\frac{n_0}{\sqrt{n}} + \varepsilon\right)$$

i.e.

$$\frac{N_0}{\sqrt{N}} < \frac{n_0}{\sqrt{n}} + \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for arbitrary small, fixed $\varepsilon > 0$, if only $N \geq N_3\,(\varepsilon, n)$.

III. From the inequality (9) we have at once the estimate

$$\overline{\lim}\,\frac{N_0}{\sqrt{N}} \leqslant \frac{n_0}{\sqrt{n}} + \varepsilon$$

for arbitrary positive ε. Thus it follows

$$\overline{\lim}\,\frac{N_0}{\sqrt{N}} \leqslant \frac{n_0}{\sqrt{n}}$$

and since the integer n is arbitrary we have

$$\lim\frac{N_0}{\sqrt{N}} \leqslant \inf\frac{n_0}{\sqrt{n}} \leqslant \underline{\lim}\,\frac{N_0}{\sqrt{n}}.$$

Therefore

$$\overline{\lim}\,\frac{N_0}{\sqrt{N}} = \underline{\lim}\,\frac{N_0}{\sqrt{N}} = \lim\frac{n_0}{\sqrt{n}} = \inf\frac{n_0}{\sqrt{n}}.$$

Thus 1°) and 2°) is proved, the proof of 1*) and 2*) is clearly the same except that the condition $0 \leq a_i \leq n$ has to be omitted.