INTEGRAL DISTANCES

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In the present note we are going to prove the following result:

For any n we can find n points in the plane not all on a line such that their distances are all integral, but it is impossible to find infinitely many points with integral distances (not all on a line).[1]

PROOF. Consider the circle of diameter 1, $x^2+y^2=1/4$. Let $p_1, p_2, \cdots$ be the sequence of primes of the form $4k+1$. It is well known that

$$p_i^2 = a_i^2 + b_i^2, \qquad a_i \neq 0, \qquad b_i \neq 0,$$

is solvable. Consider the point (on the circle $x^2+y^2=1/4$) whose distance from $(-1/2, 0)$ is b_i/p_i. Denote this point by (x_i, y_i). Consider the sequence of points $(-1/2, 0)$, $(1/2, 0)$, (x_i, y_i), $i=1, 2, \cdots$. We shall show that any two distances are rational. Suppose this has been shown for all $i<j$. We then prove that the distance from (x_j, y_j) to (x_i, y_i) is rational. Consider the 4 concyclic points $(-1/2, 0)$, $(1/2, 0)$, (x_i, y_i), (x_j, y_j); 5 distances are clearly rational, and then by Ptolemy's theorem the distance from (x_i, y_i) to (x_j, y_j) is also rational. This completes the proof. Thus of course by enlarging the radius of the circle we can obtain n points with integral distances.

It is very likely that these points are dense in the circle $x^2+y^2=1/4$, but this we can not prove. It is easy to obtain a set which is dense on $x^2+y^2=1/4$ such that all the distances are rational. Consider the

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[1] Anning gave 24 points on a circle with integral distances. Amer. Math. Monthly vol. 22 (1915) p. 321. Recently several authors considered this question in the Mathematical Gazette.

point x_1 whose distance from $(-1/2, 0)$ is $3/5$; the distance from $(0, 1/2)$ is of course $4/5$. Denote $(-1/2, 0)$ by P_1, $(1/2, 0)$ by P_2, and let α be the angle $P_2P_1X_1$. α is known to be an irrational multiple of π. Let x_i be the point for which the angle $P_1P_2X_i$ equals $i\alpha$; the points X_i are known to be dense on the circle $X^2+y^2=1/2$, and all distances between x_i and x_j are rational because if $\sin\alpha$ and $\cos\alpha$ are rational, clearly $\sin i\alpha$ and $\cos i\alpha$ are also rational.

To give another configuration of n points with integral distances, let m^2 be an odd number with d divisors, and put

$$m^2 = x_i^2 - y_i^2.$$

This equation has clearly d solutions. Consider now the points

$$(m, 0), \qquad (0, y_i) \qquad\qquad i = 1, 2, \cdots .$$

It is immediate that all the distances are integral.

These configurations are all of very special nature. Several years ago Ulam asked whether it is possible to find a dense set in the plane such that all the distances are rational. We do not know the answer.

Now we prove that we cannot have infinitely many points $P_1, P_2, \cdots$ in the plane not all on a line with all the distances P_iP_j being integral.

First we show that no line L can contain infinitely many points $Q_1, Q_2, \cdots$. Let P be a point not on L, Q_i and Q_j two points very far away from P and very far from each other. Put $d(PQ_i)=a$, $d(Q_iQ_j)=b$, $d(PQ_j)=c$. ($d(A, B)$ denotes the distance from A to B.)

$$c \leqq a + b - 1. \tag{1}$$

Let Q_iR be perpendicular to PQ_j. We have

$$a < d(PR) + (d(Q_iR))^2/d(PR), \qquad b < d(Q_jR) + (d(Q_iR))^2/d(Q_jR).$$

Thus from (1)

$$(d(Q_iR))^2\left(\frac{1}{d(PR)} + \frac{1}{d(Q_jR)}\right) > 1$$

which is clearly false for a and b sufficiently large. ($d(Q_iR)$ is clearly less than the distance of P from L.) This completes the proof.

There clearly exists a direction P_1X such that in every angular neighborhood of P_1X there are infinitely many P_i.

Let P_2 be a point not on the line P_1X.

Denote the angle XP_1P_2 by α, $0<\alpha<\pi$. Evidently the P_i cannot form a bounded set. Let Q be one of the P_i sufficiently far away from

P_1, where the angle QP_1X equals ϵ (ϵ sufficiently small). Denote $d(P_1, P_2)=a$, $d(P_1, Q)=b$, $d(P_2, Q)=c$. We evidently have

$$c^2 = a^2 + b^2 - 2ab\cos(\alpha - \epsilon).$$

a, b, c all are integers. From this we shall show that if b and c are sufficiently large, ϵ sufficiently small, then

$$c = b - a\cos\alpha. \tag{2}$$

Put

$$c = b - a\cos\alpha + \delta, \qquad \delta > 0.$$

Then

$$(b - a\cos\alpha + \delta)^2 = b^2 - 2ab\cos\alpha + a^2\cos^2\alpha + 2\delta(b - a\cos\alpha) + \delta^2 > a^2 + b^2 - 2ab\cos(\alpha - \epsilon)$$

if b is sufficiently large and ϵ sufficiently small. Similarly we dispose of the case $\delta<0$. Thus (2) is proved.

From (2) we have

$$a^2 + b^2 - 2ab\cos(\alpha - \epsilon) = b^2 - 2ab\cos\alpha + a^2\cos^2\alpha$$

or

$$\cos(\alpha - \epsilon) - \cos\alpha = \frac{a^2\sin^2\alpha}{2b}.$$

Thus we clearly obtain

$$\epsilon < c_1/b.$$

Thus clearly all the points Q_i have distance less than c_2 from the line P_1X. Let Q_1, Q_2, Q_3 be three such points *not on a line*, where $d(Q_iQ_j)$ are large. Let Q_1Q_3 be the largest side of the triangle $Q_1Q_2Q_3$. Let Q_2R be perpendicular to Q_1Q_3. We have as before

$$d(Q_1Q_3) \leqq d(Q_1, Q_2) + d(Q_2Q_3) - 1;$$

also

$$d(Q_1Q_2) - d(Q_1R) < \epsilon, \qquad d(Q_2Q_3) - d(Q_3R) < \epsilon$$

an evident contradiction; this completes the proof.

By a similar argument we can show that we cannot have infinitely many points in n-dimensional space not all on a line, with all the distances being integral.

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