ON THE HAUSDORFF DIMENSION OF SOME SETS IN EUCLIDEAN SPACE

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Let E be a closed set in n-dimensional space, x a point not in E. Denote by $S(x)$ the largest sphere of center x which does not contain any point of E in its interior. Put $\phi(x)=E\cap\bar{S}(x)$. ($\bar{A}$ denotes the closure of A.) Denote by M_k the set of points for which $\phi(x)$ contains k or more linearly independent points (that is, points which do not lie in any $(k-2)$-dimensional hyperplane). M_k is defined for $k\leqq n+1$. In a previous paper I proved that M_2 has n-dimensional measure 0 and conjectured that M_k has Hausdorff dimension not greater than $n+1-k$. In the present note we shall prove this conjecture. In my previous paper I also proved that M_{n+1} is countable, but the proof there given applied only for the case $n=2$; now we are going to give a general proof.

Let R be any set in n-dimensional space. Let $x\in R$. We define the contingent[1] of R at x ($\text{contg}_R\, x$) as follows: The contingent will be a subset of the unit sphere. A point z of the unit sphere belongs to $\text{contg}_R\, x$ if and only if there exists a sequence of points $y_1, y_2, \cdots$ in R converging to x so that the direction of the vector connecting x with y_i tends to the direction of the vector connecting the center of the unit sphere with z. First we state the following lemma.

LEMMA. *Let there be given a set R in n-dimensional space. Assume that for every x,* $\text{contg}_R\, x$ *does not contain any point of the intersection of the unit sphere with a k-dimensional hyperplane going through its center (the hyperplane can depend on x). Then R is contained in the sum of countably many surfaces of finite $(n-k)$-dimensional measure.*

This lemma is well known.[2]

THEOREM 1. *Let $k<n+1$. Then M_k is contained in the sum of countably many surfaces of finite $(n+1-k)$-dimensional measure. If $k=n+1$, then M_k is countable.*[3]

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[1] G. Bouligand, *Introduction à la géometrie infinitésimale directe.* Also Saks, *Theory of the integral.*

[2] Saks, ibid. pp. 264–266 and pp. 304–307. Also Roger, C. R. Acad. Sci. Paris vol. 201 (1935) pp. 871–873.

[3] For $n=2$ this theorem is proved by C. Pauc, Revue Scientifique, August, 1939.

Remark. This clearly means that the Hausdorff dimension of M_k $(k \leqq n+1)$ is not greater than $n+1-k$.

Let us first consider the case $k=n+1$. Assume that $x \in M_{n+1}$. Let $z_i \in \phi(x)$, $i=1, 2, \cdots, n+1$, and assume that the z's are linearly independent. Denote by $f(x)$ the maximum value of the volume of the simplices determined by the z's (since $\phi(x)$ is closed the maximum is attained). Define now $N_{n+1}{}^{(c)}=N$ to consist of all the points $x \in M_{n+1}$ for which $f(x) \geqq c$. It clearly will be sufficient to show that N is countable (for every c). In fact we shall show that N is isolated (in other words no $x \in N$ is a limit point of $N-x$), that is, we shall prove that for every $x \in N$ $\text{contg}_N\, x$ is empty. If this would not hold then N would contain an infinite sequence of points y_j coverging to x so that the direction of the line connecting x with y_j would converge to a fixed direction. Let Z_j be a point of $\phi(x)$ which is closest to y_j, and let A_j be the (unique) hyperplane through Z_j perpendicular to the segment xy_j. It is easy to see that as $j \to \infty$, A_j converges to a limiting hyperplane A. Moreover it is easily seen that the set $\phi(y_j)$ is ultimately contained in any preassigned neighborhood of A. Thus for large enough j, the volume $f(y_j)$ must be less than c, an evident contradiction; this completes our proof.

Next we prove our theorem in the general case. Let $k \leqq n$ and define M_k' to be the set of all points x for which the maximum number of linearly independent points in $\phi(x)$ is exactly k. It will clearly be sufficient to show that M_k' is contained in the sum of countably many surfaces of finite $(n+1-k)$-dimensional measure. Let $x \in M_k'$, and let $f(x)$ be the maximum volume of the k-dimensional simplices formed from the points z_i, $i \leqq k+1$, where $z_i \in \phi(x)$. $x \in M_k{}'^{(c)}=N'$ if $f(x) \geqq c$. Let $x \in N'$, and z_i, $i \leqq k+1$, be the points which determine a simplex of maximal volume. Then a simple geometrical argument (similar to the previous one) shows that $\text{contg}_{N'}\, x$ consists only of the directions through x which are perpendicular to the hyperplane determined by the z_i's, $i \leqq k+1$. Thus our theorem follows from the lemma.

Let E be a closed set, $x \notin E$. Denote by $g(x)$ the distance of x from E. It has been proved[4] that $g(x)$ has a derivative $-\cos \alpha$ in every direction (x, y), where α is the smallest angle formed by the direction (x, y) with the direction (x, z), z in $\phi(x)$. Clearly if $x \in E$ the derivative of $g(x)$ can be 0. We shall show that the derivative of $g(x)$ is 0 for almost all points of E.

[4] Mises, C. R. Acad. Sci. Paris vol. 205 (1937) pp. 1353–1355. See also Golab, ibid. vol. 206 (1938) pp. 406–408 and Bouligand, ibid. vol. 206 (1938) pp. 552–554.

Let $x \in E$. Denote by $S(x, \epsilon)$ the sphere of center x and radius ϵ. Denote by $G(x, \epsilon)$ the greatest distance of the points of $\overline{S}(x, \epsilon)$ from E. We are going to prove the following theorem.

THEOREM 2. *For almost all points of E (that is, for all points of E except a set of n-dimensional measure* 0)

$$\lim G(x, \epsilon)/\epsilon = 0.$$

It is well known that almost all points of E are points of Lebesgue density 1. Let x be such a point, and suppose that

$$\lim G(x, \epsilon)/\epsilon \neq 0.$$

This means that there exists an infinite sequence ϵ_i and points z_i, $z_i \in \overline{S}(x, \epsilon_i)$, $\epsilon_i \to 0$, such that the distance of z_i from E is greater than $c\epsilon_i$, where $c > 0$. But this clearly means that x can not have Lebesgue density 1. This contradiction establishes our theorem.

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