APPROXIMATION BY POLYNOMIALS

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1. Let $\{n_i\}$ be a set of distinct positive integers. According to a theorem of Müntz and Szász, the condition $\sum n_i^{-1} = \infty$ is necessary and sufficient in order that *polynomials in the powers* x^{n_i} *and 1 suffice to approximate uniformly an* arbitrary continuous function in the interval $0 \leq x \leq 1$, i.e., that these powers span the space C of continuous functions in that interval.

If the series converges, these powers, then, will span a certain closed linear proper sub-manifold M of the space C. It is clear from the Müntz-Szász result that if M' and M'' are two such manifolds, then their union $M' + M''$ cannot span the space C either, although the latter is quite possible in general for two proper closed linear sub-manifolds. Such a manifold M is then in a dimensional sense "small", and it is not unreasonable to conjecture that the functions comprising M are subject to quite restrictive conditions. We shall show, in fact, that every such function may be extended to be analytic in the interior of the unit circle. The power series for this function contains only powers x^{n_i}, and it may diverge at the point $x = 1$. It will converge at that point, however, if the sequence $\{n_i\}$ is lacunary $(n_{i+1}/n_i > c > 1)$.

Our results also enable us to extend the Müntz-Szász theorem to an interval excluding the origin.

2. Let S stand for an increasing sequence of distinct positive integers $\{n_i\}$ with $\sum_{i=1}^{\infty} n_i^{-1} < \infty$. Denote by $M(S)$ the set of polynomials with real coefficients containing powers x^{n_i}.

The following estimate is due to Müntz [2]:

THEOREM 1. *If* $m \,\epsilon'\, S$, $P(x) \,\epsilon\, M(S)$, *then*

$$\int_0^1 | x^m - P(x) |^2 \, dx \geq \frac{1}{2m+1} \prod_{i=1}^{\infty} \left(1 - \frac{2m+1}{n_i + m + 1}\right)^2 = A(m, S) > 0.$$

This estimate is the basis for our results. Our first step is to derive some properties of the function $A(m, S)$. From the theorem itself it is clear that for S fixed, $m \,\epsilon'\, S$, $\lim_{m \to \infty} A(m, S) = 0$; we shall show that A cannot converge to zero as fast as an exponential function of m. Our application, moreover, demands a certain uniformity with respect to the sequence S. Define, for $m \,\epsilon'\, S$,

$$\Phi(m, S) = \sum_{m < n_i} n_i^{-1}$$

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and

$$\Psi(m, S) = \frac{1}{m} \sum_{n_i \leq m} 1.$$

Our assumptions imply that both Φ and Ψ approach zero with m^{-1}.

We now prove

THEOREM 2. *For S and $\epsilon > 0$ fixed, there is an $m_0(\epsilon, S)$ such that $m > m_0(\epsilon, S)$, $m \,\epsilon'\, S$ imply*

$$A(m, S) > (1 + \epsilon)^{-m}. \tag{1}$$

Moreover, this m_0 may be chosen simultaneously for any aggregate of sequences S provided only that $\Phi(m, S)$ and $\Psi(m, S)$ tend to zero as $m \to \infty$ uniformly for S in this aggregate.

Proof. We write

$$A(m, S) = (2m + 1)^{-1} B^2(m, S) C^2(m, S),$$

where

$$B(m, S) = \prod_{3m+1<n_i} \left(1 - \frac{2m + 1}{n_i + m + 1}\right),$$

and $C(m, S)$ is the analogous product with $n_i \leq 3m + 1$. We shall show that the functions B and C possess the property described in our theorem, from which it is easily seen that our conclusion follows. Turning first to $B(m, S)$, we note that $n_i > 3m + 1$ implies

$$0 < \frac{2m + 1}{m + n_i + 1} < \tfrac{1}{2}$$

and that $0 < x < \frac{1}{2}$ implies $1 - x > e^{-2x}$; from these we infer that

$$\begin{aligned} B(m, S) &> \prod_{3m+1<n_i} \exp\left(-2\,\frac{2m + 1}{m + n_i + 1}\right) \\ &\geq \exp\left[-2(2m + 1) \sum_{3m+1<n_i} n_i^{-1}\right] \geq \exp\left[-6m\Phi(3m + 1, S)\right], \end{aligned}$$

so that we need only select m_0 such that $\exp\left[6\Phi(3m_0 + 1, S)\right] < 1 + \epsilon$ in order to have condition (1) satisfied by $B(m, S)$ whenever $m > m_0$.

To deal with $C(m, S)$ we first note that

$$\left|1 - \frac{2m + 1}{m + n_i + 1}\right| \geq \frac{1}{6m}$$

for $n_i \leq 3m + 1$. Moreover, all factors in $C(m, S)$ are distinct, and none vanish. There are $N(3m + 1, S)$ of these factors, where $N(m, S) = m\Psi(m, S)$. Thus,

$$| C(m, S) |^{\frac{1}{2}} \geq [N(3m + 1, S)]!\, (6m)^{-N(3m+1,S)},$$

and this last expression, if we recall that $e^k > k^k(k!)^{-1}$ for positive integral k, is readily seen to exceed

$$\frac{N(3m+1, S)^{N(3m+1,S)}}{(6m)^{N(3m+1,S)}e^{N(3m+1,S)}} = \left\{\left[\frac{6em}{N(3m+1, S)}\right]^{N(3m+1,S)/m}\right\}^{-m}.$$

The expression within the brace approaches unity from above as $m \to \infty$, and indeed uniformly so for any set of S's which satisfy our supplementary conditions; this concludes the proof of Theorem 2.

We remark that any aggregate of sequences S all of which are obtained from a fixed S by omitting integers will be an aggregate for which the hypotheses of Theorem 2 are satisfied.

We may now prove the first of our main results.

THEOREM 3. *Let $P_k(x) \in M(S)$ tend uniformly to $f(x)$ in $0 \leq x \leq 1$ as $k \to \infty$. Then $f(x)$ can be extended to be analytic throughout the interior of the unit circle. Its power series involves only powers x^{n_i}; and the coefficients of $P_k(x)$ tend to the coefficients in this power series.*

Proof. We first show that $\lim_{k\to\infty} a_i^{(k)}$ exists, where $a_i^{(k)}$ is the coefficient of x^{n_i} in $P_k(x)$.

Suppose, for a fixed i, that this is not true. Then, for some $\epsilon > 0$, there would be a sequence of pairs of integers $\{k_j, k_j'\}$, increasing without limit, such that

$$| a_i^{(k_j)} - a_i^{(k_j')} | > \epsilon \qquad (j = 1, 2, 3, \cdots).$$

Then,

$$P_{k_j}(x) - P_{k_j'}(x) = (a_i^{(k_j)} - a_i^{(k_j')})[x^{n_i} - Q(x)],$$

so that

$$\int_0^1 | P_{k_j}(x) - P_{k_j'}(x) |^2 \, dx \geq \epsilon^2 \int_0^1 | x^{n_i} - Q(x) |^2 \, dx \geq \epsilon^2 A(n_i, S_i),$$

where $S_i = S - n_i$. But our hypothesis clearly implies that the integral on the left tends to zero as $j \to \infty$, from which contradiction our first assertion follows. Let

$$\lim_{k\to\infty} a_i^{(k)} = A_i \qquad (i = 1, 2, 3, \cdots).$$

We next show that the $a_i^{(k)}$ may be estimated with respect to their growth as i increases, uniformly in k, whence we shall derive a like estimate for the A_i.

Suppose, indeed, that $\epsilon > 0$ is fixed. For any i, k we may write $P_k(x) = a_i^{(k)}[x^{n_i} + Q_{ki}(x)]$ (unless $a_i^{(k)} = 0$). $Q_{ki}(x)$ has no term in x^{n_i}, so $Q_{ki} \in M(S_i)$, $S_i = S - n_i$. Now all these S_i's, by our previous remark, form an aggregate which satisfies the requirements of Theorem **2**; hence there is a fixed $n_0(\epsilon)$ such that when $n_i > n_0(\epsilon)$ and for all k,

$$\int_0^1 | x^{n_i} + Q_{ki}(x) |^2 \, dx \geq (1 + \epsilon)^{-n_i}.$$

Now, from our hypothesis, $\{P_k(x)\}$ must be uniformly bounded: suppose $| P_k(x) | \leq T$ for all x and k. Then we may conclude

$$T^2 \geq [a_i^{(k)}]^2 \int_0^1 | x^{n_i} + Q_{ki}(x) |^2 \, dx \geq [a_i^{(k)}]^2 (1 + \epsilon)^{-n_i}$$

for all k, and $n_i > n_0(\epsilon)$, whence, under the same conditions,

$$| a_i^{(k)} | \leq T(1 + \epsilon)^{n_i}. \tag{2}$$

Thus, given $\epsilon > 0$, we may first select n_i large enough and then by letting $k \to \infty$ obtain the result

$$| A_i | \leq T(1 + \epsilon)^{n_i} \qquad (n_i > n_0(\epsilon)). \tag{3}$$

This clearly implies that $\overline{\lim}_{i \to \infty} | A_i |^{1/n_i} \leq 1$ so that the power series $\sum A_i x^{n_i}$ has a radius of convergence at least unity. We denote the sum of this series by $g(x)$; we now have only to identify $f(x)$ and $g(x)$ for $0 \leq x < 1$. Suppose then that x is fixed in this range: we show that

$$\lim_{k \to \infty} | P_k(x) - g(x) | = 0. \tag{4}$$

Indeed, suppose an $\eta > 0$ is given; we write

$$\begin{aligned} | P_k(x) - g(x) | &= | \sum_{i=1}^{\infty} A_i x^{n_i} - \sum_{i=1}^{\infty} a_i^{(k)} x^{n_i} | \\ &\leq \sum_{i=1}^{N} | A_i - a_i^{(k)} | x^{n_i} + \sum_{i=N+1}^{\infty} (| a_i^{(k)} | + | A_i |) x^{n_i}. \end{aligned} \tag{5}$$

We first select N large enough so that $| a_i^{(k)} |, | A_i | < T(2/(1 + x))^{n_i}$ whenever $i > N$, which is possible by (2) and (3), and then further increase N so that the second term in (5), which is now dominated by a geometric series remainder, is less than $\frac{1}{2}\eta$. This N being fixed, the first term may be made $< \frac{1}{2}\eta$ by choosing k sufficiently large. Hence, (4) is now demonstrated, and as $P_k(x) \to f(x)$, the proof is finished.

It is not true, in general, that the partial sums of the power series for $f(x)$ will serve as approximating polynomials for this function in the closed unit interval, since this power series may not converge at the point $x = 1$. Consider a sequence of positive integers

$$n_1 < n_1' < n_2 < n_2' < \cdots < n_k < n_k' < \cdots,$$

such that $1 < n_k'/n_k < 1 + 2^{-k}$. It may easily be verified that the k-th term of the series $\sum_{k=1}^{\infty} (x^{n_k'} - x^{n_k})$ has absolute value less than 2^{-k} for $0 \leq x \leq 1$, so that $f(x)$, the sum of this series, is a continuous function in that interval. Moreover, since our condition implies $\sum n_k^{-1}, \sum (n_k^1)^{-1} < \infty$, we infer by our previous result that $f(x)$ can be continued to an analytic function within the unit circle, whence it follows at once that the power series for $f(x)$ is $-x^{n_1} + x^{n_1'} - x^{n_2} + x^{n_2'} - \cdots$, which diverges at $x = 1$.

On the other hand, Hardy and Littlewood [1] have shown that if the power series $\sum a_k x^{n_k}$, with radius of convergence unity, is lacunary $(n_{k+1}/n_k > c > 1)$ and $\lim_{x \to 1} \sum a_k x^{n_k} = a$, then $\sum a_k = a$. From this it follows immediately that in the lacunary case the divergence at $x = 1$ shown in our last example cannot occur, and that the power series will converge uniformly in the closed unit interval.

3. By utilizing Theorem 2 we are able to extend the theorem of Müntz and Szász to any closed interval (a, b), where $a > 0$. In this case the presence of the constant power x^0 is irrelevant. We have chosen to omit a discussion of the corresponding extension to an interval containing the origin in its interior; this is of a more routine nature, involving only a natural distinction between even and odd powers.

THEOREM 4. *Let $S = \{n_i\}$ be an increasing sequence of distinct positive integers, and let I be the closed interval $[a, b]$, $0 < a < b$. Any real function continuous on I can be uniformly approximated on I by polynomials involving only powers x^{n_i} if and only if $\sum_{i=1}^{\infty} n_i^{-1}$ diverges.*

Proof. The sufficiency being clear from known results, we consider only the necessity. Without loss of generality, we assume $b = 1$.

Suppose, then, that $\sum_{i=1}^{\infty} n_i^{-1} < \infty$, and that corresponding to each $m \varepsilon' S$ there exists a polynomial $P_m(x)$ in the x^{n_i} such that $\| x^m - P_m \| < 2^{-m}$, where for any function f the notations $\| f \|$, $\| f \|'$, $\| f \|''$ denote respectively the maxima of f in $(a, 1)$, $(0, a)$, and $(0, 1)$.

We first show that $\| P_m \|''$ tends to infinity exponentially as m increases. Let S_l, for any positive integer l, be the sequence $\{n_i + l\}$. We remark that the supplementary conditions of uniformity laid down in Theorem 2 are satisfied by the set of sequences S_l $(l = 1, 2, 3, \cdots)$. Indeed, that the functions $\Psi(m, S_l)$ tend to zero with m^{-1}, uniformly in l, is obvious from their definition; and, from the relations

$$\Phi(m, S_l) = \sum_{m < n_i + l} (n_i + l)^{-1} = \sum_{m < n_i} (n_i + l)^{-1} + \sum_{m - l < n_i \leq m} (n_i + l)^{-1}$$
$$\leq \sum_{m < n_i} n_i^{-1} + m^{-1} \sum_{n_i \leq m} 1 = \Phi(m, S) + \Psi(m, S),$$

our remark follows.

We now select an integer m_0 subject to several conditions, the first of which is that $m \epsilon' S$ and $m > m_0$ will imply $\| x^{2m} - x^m P_m(x) \|'' > (2 - a)^{-m}$, which, by our remark above and Theorem 2, is certainly possible. Since $\| x^{2m} - x^m P_m \| < 2^{-m}$, this implies $\| x^{2m} - x^m P_m \|' > (2 - a)^{-m}$, whence $\| x^m - P_m \|' > [a(2 - a)]^{-m}$. Since $a(2 - a) < 1$, this shows that (with a possible further increase of m_0) when $m > m_0$ and $m \epsilon' S$, $A_m \equiv \| P_m \|'' > c^m$ $(c > 1)$.

We further restrict m_0 to satisfy the two following conditions:

$$\text{(i)} \qquad \sum_{m_0 < r} [a(2 - a)]^r < \tfrac{1}{2};$$

$$\text{(ii)} \qquad \| x^{n_k} - \sum_{j \neq k} a_j x^{n_j} \|'' > (2 - a)^{-n_k} \qquad \text{for } n_k > m_0 .$$

Here (ii) is possible once more by Theorem 2.

We now write $P_m(x) = \sum_j b_{mj} x^{n_j}$, and note that because of (ii), whenever $n_j > m_0$, and unless $b_{mj} = 0$,

$$\left\| \frac{P_m(x)}{b_{mj}} \right\|'' = \| x^{n_j} + \cdots \|'' > (2 - a)^{-n_j},$$

from which $| b_{mj} | < A_m (2 - a)^{n_j}$ when $n_j > m_0$.. Thus, for any m,

$$\| \sum_{m_0 < n_j} b_{mj} x^{n_j} \|' < A_m \sum_{m_0 < r} [a(2 - a)]^r < \tfrac{1}{2} A_m .$$

Increasing m_0 so that $m > m_0$ will imply $\| P_m \|'' = \| P_m \|'$, we then conclude from our last result that for $m > m_0$, $\sum_{n_j \leq m_0} | b_{mj} | > \frac{1}{2} A_m$. For each $m > m_0$ there must be, then, a corresponding $j(m)$ such that $n_{j(m)} \leq m_0$ and $| b_{mj(m)} | > A_m / 2(m_0 + 1)$. We further demand of m_0 that $m > m_0$ imply $| b_{mj(m)} | > 1$.

Now, for $m > m_0$, we write

$$b_{mj(m)}^{-1} | x^{2m} - x^m P_m(x) | = | x^{n_{j(m)} + m} + Q_m(x) |,$$

where Q_m is a polynomial in x^{2m} and $x^{n_i + m}$, $i \neq j(m)$. This expression is less than $\alpha^{n_{j(m)} + m}$, where $0 < \alpha < 1$, for the entire interval $0 \leq x \leq 1$. For the interval $(a, 1)$ this is at once clear from the definition of P_m. In $(0, a)$ we write

$$b_{mj(m)}^{-1} | x^{2m} - x^m P_m | \leq a^{2m} + 2(m_0 + 1) a^m,$$

which again tends to zero exponentially.

If we write

$$| x^{n_{j(m)} + m} + Q_m(x) | = | x^{n_{j(m)} + m} + b_{mj(m)}^{-1} x^{2m} + R_m(x) |,$$

and recall that $b_{mj(m)}$ tends to infinity as an exponential in m, it is clear that for an appropriate β, $0 < \beta < 1$, we shall have for $0 \leq x \leq 1$, $m > m_0$,

$$| x^{n_{j(m)} + m} + R_m(x) | < \beta^{n_{j(m)} + m},$$

where the polynomial R_m involves only powers x^{ν_i+m}, $i \neq j(m)$. Since this contradicts Theorem 2, our proof is now concluded.

BIBLIOGRAPHY

1. G. H. HARDY AND J. E. LITTLEWOOD, *A further note on the converse of Abel's theorem*, Proceedings of the London Mathematical Society, vol. 25(1926), pp. 219–236.
2. H. A. SCHWARZ, *Festschrift*, Mathematische Abhandlungen, Berlin, 1914, pp. 303–312.

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