ON A PROBLEM OF SIDON IN ADDITIVE NUMBER THEORY, AND ON SOME RELATED PROBLEMS

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Let $a_1 < a_2 < \ldots$ be a sequence of positive integers, and suppose that the sums $a_i + a_j$ (where $i \leqslant j$) are all different. Such sequences, called B_2 sequences by Sidon†, occur in the theory of Fourier series. Suppose that n is given, and that $a_x \leqslant n < a_{x+1}$; the question was raised by Sidon how large x can be; that is, how many terms not exceeding n a B_2 sequence can have. Put $x = \phi(n)$, and denote by $\Phi(n)$ the maximum of $\phi(n)$ for given n. Sidon observed that $\Phi(n) > cn^{\frac{1}{4}}$, where c is a positive constant. In the present note we prove that

$$\Phi(n) > \left(\frac{1}{\sqrt{2}} - \epsilon\right) \sqrt{n}$$

for any positive ϵ and all $n > n_0(\epsilon)$. In the opposite direction, it is clear that $\Phi(n) < \sqrt{(2n)} + 1$ [for the numbers $a_i - a_j$, where $1 \leqslant j < i \leqslant x$, must all be different, whence $\frac{1}{2}x(x-1) \leqslant n-1$]. We prove that

$$\Phi(n) < (1+\epsilon)\sqrt{n}$$

for any positive ϵ and all $n > n_0(\epsilon)$. Thus

$$\frac{1}{\sqrt{2}} \leqslant \varliminf \frac{\Phi(n)}{\sqrt{n}} \leqslant \varlimsup \frac{\Phi(n)}{\sqrt{n}} \leqslant 1.$$

It is very likely that $\lim \Phi(n)/\sqrt{n}$ exists, but this we have not been able to prove.

We also prove the following result: let $f(n)$ denote the number of representations of n as $a_i + a_j$, where the a's are an arbitrary sequence of positive integers; then it is impossible that $f(n)$ should be constant for all $n \geqslant n_0$.

I. Let p be a prime, and let

$$a_k = 2pk + (k^2) \quad \text{for} \quad k = 1, 2, \ldots, (p-1),$$

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† S. Sidon, *Math. Annalen*, 106 (1932), 539.

where (k^2) denotes the unique integer u satisfying $k^2 \equiv u \pmod{p}$, $1 \leqslant u \leqslant p-1$. Clearly the a's are all less than $2p^2$. We show that

$$a_i + a_j \neq a_k + a_l \tag{1}$$

if the pairs (i, j) and (k, l) are different. If (1) does not hold, we clearly have

$$i+j = k+l, \quad i^2+j^2 \equiv k^2+l^2 \pmod{p}, \tag{2}$$

and hence $i-k = l-j$, $i^2-k^2 \equiv l^2-j^2 \pmod{p}$. Thus either $i-k = l-j = 0$, or $i+k \equiv l+j \pmod{p}$. In the latter case, it follows from (2) that $i \equiv l \pmod{p}$ and $k \equiv j \pmod{p}$, whence $i = l$ and $k = j$, and the pairs (i, j) and (k, l) are not different.

Since the a's satisfy (1), we have $\Phi(2p^2) \geqslant p-1$; and, since the quotient of consecutive primes tends to 1, it follows that

$$\varliminf \frac{\Phi(n)}{\sqrt{n}} \geqslant \frac{1}{\sqrt{2}}.$$

II. Let $a_1 < a_2 < \ldots < a_x \leqslant n$ be positive integers such that the sums $a_i + a_j$ $(i \leqslant j)$ are all different. Let m be a positive integer less than n, and consider the intervals

$$(-m+1, 1), \quad (-m+2, 2), \quad \ldots, \quad (n, n+m).$$

Let A_u denote the number of a's in the interval $-m+u \leqslant a_i < u$. Since each a_i occurs in exactly m intervals, we have

$$\sum_{u=1}^{m+n} A_u = mx.$$

The number of pairs a_i, a_j $(j > i)$ which lie in the above interval is

$$\tfrac{1}{2} A_u (A_u - 1).$$

The total number of these is

$$\sum_{u=1}^{m+n} \tfrac{1}{2} A_u (A_u - 1),$$

and, by an elementary inequality, this is greater than or equal to

$$\tfrac{1}{2}(m+n) \left(\frac{mx}{m+n}\right) \left(\frac{mx}{m+n} - 1\right).$$

For any such pair, $a_j - a_i$ is an integer r satisfying $1 \leqslant r \leqslant m-1$, and to each value of r there corresponds at most one such pair, since the numbers $a_j - a_i$ are all different. The pair which corresponds to r occurs in exactly $m-r$ of the intervals. Hence the total number of pairs is less than or equal to

$$\sum_{r=1}^{m-1}(m-r)=\tfrac{1}{2}m(m-1).$$

Comparing these results, we have

$$\tfrac{1}{2}mx(mx-m-n)\leqslant\tfrac{1}{2}m(m-1)(m+n),$$

whence

$$x(mx-2n)<m(m+n),$$

and

$$x<\frac{n}{m}+\left(n+m+\frac{n^2}{m^2}\right)^{\frac{1}{2}}.$$

Taking $m=[n^{\frac{3}{4}}]$, we obtain $x<n^{\frac{1}{2}}+O(n^{\frac{1}{4}})$. This proves the second result.

It is easy to see that, for every *infinite* B_2 sequence, $\underline{\lim}\,\phi(n)/\sqrt{n}=0$. On the other hand, it is not difficult to give an example of a B_2 sequence with $\overline{\lim}\,\phi(n)/\sqrt{n}>0$.

III. Let $a_1, a_2, \ldots$ be an arbitrary sequence of positive integers, and suppose that $f(n)=k$ for $n\geqslant n_0$, where $f(n)$ denotes the number of representations of n as a_i+a_j. Clearly $\phi(n)=o(n)$. For, if not, there would be arbitrarily large values of n for which the number of pairs a_i, a_j both less than n would be greater than cn^2, and so there would be a number $m<2n$ for which $f(m)>cn^2/2n$, which is contrary to hypothesis.

Therefore, by Fabry's gap theorem, the power series $\sum_{i=1}^{\infty} z^{a_i}$ has the unit circle as its natural boundary. But

$$\left(\sum_{i=1}^{\infty} z^{a_i}\right)^2=\sum_{n=1}^{\infty} f(n)\,z^n=\psi(z)+\frac{kz^{n_0}}{1-z}, \tag{4}$$

where $\psi(z)$ is a polynomial of degree not exceeding n_0-1. Clearly (4) gives a continuation of Σz^{a_i} over the whole plane as an algebraic function, which is an obvious contradiction. This proves the result.

It would be of interest to have an elementary proof of this result, but we have not succeeded in finding one. Perhaps the following conjectures on the behaviour of $f(n)$ may be of some interest.

(1) It is impossible that

$$\sum_{m=1}^{n} f(m)=cn+O(1),$$

where c is a constant. If, for example, $a_i=i^2$, the error term is known not to be even $O(n^{\frac{1}{4}})$.

(2) If $f(n) > 0$ for $n > n_0$, then $\overline{\lim} f(n) = \infty$. Here we may mention that the corresponding result for $g(n)$, the number of representations of n as $a_i a_j$, can be proved*.

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* The proof is similar to that used by P. Erdös in *Mitt. Tomsk Univ.*, 2 (1938), 74–82, but is considerably more complicated.

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