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ON AN ELEMENTARY PROOF OF SOME ASYMPTOTIC FORMULAS IN THE THEORY OF PARTITIONS

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Denote by $p(n)$ the number of partitions of n. Hardy and Ramanujan[1] proved in their classical paper that

$$p(n) \sim \frac{1}{4n3^{\frac{1}{2}}} e^{cn^{\frac{1}{2}}}, \qquad c = \pi(\tfrac{2}{3})^{\frac{1}{2}},$$

using complex function theory. The main purpose of the present paper is to give an elementary proof of this formula. But we can only prove with our elementary method that

$$p(n) \sim \frac{a}{n} e^{cn^{\frac{1}{2}}} \tag{1}$$

and are unable to prove that $a = 1/4.3^{\frac{1}{2}}$.

Our method will be very similar to that used in a previous paper.[2] The starting point will be the following identity:

$$np(n) = \sum_{v=1} \sum_{k=1} vp(n - kv), \qquad p(0) = p(-m) = 0. \tag{2}$$

(We easily obtain (2) by adding up all the $p(n)$ partitions of n, and noting that v occurs in $p(n - v)$ partitions.) (2) is of course well known. In fact, Hardy and Ramanujan state in their paper[3] that by using (2) they have obtained an elementary proof of

$$\log p(n) \sim cn^{\frac{1}{2}}. \tag{3}$$

The proof of (3) is indeed easy. First we show that

$$p(n) < e^{cn^{\frac{1}{2}}}. \tag{4}$$

We use induction. (4) clearly holds for $n = 1$. By (2) and the induction hypothesis we have

$$np(n) < \sum_{\substack{v=1 \\ kv<n}} \sum_{k=1} ve^{c(n-kv)^{\frac{1}{2}}} < \sum_{v=1}^{\infty} \sum_{k=1}^{\infty} ve^{cn^{\frac{1}{2}} - ckv/2n^{\frac{1}{2}}} = e^{cn^{\frac{1}{2}}} \sum_{k=1}^{\infty} \frac{e^{-kc/2n^{\frac{1}{2}}}}{(1 - e^{kc/2n^{\frac{1}{2}}})^2}.$$

[1] Hardy, Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. 17, (1918), pp. 75–115.

[2] Erdös, *On some asymptotic formulas in the theory of factorisatio numerorum*, these Annals 42, (1941), pp. 989–993.

[3] Hardy, Ramanujan, ibid, p. 79.

Now it is easy to see that for all real x, $\dfrac{e^{-x}}{(1-e^{-x})^2} < \dfrac{1}{x^2}$. Thus

$$np(n) < e^{cn^{\frac{1}{2}}} \sum_{k=1}^{\infty} \frac{4n}{c^2 k^2} = ne^{cn^{\frac{1}{2}}},$$

which proves (4).

Similarly but with slightly longer calculations, we can prove that for every $\epsilon > 0$ there exists an $A > 0$ such that

$$p(n) > \frac{1}{A} e^{(c-\epsilon)n^{\frac{1}{2}}}. \tag{5}$$

(4) and (5) clearly imply (3).

To prove (1) we need the following

LEMMA 1:

$$\sum = \sum_{\substack{v=1 \\ kv<n}}^{\infty} \sum_{k=1}^{\infty} \frac{ve^{c(n-kv)^{\frac{1}{2}}}}{n-kv} = e^{cn^{\frac{1}{2}}}\left[1 + O\left(\frac{1}{n^{\frac{1}{2}+\epsilon}}\right)\right], \tag{6}$$

for some fixed $\epsilon > 0$.

PROOF. We omit as many details as possible, since the proof is quite straight forward and uninteresting. We evidently have by expanding $1/(n-kv)$ and omitting the terms with $kv > n^{\frac{1}{2}+\epsilon}$

$$\sum_{\substack{v=1 \\ kv<n}}^{n} \sum_{k=1}^{n} \frac{ve^{c(n-kv)^{\frac{1}{2}}}}{n-kv} = \frac{1}{n} \sum_{\substack{v=1 \\ kv<n}}^{\infty} \sum_{k=1}^{\infty} ve^{c(n-kv)^{\frac{1}{2}}} + \frac{1}{n^2} \sum_{\substack{v=1 \\ kv<n}}^{\infty} \sum_{k=1}^{\infty} kv^2 e^{c(n-kv)^{\frac{1}{2}}}$$

$$+ O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right) = \sum\nolimits_1 + \sum\nolimits_2 + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right).$$

Now

$$\sum\nolimits_2 = \frac{e^{cn^{\frac{1}{2}}}}{n^2} \sum_{v=1}^{\infty} \sum_{k=1}^{\infty} kv^2 e^{-ckv/2n^{\frac{1}{2}}} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right).$$

(It is easy to see that the other terms of $e^{c(n-kv)^{\frac{1}{2}}}$ can be neglected and that the summation for v and k can be extended to ∞.) Thus

$$\sum\nolimits_2 = \frac{e^{cn^{\frac{1}{2}}}}{n^2} \sum_{k=1}^{\infty} \frac{2k}{(1-e^{-kc/2n^{\frac{1}{2}}})^3} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right) = e^{cn^{\frac{1}{2}}} \sum_{k=1}^{\infty} \frac{2k \cdot 8n^{\frac{3}{2}}}{k^3 c^3}$$

$$+ O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right) = \frac{4}{c} \frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}}} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right).$$

On the other hand

$$\sum\nolimits_1 = \frac{e^{cn^{\frac{1}{2}}}}{n}\sum_{v=1}^{\infty}\sum_{k=1}^{\infty} ve^{-ckv/2n^{\frac{1}{2}}-ck^2v^2/8n^{\frac{3}{2}}} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right)$$

$$= \frac{e^{cn^{\frac{1}{2}}}}{n}\left(\sum_{v=1}^{\infty}\sum_{k=1}^{\infty} ve^{-ckv/2n^{\frac{1}{2}}} - \frac{ck^2v^3}{8n^{\frac{3}{2}}}e^{-ckv/2n^{\frac{1}{2}}}\right) = \sum\nolimits_1' - \sum\nolimits_1'' + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right).$$

$$\sum\nolimits_1'' = \frac{ce^{cn^{\frac{1}{2}}}}{8n^{\frac{5}{2}}}\sum_{v=1}^{\infty}\sum_{k=1}^{\infty} k^2v^3e^{-ckv/2n^{\frac{1}{2}}} = \frac{ce^{cn^{\frac{1}{2}}}}{8n^{\frac{5}{2}}}\sum_{k=1}^{\infty}\frac{6k^2}{(1-e^{-ck/2n^{\frac{1}{2}}})^4} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right)$$

$$= \frac{ce^{cn^{\frac{1}{2}}}}{8n^{\frac{5}{2}}}\sum_{k=1}^{\infty}\frac{6k^2\cdot 16n^2}{k^4c^4} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right) = \frac{3}{c}\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}}} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right).$$

$$\sum\nolimits_1' = \frac{e^{cn^{\frac{1}{2}}}}{n}\sum_{v=1}^{\infty}\sum_{k=1}^{\infty} ve^{-ckv/2n^{\frac{1}{2}}} = \frac{e^{cn^{\frac{1}{2}}}}{n}\sum_{k=1}^{\infty}\frac{e^{-ck/2n^{\frac{1}{2}}}}{(1-e^{-ck/2n^{\frac{1}{2}}})^2}.$$

A simple calculation shows that

$$\frac{e^{-x}}{(1-e^{-x})^2} = \frac{1}{x^2} + O(1), \quad \text{i.e.} \quad \frac{e^{-ck/2n^{\frac{1}{2}}}}{(1-e^{-ck/2n^{\frac{1}{2}}})^2} = \frac{4n}{c^2k^2} + O(1).$$

Hence

$$\sum\nolimits_1' = \frac{e^{cn^{\frac{1}{2}}}}{n}\sum_{k=1}^{u}\frac{4n}{c^2k^2} + \sum_{k>u}\frac{e^{-ck/2n^{\frac{1}{2}}}}{(1-e^{-ck/2n^{\frac{1}{2}}})^2} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right), \qquad u = [n^{\frac{3}{4}}].$$

But

$$\sum_{k=1}^{u}\frac{4n}{c^2k^2} = \frac{4n}{c^2}\frac{\pi^2}{6} - \frac{4n}{c^2}\sum_{k>u}\frac{1}{k^2} = n - \frac{4n}{c^2u} + O\left(\frac{n}{u^2}\right).$$

And

$$\sum_{k>u}\frac{e^{-ck/2n^{\frac{1}{2}}}}{(1-e^{-ck/2n^{\frac{1}{2}}})^2} = \int_u^{\infty}\frac{e^{-cx/2n^{\frac{1}{2}}}}{(1-e^{-cx/2n^{\frac{1}{2}}})^2}\,dx + O\left(\frac{1}{u^2}\right)$$

$$= \frac{2n^{\frac{1}{2}}}{c(1-e^{-cu/2n^{\frac{1}{2}}})} - \frac{n^{\frac{1}{2}}}{c} + O\left(\frac{1}{u^2}\right) = \frac{4n}{c^2u} - \frac{n^{\frac{1}{2}}}{c} + O\left(\frac{1}{n^{\frac{1}{2}+\epsilon}}\right).$$

Thus finally

$$\sum\nolimits_1' = e^{cn^{\frac{1}{2}}} - \frac{e^{cn^{\frac{1}{2}}}}{cn^{\frac{1}{2}}} + O\left(\frac{e^{cn^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}}\right).$$

Hence

$$\sum = \sum\nolimits_1' - \sum\nolimits_1'' + \sum\nolimits_2 = e^{cn^{\frac{1}{2}}}\left[1 + O\left(\frac{1}{n^{\frac{1}{2}+\epsilon}}\right)\right]$$

which proves the lemma.

Next we show that

$$0 < \liminf \frac{np(n)}{e^{cn^{\frac{1}{2}}}} \leqq \limsup \frac{np(n)}{e^{cn^{\frac{1}{2}}}} < \infty . \tag{7}$$

To prove (7) write

$$c_1^{(n)} = \max_{m \leqq n} \frac{mp(m)}{e^{cm^{\frac{1}{2}}}} . \tag{8}$$

Clearly by (8) and (6) and (2)

$$(n+1)p(n+1) \leqq c_1^{(n)} \sum_{\substack{v=1 \\ kv<n}} \sum_{k=1} \frac{ve^{c(n+1-kv)^{\frac{1}{2}}}}{n+1-kv} < c_1^{(n)} e^{c(n+1)^{\frac{1}{2}}} \left(1 + \frac{b_1}{n^{\frac{1}{2}+\epsilon}}\right).^{4}$$

Write

$$\frac{(n+j)p(n+j)}{e^{c(n+j)^{\frac{1}{2}}}} = c_1^{(n)} \left(1 + \frac{b_j}{n^{\frac{1}{2}+\epsilon}}\right), \qquad j = 1, 2, \cdots .$$

Then

$$(n+r+1)p(n+r+1) < c_1^{(n)} \sum_{\substack{v=1 \\ kv \leqq n+r}} \sum_{k=1} \frac{ve^{c(n+r+1-kv)^{\frac{1}{2}}}}{n+r+1-kv}$$

$$+ c_1^{(n)} \frac{\max_{j \leqq r} b_j}{n^{\frac{1}{2}+\epsilon}} \sum_{\substack{v=1 \\ kv \leqq r}} \sum_{k=1} \frac{ve^{c(n+r+1-kv)^{\frac{1}{2}}}}{n+r+1-kv}$$

$$< c_1^{(n)} e^{c(n+r+1)^{\frac{1}{2}}} \left(1 + \frac{b_1}{n^{\frac{1}{2}+\epsilon}} + \frac{\max_{j \leqq r} b_j}{n^{\frac{1}{2}+\epsilon}} \frac{r^2 e^{c(n+r+1)^{\frac{1}{2}}}}{n}\right),$$

since

$$\sum_{kv \leqq r} v \leqq r^2.$$

Hence

$$b_{r+1} < b_1 + \frac{r^2 \max_{j \leqq r} b_j}{n} .$$

We show that, for $r^2 \leqq n/2$, $b_{r+1} \leqq 2b_1$. We use induction. We have

$$b_{r+1} < b_1 + \frac{r^2 \cdot 2b_1}{n} \leqq 2b_1 .$$

[4] b_1 is chosen such that for every $m > 0$

$$\sum_{\substack{v \\ m-kv>0}} \sum_{k} \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m-kv} < e^{c(m^{\frac{1}{2}})} \left(1 + \frac{b_1}{m^{\frac{1}{2}+\epsilon}}\right).$$

Thus

$$c_1^{[n+(\frac{1}{2}n)^{\frac{1}{2}}]} \leqq c_1^{(n)}\left(1+\frac{2b_1}{n^{\frac{1}{2}+\epsilon}}\right).$$

Or

$$c_1^{((m+1)^2)} < c_1^{(m^2)}\left(1+\frac{10b_1}{n^{\frac{1}{2}+\epsilon}}\right);$$

and since $\sum m^{1/1+\epsilon}$ converges we see that $\limsup c_1^{(n)} < \infty$; i.e. $\limsup np(n)/e^{cn^{\frac{1}{2}}} < \infty$. Similarly we can show that $\liminf np(n)/e^{cn^{\frac{1}{2}}} > 0$, which completes the proof of (7).

Next we prove that

$$\liminf \frac{np(n)}{e^{cn^{\frac{1}{2}}}} = \limsup \frac{np(n)}{e^{cn^{\frac{1}{2}}}} \tag{9}$$

and this will complete the proof of (1).

Suppose that (9) does not hold; write

$$\liminf \frac{np(n)}{e^{cn^{\frac{1}{2}}}} = d, \qquad \limsup \frac{np(n)}{e^{cn^{\frac{1}{2}}}} = D. \tag{10}$$

Now choose n large and such that

$$\frac{np(n)}{e^{cn^{\frac{1}{2}}}} > D - \epsilon.$$

Then since $p(n)$ is an increasing function of n there exists a c_2 such that for every m in the range $n \leqq m \leqq n + c_2 n^{\frac{1}{2}}$

$$\frac{mp(m)}{e^{cm^{\frac{1}{2}}}} > \frac{d+D}{2}.$$

Now we claim that for every r_1 there exists a $\delta_{r_1} = \delta(r_1)$ such that, for $n \leqq m \leqq n + r_1 n^{\frac{1}{2}}$,

$$\frac{mp(m)}{e^{cm^{\frac{1}{2}}}} > d + \delta_{r_1}. \tag{11}$$

We prove (11) as follows: We evidently have by our lemma

$$mp(m) \geqq d \sum_{\substack{v=1 \\ kv<m}} \sum_{k=1} \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m-kv} + \frac{D-d}{2} \sum_{\substack{v=1 \\ n \leqq m-kv \leqq n+c_2 n^{\frac{1}{2}}}} \sum_{k=1} \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m-kv} - o(e^{cm^{\frac{1}{2}}}) \text{ [5]}$$

[5] The term $o(e^{cm^{\frac{1}{2}}})$ is present because d is the lower limit and not the lower bound of $\dfrac{mp(m)}{e^{cm^{\frac{1}{2}}}}$.

$$> de^{cm^{\frac{1}{2}}} + \frac{D-d}{2}\frac{e^{cn^{\frac{1}{2}}}}{m}\sum_{n\leqq m-v\leqq n+c_2 n^{\frac{1}{2}}} v - o(e^{cm^{\frac{1}{2}}}) > de^{cm^{\frac{1}{2}}} + c_3 e^{cn^{\frac{1}{2}}} - o(e^{cm^{\frac{1}{2}}})$$

$$> (d+\delta_{r_1})e^{cm^{\frac{1}{2}}}, \qquad \left(\text{i.e. } \frac{e^{cn^{\frac{1}{2}}}}{e^{cm^{\frac{1}{2}}}} > c_4\right).$$

which proves (11).

Suppose $2n \geqq m \geqq n + sn^{\frac{1}{2}}$, s sufficiently large; we show that

$$\sum_{\substack{v=1\\ m-kv<n}}\sum_{k=1}\frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m-kv} < \frac{e^{cm^{\frac{1}{2}}}}{s^{10}}. \tag{12}$$

Clearly

$$\sum_{\substack{v\\ 0<m-kv<n}}\sum_{k}\frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m-kv} \leqq \sum_{\substack{v\\ kv>sn^{\frac{1}{2}}}}\sum_{k}\frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m-kv}$$

$$< e^{cm^{\frac{1}{2}}}\sum_{\substack{v\\ \frac{1}{2}m>kv>sn^{\frac{1}{2}}}}\sum_{k}\frac{2ve^{-ckv/2m^{\frac{1}{2}}}}{m} + \sum_{\substack{v\\ m>kv\geqq\frac{1}{2}m}}\sum_{k}\frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m-kv}$$

$$< e^{cm^{\frac{1}{2}}}\sum_{\substack{v\\ \frac{1}{2}m>kv>sn^{\frac{1}{2}}}}\sum_{k}\frac{2ve^{-ckv/2m^{\frac{1}{2}}}}{m} + m^2 e^{c(\frac{1}{2}m)^{\frac{1}{2}}},$$

since

$$\sum_{\substack{v\\ kv<x}}\sum_{k} v \leqq x^2.$$

Further

$$\sum_{\substack{v\\ \frac{1}{2}m>kv>sn^{\frac{1}{2}}}}\sum_{k} ve^{-ckv/2m^{\frac{1}{2}}} < \sum_{u=1}^{m}\sum_{\substack{v\\ (u+1)sn^{\frac{1}{2}}\geqq kv>usn^{\frac{1}{2}}}}\sum_{k} ve^{-cusn^{\frac{1}{2}}/2m^{\frac{1}{2}}}$$

$$< \sum_{u=1}^{m}\sum_{\substack{v=1\\ kv\leqq(u+1)sn^{\frac{1}{2}}}}\sum_{k=1} ve^{-cus/4} < \sum_{u=1}^{m}(u+1)^2 s^2 ne^{-cus/4}.$$

Thus

$$\sum_{\substack{v\\ \frac{1}{2}m>kv>sn^{\frac{1}{2}}}}\sum_{k} ve^{-ckv/2m^{\frac{1}{2}}} < ms^2\sum_{u=1}^{\infty}(u+1)^2 e^{-cus/4} < \frac{m}{4s^{10}}$$

for sufficiently large s. Hence finally

$$\sum_{\substack{v=1\\ m-kv<n}}\sum_{k=1}\frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m-kv} < \frac{e^{cm^{\frac{1}{2}}}}{2s^{10}} + m^2 e^{c(\frac{1}{2}m)^{\frac{1}{2}}} < \frac{e^{cm^{\frac{1}{2}}}}{s^{10}}$$

for sufficiently large m and s (since $s < n^{\frac{1}{2}}$).

Consider now the intervals $n + tn^{\frac{1}{2}}, n + (t + 1)n^{\frac{1}{2}}, t > r_1, t + 1 < n^{\frac{1}{2}}$. Split it into t^2 equal parts. Write

$$\min \frac{mp(m)}{e^{cm^{\frac{1}{2}}}} = d + \delta_t^u, \qquad n \leqq m \leqq n + \left(t + \frac{u+1}{t^2}\right)n^{\frac{1}{2}}$$

and put $\delta_t^{t^2-1} = \delta_t$. Now let $n + (t + u/t^2)n^{\frac{1}{2}} \leqq m \leqq n + (t + (u + 1)/t^2)n^{\frac{1}{2}}$; then we have

$$mp(m) > d \sum_{\substack{v=1 \\ kv<m}} \sum_{k=1} \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv} + \delta_t^{(u-1)} \sum_v{}' \sum_k{}' \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv} - o(e^{cm^{\frac{1}{2}}}),$$

where the primes indicate that the summation is extended only over those v and k for which $n \leqq m - kv \leqq n + (t + u/t^2)n^{\frac{1}{2}}$. Further by Lemma 1

$$mp(m) \geqq (d + \delta_t^{(u-1)})e^{cm^{\frac{1}{2}}} - \delta_t^{(u-1)} \sum{}'' \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv}$$

$$- \delta_t^{(u-1)} \sum{}''' \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv} - o(e^{cm^{\frac{1}{2}}}),$$

where in $\sum''$ the summation is extended only over those v and k for which $m - kv \leqq n$, and in $\sum'''$ the summation is extended only over those v and k for which $m - kv \geqq n + (t + u/t^2)n^{\frac{1}{2}}$. We have by (11)

$$\sum{}'' < \frac{e^{cm^{\frac{1}{2}}}}{t^{10}}.$$

Further we have

$$\sum{}''' < \frac{n}{t^4} \frac{2e^{cm^{\frac{1}{2}}}}{m} < \frac{2e^{cm^{\frac{1}{2}}}}{t^4}.$$

Hence finally

$$mp(m) > e^{cm^{\frac{1}{2}}} \left(d + \delta_t^{(u-1)} - \frac{3\delta_t^{(u-1)}}{t^4}\right) - o(e^{cm^{\frac{1}{2}}}).$$

Hence

$$\delta_t^{(u)} > \delta_t^{(u-1)} \left(1 - \frac{3}{t^4}\right) - o(1).$$

Thus if t is fixed, independent of n, we have

$$\delta_{t+1} > \delta_t \left(1 - \frac{3}{t^4}\right)^{t^2} - o(1),$$

therefore

$$\delta_t > \delta_{r_1} \prod_{u > r_1} \left(1 - \frac{3}{u^4}\right)^{u^2} - o(1).$$

But $\prod_u (1 - 3/u^4)^{u^2}$ converges; thus, if r_1 was sufficiently large, we have $\delta_t > \delta_{r_1}/2$. Now choose r_2 sufficiently large; then we have $\delta_{r_2} > \delta_{r_1}/2$, i.e. for $n \leqq m \leqq n + r_2 n^{\frac{1}{2}}$,

$$\frac{mp(m)}{e^{cm^{\frac{1}{2}}}} > d + \frac{\delta_{r_1}}{2}.$$

Consider the interval $n + tn^{\frac{1}{2}}, n + (t+1)n^{\frac{1}{2}}, t > r_2$. Split it into t^2 equal parts. $\delta_t^{(u)}$ and δ_t have the same meaning as before. Suppose $n + (t + u/t^2)n^{\frac{1}{2}} \leqq m \leqq n + (t + (u+1)/t^2)n^{\frac{1}{2}}$; then evidently

$$mp(m) > (d + \delta_t^{(u-1)}) \sum_v{}' \sum_k{}' \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv},$$

where the primes indicate that the summation is extended only over those v and k for which $n \leqq m - kv \leqq n + n^{\frac{1}{2}}(t + u/t^2)$.
Now

$$\sum_v{}' \sum_k{}' \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv} = \sum_{v=1}\sum_{\substack{k=1\\ kv<m}} \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv} - \sum{}'' - \sum{}''',$$

where $\sum''$ and $\sum'''$ are defined as before. By (12) and the previous estimate of $\sum'''$ we have

$$\sum{}'' < \frac{e^{cm^{\frac{1}{2}}}}{t^{10}}, \qquad \sum{}''' < \frac{2e^{cm^{\frac{1}{2}}}}{t^4}.$$

Hence by Lemma 1

$$mp(m) > e^{cm^{\frac{1}{2}}}(d + \delta_t^{(u-1)})\left(1 - \frac{3}{t^4}\right) - \frac{b_1(d + \delta_t^{(u-1)})e^{cm^{\frac{1}{2}}}}{n^{\frac{1}{2}+\epsilon}};$$

i.e.

$$d + \delta_t^{(u)} > (d + \delta_t^{(u-1)})\left(1 - \frac{3}{t^4}\right) - \frac{b_1(d + \delta_t^{(u-1)})}{n^{\frac{1}{2}+\epsilon}},$$

and

$$d + \delta_{t+1} > (d + \delta_t)\left(1 - \frac{3}{t^4}\right)^{t^2} - \frac{b_1 t^2(d + \delta_t^{(u-1)})}{n^{\frac{1}{2}+\epsilon}},$$

or

$$d + \delta_s > \left(d + \frac{\delta_{r_1}}{2}\right)\prod_{t>r_2}\left(1 - \frac{3}{t^4}\right)^{t^2} - \frac{b_2 s^3}{n^{\frac{1}{2}+\epsilon}}.$$

For sufficiently large r_2 we have,

$$\left(d + \frac{\delta_{r_1}}{2}\right)\prod_{t>r_2}\left(1 - \frac{3}{t^4}\right)^{t^2} > d + \frac{\delta_{r_1}}{4},$$

and if $s \leqq (\log n)^2$ and n is sufficiently large,

$$\delta_s > \frac{\delta_{r_1}}{8};$$

that is, for $n \leqq m \leqq n + n^{\frac{1}{2}}(\log n)^2$

$$\frac{mp(m)}{e^{cm^{\frac{1}{2}}}} > d + \frac{\delta_{r_1}}{8}.$$

Now suppose $m > n + n^{\frac{1}{2}}(\log n)^2$; we shall show that

$$\sum = \sum_{\substack{v \quad k \\ 0 < m-kv < n}} \sum \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv} < \frac{e^{cm^{\frac{1}{2}}}}{m}.$$

We have

$$\sum < m^2 e^{cn^{\frac{1}{2}}} < m^2 e^{cm^{\frac{1}{2}} - 10c \log m} < \frac{e^{cm^{\frac{1}{2}}}}{m}$$

for sufficiently large n. Hence by Lemma 1,

$$\sum_{\substack{v \quad k \\ m-kv \geqq n}} \sum \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv} > e^{cm^{\frac{1}{2}}}\left(1 - \frac{b_1'}{n^{\frac{1}{2}+\epsilon}}\right).$$ [6]

Now we continue as in the proof of (7). Suppose $t > n + n^{\frac{1}{2}}(\log n)^2$; write

$$d + \delta_t = \min \frac{mp(m)}{e^{cm^{\frac{1}{2}}}}, \qquad n \leqq m \leqq t.$$

Then

$$(t + 1)p(t + 1) \geqq (d + \delta_t) \sum_{\substack{v \quad k \\ t-kv \geqq n}} \sum \frac{ve^{c(t-kv)^{\frac{1}{2}}}}{t - kv} > (d + \delta_t)e^{ct^{\frac{1}{2}}}\left(1 - \frac{b_1'}{t^{\frac{1}{2}+\epsilon}}\right).$$

Write

$$\frac{(t + r)p(t + r)}{e^{c(t+r)^{\frac{1}{2}}}} = (d + \delta_t)\left(1 - \frac{b_r'}{t^{\frac{1}{2}+\epsilon}}\right).$$

Then as in the proof of (7) we have

$$\begin{aligned}(t + j + 1)p(t + j + 1) &> (d + \delta_t) \sum_{\substack{v \quad k \\ t+j+1-kv \geqq n}} \sum \frac{ve^{c(t+j+1-kv)^{\frac{1}{2}}}}{t + j + 1 - kv} \\ &\qquad - (d + \delta_t) \frac{\max_{r \leqq j} b_r'}{t^{\frac{1}{2}+\epsilon}} \frac{j^2}{t} e^{c(t+j+1)^{\frac{1}{2}}} \\ &> (d + \delta_t)e^{c(t+j+1)^{\frac{1}{2}}}\left(1 - \frac{b_1'}{(t + j + 1)^{\frac{1}{2}}}\right) - (d + \delta_t) \frac{\max_{r \leqq j} b_r'}{t^{\frac{1}{2}+\epsilon}} \frac{j^2}{t} e^{c(t+j+1)^{\frac{1}{2}}} \\ &= (d + \delta_t)e^{c(t+j+1)^{\frac{1}{2}}}\left(1 - \frac{b_{j+1}'}{t^{\frac{1}{2}+\epsilon}}\right),\end{aligned}$$

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[6] As in footnote 4 b_1' is chosen such that for every $m > n + n^{\frac{1}{2}}(\log n)^2$

$$\sum_{\substack{v \quad k \\ n < m-kv}} \sum \frac{ve^{c(m-kv)^{\frac{1}{2}}}}{m - kv} > e^{cm^{\frac{1}{2}}}\left(1 - \frac{b_1'}{m^{\frac{1}{2}+\epsilon}}\right).$$

where

$$b'_{j+1} < b'_1 + \max_{r \leq j} b'_r \cdot \frac{j^2}{t}.$$

We show that for $j^2 < t/2$ we have, $b'_{j+1} < 2b'_1$. We use induction; we have

$$b'_{j+1} < b'_1 + \frac{2b'_1}{2} = 2b'_1.$$

Thus

$$d + \delta_{[t+\frac{1}{2}t^{\frac{1}{2}}]} > (d + \delta_t)\left(1 - \frac{2b'_1}{t^{\frac{1}{2}+\epsilon}}\right).$$

That is,

$$d + \delta_{(s+1)^2} > (d + \delta_{s^2})\left(1 - \frac{10b'_1}{s^{1+\epsilon}}\right).$$

Therefore

$$d + \delta_{u^2} > \left(d + \frac{\delta_{r_1}}{8}\right)\prod_{v > \log n}\left(1 - \frac{10b'_1}{v^{1+\epsilon}}\right) > d + \frac{\delta_{r_1}}{10},$$

which contradicts (10); and this completes the proof of (1).

As can be seen, the main idea of our proof is rather simple; unfortunately the details are long and cumbersome. By the same method we can prove the following result: Let m be a fixed integer. Denote by $p^{(m)}_{a_1, a_2, \cdots, a_r}(n)$ the number of partitions of n into integers congruent to one of the numbers $a_1, a_2, \cdots a_r$ (mod m). Then

$$(13) \qquad p^{(m)}_{a_1, a_2, \cdots, a_r}(n) \sim \frac{a}{n^\alpha} e^{cn^{\frac{1}{2}}}, \qquad ((a_1, a_2, \cdots, a^m_r) = 1)$$

where C depends on m and r, and α and a depend on $m, a_1, a_2, \cdots a_r$.

The same method will work if we consider partitions of n into rth powers. Denote the number of partitions of n into rth powers by $p_r(n)$, Hardy, Ramanujan and Wright[7] proved that

$$(14) \qquad p_r^{(n)} \sim c_1 n^{1/(r+1) - \frac{3}{2}} e^{c_2 n^{1/(r+1)}}.$$

Clearly as in the case of $p(n)$ we have

$$np_r(n) = \sum_{\substack{v \\ vk < n}} \sum_{k} v^r p_r(n - kv^r).$$

[7] Hardy, Ramanujan, ibid. p. 111. Maitland Wright, Acta Math. 63, (1934), pp. 143–191. Wright proves a very much sharper result than (13).

To prove (14) we should only have to prove the analogue of our lemma, namely

$$\sum_{\substack{v \quad k \\ v^r k < n}} \sum (n - v^r k)^{1/(r+1) - \frac{1}{2}} e^{c_2 (n - v^r k)^{1/(r+1)}} \tag{15}$$
$$= n^{1/(r+1) - \frac{1}{2}} e^{c_2 n^{1/(r+1)}} \left[1 + O\left(\frac{1}{n^{1 - (1/(r+1)) + \epsilon}} \right) \right].$$

If (15) is proved the proof of (14) proceeds as in the case of $p(n)$.

I have not worked out a proof of (15); it seems likely that a proof would be longer than that of Lemma 1, but would not present any particular difficulties.

Recently Ingham[8] proved a Tauberian theorem from which (1) and (14) follow as corollaries. In fact his Theorem 2 gives a more general result, from which (13) also follows as a very special case.

Denote by $P_r(n)$ the number of partitions of n into powers of r. Clearly

$$nP_r(n) = \sum_{\substack{v \quad k \\ r^v k < n}} \sum r^v P_r(n - r^v k).$$

It might be possible to get an asymptotic formula for $P_r(n)$ by our method. I have not succeeded so far. But we can show without difficulty that

$$\log P_r(n) \sim \frac{(\log n)^2}{2 \log r}. \tag{16}$$

We have

$$f(x) = \sum_{n=0}^{\infty} P_r(n) x^n = \prod_{v=1}^{\infty} \frac{1}{1 - x^{r^v}}.$$

It is easy to see that for $0 \leqq x \leqq 1$,

$$c_1 \left(\frac{1}{1-x} \right)^{(1/(2 \log a)) \log 1/(1-x)} < f(x) < c_2 \left(\frac{1}{1-x} \right)^{(1/(2 \log a)) \log 1/(1-x)}. \tag{17}$$

Thus

$$P_r(n) \left(1 - \frac{1}{n}\right)^n < f\left(1 - \frac{1}{n}\right) < c_2 n^{(\log n)/(2 \log a)};$$

that is

$$P_r(n) < c_3 n^{(\log n)/(2 \log a)}, \qquad \log P_r(n) < (1 - \epsilon) \frac{(\log n)^2}{2 \log a} \quad \text{for} \quad n > n_0.$$

Suppose now that for a certain large n $\log (P_r(n)) < (1 - \epsilon)(\log n)^2/2 \log a$; then, since for $m < n$ $P_r(m) \leqq P_r(n)$ we have

$$f(x) < e^{(1-\epsilon) \cdot (\log n)^2/(2 \log a)} \sum_{k=0}^{n} x^k + \sum_{k > n} c_3 k^{(\log k)/(2 \log a)} x^k, \tag{18}$$

[8] A. E. Ingham, *A Tauberian Theorem for Partitions*, these Annals, 42 (1941), p. 1083.

and a simple calculation shows that (18) contradicts (17). (Choose $x = (1 - \delta)n$, $\delta = \delta(\epsilon)$). The same method would of course give

$$\log (p(n)) \sim \pi \left(\frac{2n}{3}\right)^{\frac{1}{2}}.$$

We can also prove the following results:

I. Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers of density α, such that the a's have no common factor. Denote by $p'(n)$ the number of partitions of n into the a's. Then

$$\log (p'(n)) \sim c(\alpha n)^{\frac{1}{2}}. \qquad (c = \pi(\tfrac{2}{3})^{\frac{1}{2}} \tag{19}$$

II. Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers of density α, such that every sufficiently large m can be expressed as the sum of different a's. Then denote by $P'(n)$ the number of partitions of n into different a's. Then

$$\log P'(n) \sim c\left(\frac{\alpha}{2}\, n\right)^{\frac{1}{2}}. \tag{20}$$

We shall sketch the proof of II; the proof of I is similar but simpler. Denote by $P(n)$ the number of partitions of n into different summands: it is well known that[9]

$$\log P(n) \sim c\left(\frac{n}{2}\right)^{\frac{1}{2}}. \tag{21}$$

First we show that

$$\limsup \frac{\log P'(n)}{c\left(\frac{\alpha}{2}\, n\right)^{\frac{1}{2}}} \leqq 1. \tag{22}$$

To the partition $n = a_{i_1} + a_{i_2} + \cdots + a_{i_r}$ we make correspond the partition $i_1 + i_2 + \cdots + i_r$. For $i > i_0$ we have $i < a_i(\alpha + \epsilon)$ therefore $i_1 + i_2 + \cdots + i_k < n(\alpha + \epsilon) + i_0^2$. Thus each partition of n into the a's is mapped into a partition of integers $\leqq n(\alpha + 2\epsilon)$ with all integers as summands; hence from (20) we obtain (22). Next we prove that

$$\liminf \frac{\log P'(n)}{c\left(\frac{\alpha}{2}\, n\right)^{\frac{1}{2}}} \geqq 1. \tag{23}$$

Split the sequence a_i into two disjoint sequences $b_1, b_2, \cdots$ and $c_1, c_2, \cdots$ where the b's have density 0 and every sufficiently large integer is the sum of different b's and the c's are the remaining a's. It is easy to see that we can find the b's with the required property; also the density of the c's is clearly α. Denote by $Q(n)$ the number of partitions of n into the c's. Now associate

[9] A well known result of Euler states that the number of partitions of n into odd integers equals the number of partitions of n into different summands. Thus (20) follows from i.

with the partition $n = i_1 + i_2 + \cdots + i_k$, $i_1 < i_2 < \cdots < i_k$ the partition $c_{i_1} + c_{i_2} + \cdots + c_{i_k}$; as before, we have

$$\frac{n}{\alpha + \epsilon} < c_{i_1} + c_{i_2} + \cdots + c_{i_k} < \frac{n}{\alpha - \epsilon}.$$

Hence for at least one $n/(\alpha + \epsilon) < m < n/(\alpha - \epsilon)$, $Q(m) > p(n)(\alpha - \epsilon)/n$. Thus there exists a sequence of integers $x_1 < x_2 < \cdots$ with $\lim x_{i+1}/x_i = 1$ and

$$\text{24)} \qquad \liminf \frac{\log Q(x_i)}{c\left(\frac{\alpha}{2} x_i\right)^{\frac{1}{2}}} = 1.$$

Now suppose $x_j \leqq m < x_{j+1}$. Choose x_i such that $\epsilon m > m - x_i > C$. Then $m - x_i$ is a sum of different b's, hence $P(m) \geqq Q(x_i)$. Thus (23) follows from (24), and this completes the proof of II.

If might be worth while to mention the following problem: Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers, such that $\log P(n) \sim c(\alpha n)^{\frac{1}{2}}$. Does it then follow that the density of the a's is α. I cannot decide this problem. Perhaps the following result might be of some interest in this connection: Let $a_1 < a_2 \cdots$ be an infinite sequence of integers. $f(n)$ denotes the number of a's $\leqq n$, and $\varphi(n)$ denotes the number of solutions of $a_i + a_j \leqq n$. It can be shown trivially that if $\lim f(n)/n^{\alpha} = c_1$ then $\lim \varphi(n)/n^{2\alpha} = c_2$. But the converse is also true, and can be simply proved by using a Tauberian theorem of Hardy and Littlewood.[10] We have

$$(f(z))^2 = \left(\sum_{i=1}^{\infty} z^{a_i}\right)^2 = \sum_{k=1}^{\infty} d_k z^k$$

and, since $\sum d_k = \varphi(n) \sim c_2 n^{2\alpha}$, we evidently have

$$\underset{L \to 1}{f(z)} \sim \frac{c_3}{(1 - z)^{\alpha}}$$

and hence by the theorem of Hardy and Littlewood,

$$f(n) = \sum_{a_k \leqq n} 1 \sim c_1 n^{\alpha}.$$

By the same methods that were used in proving II, we can prove the following result: Denote by $R(n)$ the number of partitions of n into integers relatively prime to n. We have

$$\log R(n) \sim c(\varphi(n))^{\frac{1}{2}}.$$

Similarly, if we denote by $R'(n)$ the number of partitions of n into different integers relatively prime to n, we have

$$\log R'(n) \sim c\left(\frac{\varphi(n)}{2}\right)^{\frac{1}{2}}.$$

[10] Hardy-Littlewood, *Tauberian Theorems*, Proc. London Math. Soc. 13, (1914), pp. 174–191.

I have not succeeded in finding asymptotic formulas for $R(n)$ and $R'(n)$. This problem seems rather difficult.

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In the meantime I have proved the above conjecture. Consider

$$f(x) = \sum_{n=1}^{\infty} P(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^{a_k}}.$$

If we assume that $\log P(n) \sim a(n)^{\frac{1}{2}}$, we obtain by a simple calculation

$$\log_{x\to 1} f(x) \sim \frac{\pi^2}{6} \frac{\alpha}{1 - x}.$$

But

$$\log f(x) = \sum x^{a_i} + \tfrac{1}{2} \sum_i x^{2a_i} + \cdots = \sum_{k=1}^{\infty} b_k x^k.$$

Denote by $A(n)$ the number of a's not exceeding n. We have

$$B(n) = \sum_{k=1}^{n} b_k = \sum_{k=1}^{\infty} \frac{1}{k} A\left(\frac{n}{k}\right).$$

Thus

$$A(n) = \sum_{k=1}^{\infty} \frac{u(k)}{k} B\left(\frac{n}{k}\right).$$

But by the well known Tauberian theorem of Hardy-Littlewood,[11] we have

$$B(n) \sim \frac{\alpha\pi^2 n}{6}.$$

Hence

$$A(n) \sim \sum_{k=1}^{\infty} \frac{u(k)}{k^2} \frac{\alpha\pi^2 n}{6} \sim \alpha n. \quad \text{q.e.d.}$$

Similarly we can show that if $\log P'(n) = c[(\alpha/2)n]^{\frac{1}{2}}$, the density of the a's is α.

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[11] Hardy-Littlewood, ibid.