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ON INTERPOLATION. III. INTERPOLATORY THEORY OF POLYNOMIALS

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Dedicated to Professor L. Fejér on the occasion of his sixtieth birthday

This paper may be read without a knowledge of our first two papers on interpolation.

Let

$$\mathfrak{M} \equiv \left\{ \begin{matrix} x_1^{(1)} & & & \\ x_1^{(2)}, & x_2^{(2)} & & \\ \vdots & & & \\ x_1^{(n)}, & x_2^{(n)}, & \cdots, & x_n^{(n)} \\ \vdots & \vdots & & \ddots \end{matrix} \right\} \tag{1}$$

be a triangular matrix where for each n

$$1 \geqq x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} \geqq -1. \tag{2}$$

All the $x_\nu^{(n)}$ may be written in the form $\cos \vartheta_\nu^{(n)}$; hence for each $\mathfrak{M}$ we may define a triangular matrix

$$\mathfrak{M}' \equiv \left\{ \begin{matrix} \vartheta_1^{(1)} & \ddots & & \\ \vdots & & & \\ \vartheta_1^{(n)}, & \vartheta_2^{(n)}, & \cdots, & \vartheta_n^{(n)} \\ \vdots & \vdots & & \ddots \end{matrix} \right\} \tag{3}$$

with

$$0 \leqq \vartheta_1^{(n)} < \cdots < \vartheta_n^{(n)} \leqq \pi. \tag{4}$$

Let $f(x)$ be defined in $[-1, +1]$; then we define the n^{th} Lagrange interpolatory polynomial of $f(x)$ with respect to $\mathfrak{M}$ as the polynomial $L_n(f)$ of degree $(n-1)$ at most taking at the points $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$ the values $f(x_1^{(n)}), f(x_2^{(n)}), \cdots, f(x_n^{(n)})$. It may be verified that

$$L_n(f) \equiv \sum_{\nu=1}^{n} f(x_\nu^{(n)}) l_{\nu,n}(x) \equiv \sum_{\nu=1}^{n} f(x_\nu) l_\nu(x) \equiv \sum_{\nu=1}^{n} f(x_\nu^{(n)}) \frac{\omega_n(x)}{\omega_n'(x_\nu^{(n)})(x - x_\nu^{(n)})}, \tag{5a}$$

[1] Some of these results have been presented before the Mathematical Association in Budapest in April, 1937.

where

$$\omega_n(x) \equiv \prod_{\nu=1}^{n} (x - x_\nu^{(n)}) \equiv \prod_{\nu=1}^{n} (x - x_\nu). \tag{5b}$$

We shall explicitly indicate the upper and double indices only when some misunderstanding may arise. The polynomials $l_\nu(x)$ (for which we omitted to indicate explicitly their dependence upon n) are independent of $f(x)$ and dependent only upon $\mathfrak{M}$, ν, and n; following Fejér they are called the fundamental functions of interpolation. For these it is easy to verify that

$$\sum_{\nu=1}^{n} l_{\nu,n}(x) \equiv \sum_{\nu=1}^{n} l_\nu(x) \equiv 1, \qquad n = 1, 2, \cdots, \tag{6a}$$

and more generally

$$L_n(r) \equiv \sum_{\nu=1}^{n} r(x_\nu) l_\nu(x) \equiv r(x), \quad n = k+1, k+2, \cdots \tag{6b}$$

where $r(x)$ denotes any polynomial of degree k. The numbers $\int_{-1}^{+1} l_\nu(x)\,dx \equiv \lambda_\nu^{(n)} \equiv \lambda_\nu$ (depending only upon $\mathfrak{M}$, ν, and n) are called the Cotes numbers belonging to $\mathfrak{M}$. From (6a) we evidently have

$$\sum_{\nu=1}^{n} \lambda_\nu^{(n)} \equiv \sum_{\nu=1}^{n} \lambda_\nu = 2. \tag{7}$$

We intend to consider chiefly the case of two general and very often used matrices. The first of them is obtained as follows: let $p(x)$ be nonnegative and integrable in Lebesgue's sense (L-integrable) for $[-1, +1]$. Then a sequence of uniquely determined polynomials $\omega_0(x), \omega_1(x), \cdots$, corresponds to $p(x)$ so that $\omega_n(x)$ is a polynomial of degree n with

$$\text{coeff. } x^n \text{ in } \omega_n(x) = 1 \tag{8a}$$

and

$$\int_{-1}^{1} \omega_n(x)\omega_m(x)p(x)\,dx = 0, \qquad n \neq m. \tag{8b}$$

The sequence of such polynomials is called orthogonal with respect to the weight function $p(x)$. The sequence of polynomials $\Omega_0(x), \cdots, \Omega_n(x) \cdots$, for which

$$\int_{-1}^{1} \Omega_n(x)\Omega_m(x)p(x)\,dx = 0 \qquad n \neq m, \tag{8c}$$

$$\int_{-1}^{1} \Omega_n(x)^2 p(x)\,dx = 1 \qquad n = 0, 1, \cdots, \tag{8d}$$

$$\text{coeff. } x^n \text{ in } \Omega_n(x) \text{ is greater than } 0, \qquad n = 0, 1, \cdots, \tag{8e}$$

we call a sequence of normal-orthogonal polynomials with respect to $p(x)$. The polynomials $\omega_n(x)$ and $\Omega_n(x)$ evidently differ only in a constant factor dependent only upon n. By (8b) it is easy to see that all roots of $\omega_n(x)$ are real and situated in $[-1, +1]$. Taking these roots for $n = 1, 2, \cdots$ we obtain the so called p-matrix. It is well known that for $p(x) \equiv 1$ we obtain the sequence of Legendre-polynomials $P_n(x)$, for $p(x) = 1/\sqrt{(1-x^2)}$ the Tchebycheff-polynomials $T_n(x)$, and in general, for $p(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha > -1, \beta > -1$, the Jacobi-polynomials $P_n^{(\alpha,\beta)}(x)$.

The second class of matrices has been found by Fejér[2] in his paper about Lagrange-interpolation. According to his notation the matrix $\mathfrak{M}$ is *normal*, if

$$(9a) \qquad v_k(x) \equiv 1 - \frac{\omega_n''(x_k)}{\omega_n'(x_k)}(x - x_k) \geqq 0,$$

$$-1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots n, \qquad n = 1, 2, \cdots,$$

and it is *strongly normal*, if

$$(9b) \qquad 1 - \frac{\omega_n''(x_k)}{\omega_n'(x_k)}(x - x_k) \geqq c_1,$$

$$-1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots n, \qquad n = 1, 2, \cdots,$$

where c_1—and later all the other c's—are positive constants independent of x, n, k. Their dependence upon accidental parameters will always be explicitly stated. Fejér proved that e.g. the sequence of Jacobi-polynomials $P_n^{(\alpha,\beta)}(x)$ presents a normal matrix if $-1 < \alpha \leqq 0, -1 < \beta \leqq 0$ and a strongly normal one, if $-1 < \alpha < 0, -1 < \beta < 0$. For this second matrix class, by

$$(10) \qquad \sum_{k=1}^{n} v_k(x) l_k(x)^2 \equiv 1,$$

we have

$$(11) \qquad | l_k(x) | \leqq \frac{1}{\sqrt{c_1}}$$

$$-1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots n, \qquad n = 1, 2, \cdots$$

Orthogonal polynomials, and especially Jacobi-polynomials, play a most important part in many problems of analysis; we mention here only the works of Legendre, Laplace, Jacobi, Bruns Tchebycheff, A. Markoff, Stieltjes, Christoffel, Darboux, Fejér, S. Bernstein and Szegö. In the general theory of orthogonal polynomials (i.e. for general $p(x)$) an important step has been made by G. Szegö.[3] He succeeded in proving for a general class of weight-functions the asymptotic formulae of Laplace-Darboux concerning Jacobi-polynomials. Thus he proved

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[2] L. Fejér: *Lagrangesche interpolation und die zugehörigen konjugierten Punkte*, Math. Ann., 1932, pp. 1–55.

[3] G. Szegö: *Über die Entwicklung einer analytischen Function usw.*, Math. Ann., 1921, pp. 188–212.

that if $p(x)$ is such that to $p(\cos\vartheta) \mid \sin\vartheta \equiv p_1(\vartheta)$ there exists a function $D(z)$ regular in $|z| < 1$, here $\neq 0$ and for almost all ϑ

$$\lim_{r\to 1-0} |D(re^{i\vartheta})|^2 = p_1(\vartheta), \tag{12}$$

then in $|z| \geqq R > 1$ for $n \to \infty$ uniformly

$$\lim_{n\to\infty} \frac{\Omega_n\left[\left(z + \frac{1}{z}\right)\frac{1}{2}\right]}{z^n} = \frac{1}{(2\pi)^{\frac{1}{2}} D\left(\frac{1}{z}\right)},$$

which determines the asymptotic behavior of the polynomials for any point of the z plane not lying in $[-1, +1]$. (12) is satisfied, if $p(x) \geqq 0$, further if $p(x)$ and $\log p(x)$ are Lebesgue-integrable in $[-1, +1]$. Further—and this is a deeper result—Szegö[4] gave for the fundamental interval itself i.e. for $[-1, +1]$ an asymptotic formula

$$\Omega_n(\cos\vartheta_0) \sim \left(\frac{2}{\pi \sin\vartheta_0\, p(\cos\vartheta_0)}\right)^{\frac{1}{2}} \cos\left[\left(n + \frac{1}{2}\right)\vartheta_0 - \frac{\pi}{4} - \alpha\right], \tag{13}$$

where α depends in a given way upon $p(x)$. In order to give a simple example, he proved this for $\epsilon \leqq \vartheta_0 \leqq \pi - \epsilon$ and $n \to \infty$, if $p(x)$ remains in $[-1, +1]$ between two positive bounds and the first and second derivatives of $p(x)$ in the same interval exist. S. Bernstein[5] proved a theorem, which is analogous to the above mentioned theorem of Szegö. He proved the asymptotic formula (13) if, in $[-1, +1]$ $p(x)\sqrt{(1 - x^2)}$ remains between two positive bounds and uniformly satisfies here a logarithmic Lipschitz condition with the exponent $1 + \epsilon$. For this theorem, Szegö gives a very simple proof in his book to be published. The papers of J. Shohat[6] also contain general results of this kind.

The problems concerning orthogonal polynomials can be divided into four classes: a) the behavior of the polynomials within the interval $[-1, +1]$ ((*internal behavior*), b) the behavior of the polynomials upon the plane cut along $[-1, +1]$ (*external behavior*), c) distance of consecutive roots (problems of the *finer distribution* of roots), d) number of roots in a fixed subinterval (problems of the *mean-distribution* of roots). Problems concerning a) are completely solved by Szegö and Bernstein for a rather general class of weight-functions; if we require the weight function only to satisfy

$$p(x) \geqq c_2 \qquad -1 \leqq x \leqq +1, \tag{14a}$$

(14b) $p(x)$ is Lebesgue-integrable in $[-1, +1]$,

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[4] G. Szegö: *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonälitatseigenschaft definiert sind*, Math. Ann., 1922, pp. 114–139.

[5] S. Bernstein: *Sur les polynomes orthogonaux on a segment fini*, Journal de Mathematiques, pp. 127–177.

[6] See J. Shohat: *Théorie générale des polynomes orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, Fasc. LXVIII.

then Shohat[6] gives an upper estimate for the orthogonal polynomials belonging to $p(x)$. As far as we know there are no other general results in this direction. Question b) is settled by Szegö for rather general weights. As for c) we obtain from Szegö's formula that if the weight function throughout a subinterval has derivatives of the first and second order and remains throughout $[-1, +1]$ between two positive bounds (Szegö gives some other, more general condition), then for the n^{th} fundamental points $x_\nu^{(n)} = \cos \vartheta_\nu^{(n)}$ lying in that subinterval, we have $\lim_{n \to \infty} n(\vartheta_{\nu+1}^{(n)} - \vartheta_\nu^{(n)}) = \pi$; we obtain the same result for consecutive fundamental points from Bernstein's theorem, if throughout the interval $[-1, +1]$

$$c_3 \geqq p(x)\sqrt{(1 - x^2)} \geqq c_4 \tag{15a}$$

and if for $p(\cos \vartheta) \sin \vartheta = t(\vartheta)$ throughout and uniformly in $[0, \pi]$

$$|t(\vartheta + h) - t(\vartheta)| < \frac{c_5}{\log^{1+\epsilon} \frac{1}{h}} \tag{15b}$$

Concerning d) Szegö[7] implicitly proved, that, if the weights are non-negative in $[-1, +1]$, and if in the same interval $p(x)$ and $\dfrac{1}{\sqrt{(1 - x^2)}} \log p(x)$ are L-integrable, then the distribution of the n^{th} fundamental points is uniform, which means, that if $0 \leqq \alpha < \beta \leqq \pi$, then with $x_\nu^{(n)} = \cos \vartheta_\nu^{(n)}$

$$\lim_{n \to \infty} \frac{1}{n} \sum_\nu 1 = \frac{\beta - \alpha}{\pi}, \qquad \alpha \leqq \vartheta_\nu^{(n)} \leqq \beta. \tag{16}$$

Szegö's and Bernstein's methods are based upon asymptotic formulae for polynomials. But it is probable that in the general case such a formula does not exist not even for continuous weights remaining between two positive bounds. Thus, in this way we cannot obtain any answer to questions such as e.g. what is the effect of the singularities of the weight function (loci of discontinuity, infinities, zeros) upon the distribution of roots, whether this effect is only local etc. The investigation of this last question will be a main object of our paper. Here we make use of a principle introduced by Fejér: we derive the structure of the matrix from the properties of interpolatory fundamental functions belonging to $\mathfrak{M}$. Fejér deals with two such properties. The first[2] is the property of being strongly normal, from which he deduces the relation

$$\lim_{n \to \infty} \max_{\nu = 1, 2, \cdots (n-1)} (x_{\nu+1}^{(n)} - x_\nu^{(n)}) = 0, \tag{17}$$

which—from what precedes—means a statement about the distribution of roots of certain Jacobi-polynomials. The second property[7] is the non-negativeness

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[7] L. Fejér: *Mechanische Quadraturen mit positiven Cotesschen Zahlen*, Math. Zeitschr., 1933, pp. 287–310. His proof gives also the following result: if there exists for the matrix

of the Cotes-numbers belonging to $\mathfrak{M}$, from which we once more obtain (17). Theorems deducing properties of $\mathfrak{M}$ from some interpolatory properties we shall call Fejérian theorems. We proved[8] two such theorems, the application of which to p-matrices gave the following two theorems.

I. *If throughout* $[-1, +1]$ $c_6 \leqq p(x) \leqq c_7$ *and* $p(x)$ *is L-integrable, then*

$$\vartheta_{\nu+1}^{(n)} - \vartheta_\nu^{(n)} \leqq \frac{c_8}{n}$$

$$\nu = 0, 1, 2, \cdots (n-1), n, \qquad n = 1, 2, \cdots, \qquad \vartheta_0^{(n)} = 0, \qquad \vartheta_{n+1}^{(n)} = \pi$$

for the corresponding matrix $\mathfrak{M}'$. (See (3).)

II. *If throughout the interval* $[-1, +1]$ $p(x) \geqq 0$, $p(x)$ *and* $\dfrac{1}{p(x)}$ *L-integrable, then*

$$\vartheta_{\nu+1}^{(n)} - \vartheta_\nu^{(n)} \leqq \frac{c_9 \log (n+1)}{n}$$

$$\nu = 0, 1, \cdots n. \qquad n = 1, 2, \cdots.$$

By systematic application of Fejér's principle we obtain Fejérian theorems for each of the four classes mentioned above, theorems, which may be applied to p-matrices as well as to strongly normal ones. Properly speaking we deduce the theory of both classes of polynomials from that of a more general class of polynomials, the roots of which form a matrix $\mathfrak{M}$, and for which the values of the fundamental functions $l_\nu(x)$ satisfy certain conditions.

In §2 we consider problem a). That will be the only section in which we shall not explicitly express a Fejérian-theorem. Our theorem I asserts for strongly normal polynomials

$$|\omega_n(x)| \leqq \frac{8}{\sqrt{c_1}} \frac{\sqrt{n}}{2^n}, \qquad -1 \leqq x \leqq +1, \qquad n = 1, 2, \cdots$$

where c_1 is any constant for which (11) is valid. This result cannot be essentially improved in $[-1, +1]$, but it is probable, that in $[-1+\epsilon, 1-\epsilon]$ the factor with $\sqrt{n}$ can be omitted and the factor $8/\sqrt{c_1}$ replaced by a $c_{10} = c_{10}(c_1, \epsilon)$. Theorem II applies to the orthogonal polynomials and it states that,

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$\mathfrak{M}$ a function $s(x)$, non-negative and L-integrable on $[-1, +1]$, positive in $[a, b]$ and such that

$$\int_{-1}^{1} l_{\nu n}(x) s(x)\, dx \geqq 0, \qquad \nu = 1, 2, \cdots, n, \quad n = 1, 2, \cdots$$

then (17) holds in $[a, b]$. This is satisfied e.g. if the matrix $\mathfrak{M}$ is a p-matrix and $s(x) \equiv p(x)$; hence the roots of the polynomials orthogonal to a weight-function, non-negative and L-integrable on $[-1, +1]$ and positive throughout the subinterval $[a, b]$ cover the interval $[a, b]$ everywhere densely.

[8] P. Erdös and P. Turán: *On Interpolation II*, Annals of Math. 1938, pp. 703–724.

if the weight-function is non-negative and L-integrable in $[-1, +1]$ and if throughout the subinterval $[a, b]$ $p(x) \geqq m > 0$, then

$$(18\text{a})\qquad |\omega_n(x)| < \left[\frac{72}{(b-a)m}\int_{-1}^{1} p(t)\,dt\right]^{\frac{1}{2}} \frac{n}{2^n} \quad \text{in} \quad a \leqq x \leqq b,\ n = 1, 2, \cdots,$$

$$(18\text{b})\qquad |\omega_n(x)| < \left[\frac{12}{m[\epsilon(b-a)]^{\frac{1}{2}}}\int_{-1}^{1} p(t)\,dt\right]^{\frac{1}{2}} \cdot \frac{\sqrt{n}}{2^n} \quad \text{in} \quad a + \epsilon \leqq x \leqq b - \epsilon,\ n = 1, 2, \cdots.$$

For the case $a = -1, b = +1$ these estimations have been presented by Shohat. By the same method we obtain lower estimates for the orthogonal polynomials $\omega_n(x)$. More exactly: if the weight $p(x)$ is non-negative and L-integrable throughout $[-1, +1]$ and $p(x) \geqq m > 0$ throughout a subinterval $[a, b]$, then for any x (real or complex)

$$(19\text{a})\qquad |\omega_n(x)| \geqq \left[c_{11}\frac{m}{(b-a)\int_{-1}^{1} p(t)\,dt}\right]^{\frac{1}{2}} \left(\frac{b-a}{4}\right)^n |x - x_\nu^{(n)}|,$$

where $x_\nu^{(n)}$ denotes the n^{th} fundamental point nearest to x. As a matter of fact this has importance only for the interval $[-1, +1]$. If in addition to the above properties throughout a subinterval $[c, d]$ of $[a, b]$ the weight is bounded, $p(x) \leqq M$, then a factor with $\sqrt{n}$ can be appended to the right side, if we take $c_{11} = c_{11}(M)$. For $c = a = -1, d = b = +1$ we find implicitly and qualitatively the same as Shohat.[9] By this and by the results of §3 we obtain e.g. that if for the L-integrable weight function $p(x) \geqq m > 0$ throughout $[-1, +1]$ and if $p(x) \leqq M$ throughout the subinterval $[e, f]$, then $\omega_n(x)$ takes in any $[x_{\nu+1}^{(n)}, x_\nu^{(n)}]$ lying in $[e + \epsilon, f - \epsilon]$ a value, greater than

$$(19\text{b})\qquad \frac{c_{12}(\epsilon, M, m, e, f)}{2^n\sqrt{n}}$$

It is probable, that in (18a) and (18b) the factor n or $\sqrt{n}$ may be improved to $c_{13}(a, b, m)\sqrt{n}$ or to $c_{14}(\epsilon, a, b, m)$ respectively—this is true in the mean—and also in (19b) we may omit from the denominator the factor with $\sqrt{n}$. If in $[-1, +1]$ $p(x) \geqq m > 0$ and in the subinterval $[a, b]$ $p(x) \leqq M$, then we proved that there exists an $\eta(n)$ such that $\eta(n) \to 0$ for $n \to \infty$ and that in $[a + \epsilon, b - \epsilon]$

$$|\omega_n(x)| < c_{15}(a, b, \epsilon, m, M)\frac{\eta(n)\sqrt{n}}{2^n}.$$

We omit the details of the proof.

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[9] See footnote 6, p. 41, formula (60).

In §3 we are concerned with b) problems. The base of the investigation is the following Fejérian theorem: If for the matrix $\mathfrak{M}$ for every $\epsilon > 0$

$$[| l_k(x) |]^{1/n} \leqq 1 + \epsilon, \tag{20}$$
$$n > c_{16}(\epsilon), \qquad k = 1, 2, \cdots n, \qquad -1 \leqq x \leqq +1,$$

then for any fixed z of the complex plane cut along $[-1, +1]$ we have

$$\lim_{n\to\infty} [\omega_n(z)]^{1/n} = \frac{z + \sqrt{(z^2 - 1)}}{2}, \tag{21}$$

where we are to take those values of the n^{th} and square roots, which are positive on the positive real axis. Condition (20) is abundantly satisfied for sequences of strongly normal polynomials. Thus the asymptotic representation (21) applies for these too.

We shall see that if in $[-1, +1]$ the L-integrable $p(x)$ is $\geqq 0$ and its roots form an aggregate of measure 0, then (20) is satisfied hence (21) holds too. Formula (21) presents less than the above quoted formula of Szegö but it refers to a wider class of weight-functions: e.g. (21) holds for the weight-function $p(x) = e^{-1/x^2}$, whereas Szegö's formula has nothing to say in this case.

We shall give a direct and elementary proof of the aforesaid theorem, but we are bound to mention that it is to be deduced indirectly from a deep theorem of L. Kalmár[10] by the following note of Polya:[11] If upon the matrix $\mathfrak{M}$ we have uniformly in $[-1, +1]$

$$\lim_{n\to\infty} [| l_1(x) | + \cdots + | l_n(x) |]^{1/n} = 1,$$

then the Lagrange parabolas taken upon $\mathfrak{M}$ of a function $f(x)$ analytic in this interval uniformly converge to $f(x)$. In order to prove this note standard theorems about approximation of analytic functions are required.

On the other hand by a further theorem of Kalmár[10] it follows, that the ele-

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[10] L. Kalmár: *Az interpolátióról*, Mathematikai és Fizikai Lapok, 1927, pp. 120–149 (Hungarian). This gives the following result: Let $\mathfrak{B}$ be the closed interior of Jordan-curve l on the complex z-plane and let $x = \varphi(z)$ be regular on the exterior of l and continuous on the closed exterior of l, which maps $\mathfrak{B}$ upon the exterior of a circle $| x | \leqq c$ with $\lim_{|z|\to\infty} \frac{\varphi(z)}{z} = 1$. Let the matrix $\mathfrak{M}$ be given in $\mathfrak{B}$ and $\omega_n(z) = \prod_{\nu=1}^{n} (z - z_\nu^{(n)})$. Then a necessary and sufficient condition that $\lim_{n\to\infty} L_n(f) = f(z)$ uniformly in $\mathfrak{B}$ for any $f(z)$ regular in $\mathfrak{B}$ is that $\lim_{n\to\infty} [\omega_n(z)]^{1/n} = \varphi(z)$ for any z of the exterior of l. We use this only in the case if l is the interval $[-1, +1]$.

[11] G. Polya: *Über die Konvergenz von Quadraturverfahren*, Math. Zeitschrift, 1933, pp. 264–287.

ments of $\mathfrak{M}'$ belonging to $\mathfrak{M}$ are uniformly distributed in $[0, \pi]$.[12] From this we incidentally obtained the following result: if $p(x)$ is non-negative and L-integrable in $[-1, +1]$, and further, if the roots of this $p(x)$ form an aggregate of measure 0, then the elements of the matrix $\mathfrak{M}'$ belonging to the roots of the respective orthogonal polynomials are uniformly distributed. (We can prove this result in a direct and elementary way, too.)

In §4 we consider c) problems. The basis of the general consideration is given by the following Fejérian-theorem: If a matrix $\mathfrak{M}'$ is such, that for a subinterval $[\alpha, \beta]$ of $[0, \pi]$ with

$$\vartheta_{\nu-1}^{(n)} < \alpha \leqq \vartheta_\nu^{(n)} < \vartheta_{\nu+1}^{(n)} < \cdots < \vartheta_\mu^{(n)} \leqq \beta < \vartheta_{\mu+1}^{(n)}$$

we have

$$| l_k(x) | \leqq K, \qquad k = \nu, \nu + 1, \cdots \mu, \qquad a = \cos\beta \leqq x \leqq \cos\alpha = b,$$

and, in the same subinterval, the absolute value of the other n^{th} fundamental functions does not exceed $c_{16}n^{c_{17}}$, then

$$\frac{[\epsilon(b-a)]^{\frac{1}{2}}}{K}\cdot\frac{1}{n} \leqq \vartheta_{k+1}^{(n)} - \vartheta_k^{(n)} \leqq \frac{c_{18}(c_{16}, c_{17}, \epsilon, a, b)K}{n}, \tag{22}$$

if $\vartheta_k^{(n)}$ and $\vartheta_{k+1}^{(n)}$ are in $[\alpha + \epsilon, \beta - \epsilon]$. If $[\alpha, \beta] \equiv [0, \pi]$, then c_{18} is independent of ϵ and the estimate holds for all $[\vartheta_k^{(n)}, \vartheta_{k+1}^{(n)}]$ $(k = 1, 2, \cdots, n-1)$, the upper estimation holds even, as we proved[8] for $k = 0$ and $k = n$, if $\vartheta_0^{(n)} = 0$, $\vartheta_{n+1}^{(n)} = \pi$. The content of the theorem may briefly be expressed as follows: if the fundamental functions belonging to the fundamental points of a subinterval are bounded and the other fundamental functions are in the same subinterval not excessively great, then the distribution of the matrix is approximately uniform in that subinterval. In our paper cited under[8] we already proved, that the estimate of the form (22) holds in the case of strongly normal polynomials for any pair $[\vartheta_k^{(n)}, \vartheta_{k+1}^{(n)}]$ with an absolute constant c_{18}. For orthogonal polynomials we obtain that, if the L-integrable weight function is non-negative in $[-1, +1]$ and if $0 < m \leqq p(x) \leqq M$ in a subinterval $[\cos\beta, \cos\alpha]$, then for any pair $[\vartheta_k^{(n)}, \vartheta_{k+1}^{(n)}]$ in $[\alpha + \epsilon, \beta - \epsilon]$ we have

$$\frac{c_{19}(m, M, \alpha, \beta, \epsilon)}{n} \leqq \vartheta_{k+1}^{(n)} - \vartheta_k^{(n)} \leqq \frac{c_{20}(m, M, \alpha, \beta, \epsilon)}{n}.$$

For $[\alpha, \beta] \equiv [0, \pi]$ c_{19} and c_{20} are independent of ϵ; in our paper we proved the upper estimate for this case, we omit the details of the lower estimate.

If in the subinterval $[\cos\beta, \cos\alpha]$ $p(x)\sqrt{(1-x^2)} \geqq m > 0$ and besides it

[12] This means of course, that for any fixed subinterval $[\alpha, \beta]$ of $[0, \pi]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{\nu \\ \alpha \leqq \vartheta_\nu^{(n)} \leqq \beta}} 1 = \frac{\beta - \alpha}{\pi} \text{ holds.}$$

$p(x)\sqrt{(1-x^2)}$ is continuous, very much more is to be said from the n^{th} fundamental-points situated in $[\alpha+\epsilon, \beta-\epsilon]$. In this case we have for $n \to \infty$

$$\vartheta_{k+1}^{(n)} - \vartheta_k^{(n)} \sim \frac{\pi}{n}.$$

The proof is also based upon the analysis of interpolatory forms but not upon a Fejérian-theorem; it is important to notice that the formula obtained for the fundamental functions relative to $[\alpha+\epsilon, \beta-\epsilon]$ is to some extent an asymptotic one. The interval $[\alpha+\epsilon, \beta-\epsilon]$ may be replaced by $\left[\alpha + \frac{A(n)}{n}, \beta - \frac{A(n)}{n}\right]$, where $A(n)$, though arbitrarily slowly, tends to infinity, and we may postulate other, more general conditions for the weight function. In the case of strongly normal polynomials the former of us proved in another way, that for $\frac{A(n)}{n} \leqq \vartheta_k^{(n)} < \vartheta_{k+1}^{(n)} \leqq \pi - \frac{A(n)}{n}$ the difference $\vartheta_{k+1}^{(n)} - \vartheta_k^{(n)} \sim \frac{\pi}{n}$. We do not give the details of the proof.

In §5 we consider d) problems. The analysis is based upon two Fejérian-theorems. The first of them states, that the uniform distribution in the sense (16) of the matrix $\mathfrak{M}$ is a consequence of condition (20); we give for this a completely elementary direct proof. If for a matrix $\mathfrak{M}$ with the absolute constant K'

$$|\, l_\nu(x)\,| \leqq K', \qquad -1 \leqq x \leqq +1, \qquad \nu = 1, 2, \cdots n$$

then more exactly

$$(23) \quad -c_{22}(K',\epsilon)\{(\beta-\alpha)n\}^{\frac{1}{2}+\epsilon} < \sum_{\substack{\nu \\ \alpha \leqq \vartheta_\nu^{(n)} \leqq \beta}} 1 - \frac{\beta-\alpha}{\pi} n < \{(\beta-\alpha)n\}^{\frac{1}{2}+\epsilon} c_{22}(K',\epsilon)$$

for $(\beta-\alpha)n > c_{23}(K', \epsilon)$. This means, that for uniformly bounded fundamental functions the uniform distribution is already effected for very small subintervals $[\alpha, \beta]$, the size of which depend upon n. If $[\alpha, \beta]$ means any interval in $[0, \pi]$, then by the condition

$$(24) \quad |\, l_\nu(x)\,| \leqq c_{24} n^{c_{25}}, \quad -1 \leqq x \leqq +1, \quad \nu = 1, 2, \cdots n, \quad n = 1, 2, \cdots$$

we have

$$(25) \qquad \left| \sum_{\substack{\nu \\ \alpha \leqq \vartheta_\nu^{(n)} \leqq \beta}} 1 - \frac{\beta-\alpha}{\pi} n \right| < c_{26}(c_{25}, c_{24}, \epsilon) n^{\frac{1}{2}+\epsilon},$$

which establishes the uniform distribution already for intervals of the length $1/n^{\frac{1}{2}-2\epsilon}$. This is not very much weaker than the former conclusion.

By applying the above-arguments to sequences of strongly normal polynomials we immediately see that the fundamental points are distributed according to (23). Thus for orthogonal polynomials we obtained a new and strictly elementary proof of our theorem that, if the L-integrable $p(x)$ weight-function is in $[-1, +1]$ not less than 0 and the aggregate of the points x with $p(x) = 0$, is of measure 0, then the distribution of the elements of the matrix $\mathfrak{M}'$ formed of the roots of the respective orthogonal polynomials is uniformly dense in $[0, \pi]$. Here we must remark, that although our hypothesis is more general than that of Szegö, we obtained only the sufficient condition for the uniform distribution of the roots; the necessary and sufficient condition—as the first of us proved—is connected with the transfinite diameter of the aggregate of points, for which $p(x) = 0$. We omit the proof here.

Our second theorem states that, if the L-integrable weight function is $\geqq m > 0$ in $[-1, +1]$, then (25) holds for the corresponding matrix $\mathfrak{M}'$; if in addition, for $[-1, +1]$ $M \geqq p(x)\sqrt{(1 - x^2)} \geqq m$, then (23) holds too. If the aforesaid conditions are valid only for a subinterval and for the complementary subinterval of $[-1, +1]$ we postulate only the non-negativeness and the L-integrability, then nothing may be said with respect to the d) problems.

From the point of view of the theory of uniform distribution we make following remarks. Weyl's criterion for the uniform distribution of $\mathfrak{M}$ under (1) postulates, that for $n \to \infty$ the expressions $s_k \equiv \sum_{\nu=1}^{n} e^{2\pi i k \vartheta_\nu^{(n)}}$ tend to 0 for any positive integer k. Our theorems of §5 deduce the uniform distribution from the behavior of certain polynomials associated with $\mathfrak{M}$. It is to be noticed that instead of asymptotic equalities we have in the condition only inequalities and that we obtain also an error-term, that could not be obtained by Weyl's criterion. It would be plausible to ask, whether the uniform distribution with error-term is to be deduced from an inequality relative—in $[-1, +1]$—to $\omega_n(x)$ itself. The answer is affirmative; if in $[-1, +1]$ $|\omega_n(x)| \leqq \frac{A(n)}{2^n}$ with $A(n) \geqq 2$, then for a fixed subinterval $[\alpha, \beta]$ we have

$$\left| \sum_{\substack{\nu \\ \alpha \leqq \vartheta_\nu^{(n)} \leqq \beta}} 1 - \frac{\beta - \alpha}{\pi} n \right| \leqq 8[n \log A(n)]^{\frac{1}{2}}.$$

We will return to this problem on another occasion. If we disregard the error-term then, as we learned later the theorem is contained in a general theorem of Fekete[13] stating that the distribution of a matrix $\mathfrak{M}^+$ given upon any Jordan-curve l is uniform, if upon l the inequality $[|\omega_n(x)|]^{1/n} \leqq M$ holds, where M denotes the transfinite diameter of the Jordan-curve. Our argument essentially differs from his method.

From what is said before the reader may see the chief results of this paper: the uniformity of the method, the statement that the polynomials and their roots essentially depend only upon the local values of the weight function and asymptotic formulae of more general validity than before. We hope to consider the other fundamental problems in another paper.

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[13] Oral communication.

1.

THEOREM I. *For strongly normal matrices we have in* $[-1, +1]$

$$| \omega_n(x) | \leqq \frac{8}{\sqrt{c_1}} \cdot \frac{\sqrt{n}}{2^n}, \qquad n = 1, 2, \cdots.$$

PROOF. As the arithmetic mean is not less than the geometric mean we may write

$$\frac{1}{n}\frac{1}{c_1} \geqq \frac{1}{n}\sum_{\nu=1}^{n} l_\nu(x)^2 \geqq \left[\prod_{\nu=1}^{n} l_\nu(x)^2\right]^{1/n} = \frac{| \omega_n(x) |^{2-2/n}}{\left[\prod_{\nu=1}^{n} \omega_n'(x_\nu)^2\right]^{1/n}}.$$

As the x_ν's are in $[-1, +1]$, we have after Schur[14]

$$\prod_{\nu=1}^{n} | \omega_n'(x_\nu) | < \frac{c_{28} n^{n+\frac{1}{2}}}{2^{n^2-2n}},$$

i.e.

$$| \omega_n(x) |^{2-2/n} < \frac{1}{c_1 n} \cdot \frac{c_{29} n^2}{2^{2n}},$$

hence

$$| \omega_n(x) | < \frac{c_{30}\sqrt{n}}{2^n}, \qquad \text{Q.e.d.}$$

This proof is very simple, but Schur's theorem which we applied is not of interpolatory nature. Hence it will perhaps be of some interest to give another proof for it. We require

LEMMA I. *If* $1 \geqq x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} \geqq -1$, *then*

$$\sum_{\nu=1}^{n} \frac{1}{| \omega_n'(x_\nu) |} \geqq 2^{n-2}.$$

(Equality only for $\omega_n(x) = (x^2 - 1)U_{n-2}(x)$, where $U_k(\cos\vartheta) = \frac{1}{2^k}\frac{\sin (k+1)\vartheta}{\sin\vartheta}$, but for the present we shall not use this.)

PROOF. Let us fix in $[-1, +1]$ the values $\xi_1 > \xi_2 > \cdots > \xi_n$ and let us determine the polynomial $f(x)$ of degree $(n-1)$, for which coeff. $x^{n-1} = 1$ and $\max_{\nu=1,2,\cdots n} | f(\xi_\nu) |$ is minimum. According to standard theorems such $f(x)$ exists and takes at the places ξ_ν with alternating signs the same absolute values $(\nu = 1, 2, \cdots, n)$. Thus by coeff. $x^{n-1} = 1$ we have

$$f(x) = \frac{\sum_{\nu=1}^{n} \frac{(-1)^{\nu+1}}{\omega'(\xi_\nu)} \cdot \frac{\omega(x)}{x - \xi_\nu}}{\sum_{\nu=1}^{n} \frac{(-1)^{\nu+1}}{\omega'(\xi_\nu)}},$$

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[14] I. Schur: *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Zeitschrift, 1918, pp. 377–402.

where $\omega(x) = \prod_{\nu=1}^{n} (x - \xi_\nu)$. The minimum value is given by the formula

$$M_n = \frac{1}{\sum_{\nu=1}^{n} \frac{(-1)^{\nu+1}}{\omega'(\xi_\nu)}} = \frac{1}{\sum_{\nu=1}^{n} \frac{1}{|\omega'(\xi_\nu)|}},$$

i.e.

$$\frac{1}{\sum_{\nu=1}^{n} \frac{1}{|\omega'(\xi_\nu)|}} = \min_{f=x^{n-1}+\cdots} \max_{\nu=1,2,\cdots n} |f(\xi_\nu)| \leqq \min_{f=x^{n-1}+\cdots} \max_{-1\leqq x\leqq +1} |f(x)| \leqq \frac{1}{2^{n-2}},$$

since for $f(x) = T_{n-1}(x)(T_{n-1}(\cos\vartheta) = \frac{1}{2^{n-2}}\cos(n-1)\vartheta)$, in $[-1, +1]$ $\max |T_{n-1}(x)| = \frac{1}{2^{n-2}}$. By taking the reciprocals we obtain the Lemma.

By the Lemma we immediately obtain that

$$|\omega_n(x)|\, 2^{n-2} \leqq |\omega_n(x)| \sum_{\nu=1}^{n} \frac{1}{|\omega_n'(\xi_\nu)|} \leqq 2 \sum_{\nu=1}^{n} \left| \frac{\omega_n(x)}{\omega_n'(\xi_\nu)(x-\xi_\nu)} \right|$$
$$= 2 \sum_{\nu=1}^{n} |l_\nu(x)| \leqq 2\sqrt{n}\left[\sum_{\nu=1}^{n} l_\nu(x)^2\right]^{\frac{1}{2}} \leqq \frac{2}{\sqrt{c_1}}\sqrt{n},$$

which establishes the theorem. Notice that in both proofs we used only the fact that $\sum_{\nu=1}^{n} l_\nu(x)^2 < c_1$.

This result is not to be improved essentially in $[-1, +1]$, that is to be seen by the matrix given by the roots of the Jacobi-polynomial $P_n^{(-\epsilon,-\epsilon)}(x)$ for ϵ being any small fixed positive numbers. Its being strongly normal we already mentioned in the introduction. On the other hand by

$$P_n^{(\alpha,\beta)}(1) = \frac{2^n\binom{n+\alpha}{n}}{\binom{2n+\alpha+\beta}{n}}, \qquad \alpha > -1, \beta > -1$$

we have

$$P_n^{(\epsilon,-\epsilon)}(1) \sim c_{31}(\epsilon)\, \frac{n^{\frac{1}{2}-\epsilon}}{2^n}.$$

THEOREM II. *If the L-integrable weight-function $p(x)$ is non-negative in $[-1, +1]$, and for the subinterval $[a, b] \geqq m(> 0)$, then in $[a, b]$*

$$|\omega_n(x)| \leqq \left[\frac{8}{(b-a)m}\int_{-1}^{1} p(t)\,dt\right]^{\frac{1}{2}} \cdot \frac{2n+1}{2^n},$$

whereas in $[a+\epsilon, b-\epsilon]$

$$|\omega_n(x)| \leqq 2\left[\frac{1}{m[\epsilon(b-a-\epsilon)]^{\frac{1}{2}}}\int_{-1}^{1} p(t)\,dt\right]^{\frac{1}{2}} \cdot \frac{\sqrt{(2n+1)}}{2^n}.$$

PROOF. As is known—and it may easily be verified—$\omega_n(x)$ minimizes the integral $J(f) \equiv \int_{-1}^{1} f(t)^2 p(t)\,dt$, if $f(t)$ runs over the polynomials of degree n with coeff. $x^n = 1$. Thus for $a \leqq x \leqq b$ we have

$$m \int_a^b \omega_n(t)^2\,dt \leqq \int_a^b \omega_n(t)^2 p(t)\,dt \leqq \int_{-1}^{1} \omega_n(t)^2 p(t)\,dt \leqq \int_{-1}^{1} T_n(t)^2 p(t)\,dt,$$

where $T_n(\cos\vartheta) = \frac{1}{2^{n-1}} \cos n\vartheta$. Hence

$$m \int_a^b \omega_n(t)^2\,dt \leqq \frac{4}{2^{2n}} \int_{-1}^{1} p(t)\,dt. \tag{26}$$

But then, according to a theorem of A. Markoff (stating that if for $a \leqq x \leqq b$ $|F(x)| \leqq M$, then here $|F'(x)| \leqq \frac{2M}{b-a} n^2$, where n denotes the degree of $F(x)$) for $a \leqq x \leqq b$ we have

$$|\omega_n(x)|^2 \leqq \frac{8}{2^{2n} m} \int_{-1}^{1} p(t)\,dt \, \frac{(2n+1)^2}{b-a}. \qquad \text{Q.e.d.}$$

By applying to (26) the theorem of Bernstein-Fejér (stating that if for $a \leqq x \leqq b$ $|F(x)| \leqq M$, then $|F'(x)| \leqq \frac{Mn}{[(b-x)(x-a)]^{\frac{1}{2}}}$, where n denotes the degree of $F(x)$) we obtain for $a + \epsilon \leqq x \leqq b - \epsilon$

$$\omega_n(x)^2 \leqq \frac{4}{m} \int_{-1}^{1} p(t)\,dt \, \frac{1}{[\epsilon(b-a-\epsilon)]^{\frac{1}{2}}} \cdot \frac{2n+1}{2^{2n}}. \qquad \text{Q.e.d.}$$

In connection with theorem II we mentioned that it is probable that the factor $\sqrt{n}$ in (18b) is to be improved to $c_{14}(\epsilon, a, b, m)$. This conjecture may to some extent be supported by the fact that from (26)

$$\left[\int_a^b \omega(t)^2\,dt\right]^{\frac{1}{2}} < \frac{2}{2^n}\left[\frac{1}{m}\int_{-1}^{1} p(t)\,dt\right]^{\frac{1}{2}}, \tag{27}$$

i.e. for $[a, b]$ the mean value of $|\omega_n(t)|$ is $O\left(\frac{1}{2^n}\right)$.

The proof of theorem II is very simple, but it is not of interpolatory character; thus we give a proof of such kind which with a slight modification gives the lower estimate indicated in the introduction, and besides it contains many elements needed in the following investigations.

Let the numbers

$$k_\nu^{(n)} = \int_{-1}^{1} l_{\nu,n}(t) p(t)\,dt \equiv k_\nu, \qquad \nu = 1, 2, \cdots, n, \qquad n = 1, 2, \cdots, \tag{28}$$

denote the Christoffel-numbers belonging to $p(x)$, then we have

LEMMA II. *In $[-1, +1]$ suppose $p_1(x) \geqq p_2(x) \geqq 0$, both L-integrable. If $l_\nu(x)$ $(\nu = 1, 2, \cdots, n; n = 1, 2, \cdots)$ stand for the fundamental functions and k_ν for the Christoffel-numbers belonging to $p_1(x)$, $l_\nu^+(x)$ and k_ν^+ for those belonging to $p_2(x)$ respectively, then for any fixed (real or complex) x_0*

$$\sum_{\nu=1}^{n} \frac{|l_\nu(x_0)|^2}{k_\nu} \leqq \sum_{\mu=1}^{n} \frac{|l_\nu^+(x_0)|^2}{k_\nu^+}, \qquad n = 1, 2, \cdots.$$

PROOF. Let x_0 denote any fixed number and determine the polynomial $F(x)$ of degree $(n-1)$ at the utmost, for which $F(x_0) = 1$ and $I(F) \equiv \int_{-1}^{1} |F(t)|^2 p_1(t)\, dt$ is minimum.

We express $F(x)$ by the interpolatory polynomials belonging to the roots of n^{th} polynomial orthogonal to $p_1(x)$, then we have

$$F(x) = \sum_{\nu=1}^{n} d_\nu l_\nu(x),$$

i.e.

$$I(F) = \sum_{\nu=1}^{n}\sum_{\mu=1}^{n} d_\mu \bar{d}_\nu \int_{-1}^{1} l_\mu(x) l_\nu(x) p_1(x)\, dx = \sum_{\nu=1}^{n} |d_\nu|^2 k_\nu,$$

as[15] for $\mu \neq \nu$

$$\text{(29a)} \quad \int_{-1}^{1} l_\mu(t) l_\nu(t) p_1(t)\, dt = \frac{1}{\omega_n'(x_\mu)\omega_n'(x_\nu)} \int_{-1}^{1} \frac{\omega_n(x)}{(x-x_\mu)(x-x_\nu)}\, \omega_n(x) p_1(x)\, dx = 0,$$

and

$$\text{(29b)} \qquad \int_{-1}^{1} l_\mu(x)^2 p_1(x)\, dx = \int_{-1}^{1} l_\mu(x) p_1(x)\, dx = k_\mu.$$

As

$$F(x_0) = 1 = \sum_{\nu=1}^{n} d_\nu l_\nu(x_0),$$

we obtain from what precedes

$$1 = \left| \sum_{\nu=1}^{n} d_\nu l_\nu(x_0) \right|^2 = \left| \sum_{\nu=1}^{n} d_\nu \sqrt{k_\nu}\, \frac{l_\nu(x_0)}{\sqrt{k_\nu}} \right|^2 \leqq \left(\sum_{\nu=1}^{n} \frac{|l_\nu(x_0)|^2}{k_\nu} \right)\left(\sum_{\nu=1}^{n} k_\nu |d_\nu|^2 \right),$$

i.e.

$$I(F) \geqq \frac{1}{\sum_{\nu=1}^{n} \frac{|l_\nu(x_0)|^2}{k_\nu}}.$$ [16]

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[15] P. Erdös and P. Turán: *On Interpolation. I.* Annals of Math., 1937, pp. 142–155.

[16] Implicitly J. Shohat: *Théorie générale etc.*, p. 47, formula (75).

Equality is evidently to be obtained if and only if

$$F(x) = \frac{1}{\sum_{\nu=1}^{n} \frac{|l_\nu(x_0)|^2}{k_\nu}} \cdot \sum_{\nu=1}^{n} \frac{\overline{l_\nu(x_0)}}{k_\nu} l_\nu(x),$$

i.e.

$$\sum_{\nu=1}^{n} \frac{|l_\nu(x_0)|^2}{k_\nu} = \left[\min_{F(x_0)=1,\, F(x)=a_0+\cdots+a_{n-1}x^{n-1}} \int_{-1}^{1} |F(t)|^2 p_1(t)\, dt \right]^{-1}. \tag{30}$$

But then evidently

$$\begin{aligned} \sum_{\nu=1}^{n} \frac{|l_\nu(x_0)|^2}{k_\nu} &\leqq \left[\min_{F(x_0)=1,\, F(x)=a_0+\cdots+a_{n-1}x^{n-1}} \int_{-1}^{1} |F(t)|^2 p_2(t)\, dt \right]^{-1} \\ &= \sum_{\nu=1}^{n} \frac{|l_\nu^+(x_0)|^2}{k_\nu^+} \end{aligned} \tag{31}$$

Q.e.d.

In the special case of x_0 being the ν^{th} root of the n^{th} polynomial orthogonal to $p_1(x)$, then by (30) the minimum-value is k_ν and this minimum is attained only for $F(x) = l_\nu(x)$ (Corollary I).[17]

Here we remark—although we make no use of it in this paper—that the sum $\sum_{\nu=1}^{n} \frac{l_\nu^{(r)}(x)^2}{k_\nu}$ is also monotone with respect to $p_1(x)$, if $l_\nu^{(r)}(x)$ denotes the r^{th} derivative.

In Lemma II let $p_1(x) \equiv p(x)$ and $p_2(x) = m(>0)$ if $a \leqq x \leqq b$, and $p_2(x) = 0$ for the complementary intervals; furthermore suppose x_0 real. Then the explicit form of the polynomials orthogonal in $[-1, +1]$ with respect to $p_2(x)$ is given by

$$\omega_n(x) = A \cdot P_n\left(-1 + 2\frac{x-a}{b-a}\right),$$

where $P_n(x)$ denotes the n^{th} Legendre-polynomial for $[-1, +1]$, with the normalization $P_n(1) = 1$, A depending only upon n, a, b so that coeff. x^n in $\omega_n(x)$ equals 1. As in this case with $P_n(\eta_\nu) = 0$ $(\nu = 1, 2, \cdots n)$,

$$k_\nu^+ = m \int_a^b l_\nu^+(t)\, dt = \frac{m(b-a)}{2} \int_{-1}^{1} \frac{P_n(t)}{P_n'(\eta_\nu)(t-\eta_\nu)}\, dt = \frac{m(b-a)}{(1-\eta_\nu^2)P_n'(\eta_\nu)^2},$$ [18]

from Lemma II, if $\xi_1, \xi_2, \cdots, \xi_n$ denote the roots of $P_n\left(-1 + 2\frac{x-a}{b-a}\right) = 0$,

$$\begin{aligned} \sum_{\nu=1}^{n} \frac{l_\nu(x_0)^2}{k_\nu} &\leqq \frac{1}{m(b-a)} \sum_{\nu=1}^{n} (1-\eta_\nu^2)P_n'(\eta_\nu)^2 \frac{P_n\left(-1+2\frac{x_0-a}{b-a}\right)^2}{\left(\frac{2}{b-a}\right)^2 P_n'(\eta_\nu)^2 (x_0-\xi_\nu)^2} \\ &= \frac{b-a}{4m} P_n\left(-1+2\frac{x_0-a}{b-a}\right)^2 \sum_{\nu=1}^{n} \frac{1-\eta_\nu^2}{(x_0-\xi_\nu)^2}. \end{aligned} \tag{32}$$

[17] J. Shohat: *On the convergence-properties of Lagrange-interpolation etc.*, Annals of Math., 1937, pp. 758–769, formula (39).

[18] L. Fejér: *Az interpolátióról*, Akadémiai Értesitö (Hungarian), 1915.

As $-1 + 2\dfrac{\xi_\nu - a}{b - a} = \eta_\nu$ $(\nu = 1, 2, \cdots n)$, making use of the differential equation of the Legendre-polynomials we obtain by the substitution $x_0 = a + \dfrac{b-a}{2}(1 + y_0)$ in (32) for any real y_0

$$\sum_{\nu=1}^{n} \frac{l_\nu\left(a + \frac{b-a}{2}(y_0+1)\right)^2}{k_\nu} \leqq \frac{1}{m(b-a)}[(1 - y_0^2)P_n'(y_0)^2 + n^2 P_n(y_0)^2]. \tag{33}$$

Equality in (33) holds only for $p(x) \equiv m$. Suppose now $-1 \leqq y_0 \leqq +1$; then, by a well known result $|P_n(y_0)| \leqq 1$, and by the above cited theorem of Bernstein-Fejér $|P_n'(y_0)\sqrt{(1 - y_0^2)}| \leqq n$, i.e. from (33) for $a \leqq x \leqq b$ we obtain

$$\sum_{\nu=1}^{n} \frac{l_\nu(x)^2}{k_\nu} < \frac{2}{m(b-a)} n^2. \tag{34a}$$ [19]

Let now $-1 + \epsilon \leqq y \leqq 1 - \epsilon$. Then according to the classical formula of Laplace for $\epsilon' \leqq \vartheta \leqq \pi - \epsilon'$, we have

$$\left| P_n(\cos\vartheta) - \left[\frac{2}{\pi n \sin\vartheta}\right]^{\frac{1}{2}} \cos\left[\left(n + \frac{1}{2}\right)\vartheta - \frac{\pi}{4}\right]\right| < \frac{c_{31}(\epsilon')}{n^{3/2}};$$

by this and by the theorem of Bernstein-Fejér for $[-1 + \epsilon, 1 - \epsilon]$ we have

$$|P_n(x)| \leqq \frac{c_{32}(\epsilon)}{\sqrt{n}}, \qquad |P_n'(x)| \leqq c_{32}(\epsilon)\sqrt{n},$$

i.e. we obtain roughly[19] from (33) in $[a + \epsilon, b - \epsilon]$

$$\sum_{\nu=1}^{n} \frac{l_\nu(x)^2}{k_\nu} \leqq \frac{c_{33}(\epsilon)}{m(b-a)} n. \tag{34b}$$

We remark, that for the validity of (34a) and (34b) in the above intervals we require only that the L-integrable $p(x)$ is in $[-1, +1]$ not less than 0, and in $[a, b]$ $p(x) \geqq m > 0$. (Corollary II.)

Let in Lemma II $p_1(x) \equiv p(x)$, $p_2(x) = 0$ in $[-1, a][b, 1]$ and not less than $\dfrac{m}{[(x-a)(b-x)]^{\frac{1}{2}}}$ in $[a, b]$; if $\gamma_1, \gamma_2, \cdots \gamma_n$ stand for the roots of the Tchebycheff polynomial $T_n(x)(T_n(\cos\vartheta) = \cos n\vartheta)$, $\mu_1, \mu_2, \cdots \mu_n$ for the roots of $T_n\left(-1 + \dfrac{2(x-a)}{b-a}\right)$, then

$$k_\nu^+ = m\int_a^b \frac{b_\nu^+(t)}{[(t-a)(b-t)]^{\frac{1}{2}}}\,dt = m\int_{-1}^{1} \frac{T_n(t)}{T_n'(\gamma_\nu)(t - \gamma_\nu)} \frac{dt}{\sqrt{(1-t^2)}} = m\frac{\pi}{n},$$

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[19] For $a = -1$, $b = 1$, see J. Shohat: *On Interpolation*, Annals of Math., 1933.

i.e. by easy computation with real x

(34c)
$$\sum_{\nu=1}^{n} \frac{l_\nu(x)^2}{k_\nu} \leqq \frac{n}{m\pi} \sum_{\nu=1}^{n} l_\nu^+(x)^2 = \frac{n}{m\pi}\left[1 - \frac{1}{2n} + \frac{1}{2n(2n-1)} T'_{2n-1}\left(-1 + \frac{2(x-a)}{b-a}\right)\right].$$

Thus we came to the result, that if the L-integrable $p(x)$ is not less in $[-1, +1]$ than 0 and in the subinterval $[a, b]$ $p(x) \geqq \frac{m}{[(x-a)(b-x)]^{\frac{1}{2}}}$, then for $a \leqq x \leqq b$

(35)
$$\sum_{\nu=1}^{n} \frac{l_\nu(x)^2}{k_\nu} \leqq \frac{2}{\pi m} \cdot n.$$

(Corollary III). Equality holds only when in $[a, b]$ $p(x) = \frac{m}{[(x-a)(b-x)]^{\frac{1}{2}}}$, and in the complementary intervals $p(x) = 0$, further $x = a$ or $x = b$.

We deduce theorem II from (34a) and (34b) as follows. As

$$\sum_{\nu=1}^{n} k_\nu^{(n)} = \int_{-1}^{1} \left(\sum_{\nu=1}^{n} l_\nu(t)\right) p(t)\, dt = \int_{-1}^{1} p(t)\, dt$$

and as $k_\nu^{(n)} > 0$ by (6a) and (29b), we have in $[a, b]$ by (34a)

$$\left(\sum_{\nu=1}^{n} | l_\nu(x) |\right)^2 = \left(\sum_{\nu=1}^{n} \frac{| l_\nu(x) |}{\sqrt{k_\nu}} \sqrt{k_\nu}\right)^2 \leqq \sum_{\nu=1}^{n} \frac{l_\nu(x)^2}{k_\nu} \cdot \sum_{\nu=1}^{n} k_\nu < \frac{2}{m(b-a)} \int_{-1}^{1} p(t)\, dt \cdot n^2$$

and analogously in $[a + \epsilon, b - \epsilon]$

$$\left(\sum_{\nu=1}^{n} | l_\nu(x) |\right)^2 < \frac{c_{33}(\epsilon)}{m(b-a)} \int_{-1}^{1} p(t)\, dt \cdot n.$$

But then

$$| \omega_n(x) |\, 2^{n-2} \leqq 2 \sum_{\nu=1}^{n} | l_\nu(x) | \begin{cases} \leqq \left[\frac{8}{m(b-a)}\right]^{\frac{1}{2}} \left[\int_{-1}^{1} p(t)\, dt\right]^{\frac{1}{2}} \cdot n & \text{for } a \leqq x \leqq b \\ \leqq \left[\frac{4c_{33}(\epsilon)}{m(b-a)}\right]^{\frac{1}{2}} \left[\int_{-1}^{1} p(t)\, dt\right]^{\frac{1}{2}} \cdot \sqrt{n} & \text{for } a + \epsilon \leqq x \leqq b - \epsilon. \end{cases}$$

It is to be remarked, that the same argument leads to the following result: if the L-integrable $p(x)$ is not less than 0 in $[-1, +1]$ and if in the subinterval $[a, b]$

$$p(x) \geqq \frac{m}{[(x-a)(b-x)]^{\frac{1}{2}}},$$

then for $a \leqq x \leqq b$ we have

(36)
$$| \omega_n(x) | \leqq 4\left[\frac{2}{\pi m} \int_{-1}^{1} p(t)\, dt\right]^{\frac{1}{2}} \cdot \frac{\sqrt{n}}{2^n} \qquad n = 1, 2, \cdots.$$

Some further corollaries of Lemma II we shall mention later.

Let us now consider the lower estimate of the orthogonal polynomials $\omega_n(x)$.

THEOREM III. *Let the weight-function $p(x)$ be non-negative and L-integrable in $[-1, +1]$; throughout the subinterval $[a, b]$ suppose $p(x) \geqq m > 0$. Then, if $x_d^{(n)}$ denotes* the root of $\omega_n(x)$ *nearest to x, we have for real x*

$$|\omega_n(x)| \geqq \left[\frac{c_{34} m}{(b-a)\int_{-1}^{1} p(t)\,dt}\right]^{\frac{1}{2}} \left(\frac{b-a}{4}\right)^n |x - x_d^{(n)}|.$$

We require two lemmas.

LEMMA III. *In $[-1, +1]$ suppose $p_1(x) \geqq p_2(x) \geqq 0$ and both L-integrable. If $\omega_n(x)$ and $\omega_n^+(x)$ denote the corresponding orthogonal polynomials respectively, k_ν and k_ν^+ the respective Christoffel-numbers, x_ν and x_ν^+ the respective fundamental points, then*

$$\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2} \leqq \sum_{\nu=1}^{n} \frac{1}{k_\nu^+ \omega_n^{+\prime}(x_\nu^+)^2}.$$

PROOF. Let us consider the minimum of $N(F) = \int_{-1}^{1} |F(t)|^2 p_1(t)\,dt$ amongst the polynomials of degree $(n-1)$, in which coeff. $x^{n-1} = 1$. It is known that this problem has one and only one solution and that the minimum is assumed only for $F(x) = \omega_{n-1}(x)$. But we want to represent the solution in the form $F(x) = \sum_{\nu=1}^{n} d_\nu l_\nu(x)$, where the $l_\nu(x)$'s denote the fundamental functions of Lagrange-interpolation formed upon the roots of $\omega_n(x)$. It is evident that in this case we have to determine the minimum of the form $\sum_{\nu=1}^{n} k_\nu |d_\nu|^2$, if $\sum_{\nu=1}^{n} \frac{d_\nu}{\omega_n'(x_\nu)} = 1$. From this, by applying Schwarz's inequality once more, we obtain for the value of this minimum $\left[\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2}\right]^{-1}$ and equality holds only for the polynomial

$$f(x) = \frac{\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)} l_\nu(x)}{\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2}}. \tag{37}$$

Thus again

$$\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2} = \left[\min_{f=x^{n-1}+\cdots} \int_{-1}^{1} |f(t)|^2 p_1(t)\,dt\right]^{-1} \leqq \left[\min_{f=x^{n-1}+\cdots} \int_{-1}^{1} |f(t)|^2 p_2(t)\,dt\right]^{-1}$$
$$= \sum_{\nu=1}^{n} \frac{1}{k_\nu^+ \omega_n^{+\prime}(x_\nu)^2}.$$

Q.e.d.

If $p_1(x) \equiv p(x)$ and throughout the intervals $[-1, a]$, $[b, 1]$ $p_2(x) = 0$, further for $a \leqq x \leqq b$ $p_2(x) \equiv m$, then

$$\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2} \leqq \frac{1}{m}\left[\min_{f=x^{n-1}+\cdots} \int_a^b f(t)^2\,dt\right]^{-1} = \frac{1}{m}\left[\int_a^b P_{n-1}\left(-1 + 2\frac{t-a}{b-a}\right)^2 dt\right]^{-1},$$

where the integrand is the linear transform of the $(n-1)^{\text{th}}$ Legendre polynomial with the normalization coeff. $x^{n-1} = 1$. Thus

$$\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2} \leqq \frac{1}{m} \frac{(2n-1)\binom{2n-2}{n-1}^2}{(b-a)^{2n-1}}, \tag{38}$$

if throughout $[-1, +1]$ the L-integrable $p(x) \geqq 0$ and throughout $[a, b]$ $p(x) \geqq m > 0$. Equality holds only if $p(x) \equiv m$ in $[a, b]$ and $\equiv 0$ outside of $[a, b]$.

LEMMA IV. *For the fundamental functions of the Lagrange-interpolation formed upon any matrix* $\mathfrak{M}$ *we have*

$$\phi(x) = l_{kn}(x) + l_{k+1,n}(x) \geqq 1.$$
$$x_k^{(n)} \geqq x \geqq x_{k+1}^{(n)}.$$

PROOF. Let $2 < k < n-2$. Then $\phi(x)$ is a polynomial of degree $(n-1)$ at the utmost, vanishing at $x_1^{(n)}, x_2^{(n)} \cdots, x_{k-1}^{(n)}, x_{k+2}^{(n)}, \cdots x_n^{(n)}$, i.e. at $n-2$ places and equals 1 at $x_k^{(n)}$ and $x_{k+1}^{(n)}$. Consequently its first derivative has one root in each of the intervals $[x_2^{(n)}, x_1^{(n)}] \cdots [x_{k-1}^{(n)}, x_{k-2}^{(n)}]$, $[x_{k+3}^{(n)}, x_{k+2}^{(n)}], \cdots [x_n^{(n)}, x_{n-1}^{(n)}]$, which determines at least $n-4$ roots. In consequence of $\phi(x_k^{(n)}) = \phi(x_{k+1}^{(n)}) = 1$ one of the roots of this derivative lies evidently in $[x_{k+1}^{(n)}, x_k^{(n)}]$. We now show that $\phi'(x_k^{(n)}) \leqq 0$ and $\phi'(x_{k+1}^{(n)}) \geqq 0$. It will be sufficient to show the first. Suppose $\phi'(x_k) > 0$. Then $\phi'(x)$ must have at least one root in $[x_k^{(n)}, x_{k-1}^{(n)}]$. But then $\phi'(x)$ could not have more roots and thus $\phi'(x_{k+1}) < 0$; hence $\phi'(x)$ ought to have one more root in $[x_{k+2}^{(n)}, x_{k+1}^{(n)}]$, and this is impossible. $\phi'(x_{k+1}^{(n)}) \geqq 0$ is to be obtained analogously. But then, if in $(x_{k+1}^{(n)}, x_k^{(n)})$ there were a point ξ_0 with $\phi(\xi_0) = 1$ then $\phi'(x)$ would have 3 roots in $[x_{k+1}^{(n)}, x_k^{(n)}]$; an evident impossibility, which establishes the lemma. For $k = 1, 2, n-2, n-1$ the proof runs analogously.

PROOF OF THEOREM III. Here also we start from $\sum_{\nu=1}^{n} \frac{l_\nu(x)^2}{k_\nu}$. By (38) we have—$x_d^{(n)}$ has the meaning given above—

$$\begin{aligned}\omega_n(x)^2 &= \left[\sum_{\mu=1}^{n} \frac{1}{k_\mu \omega_n'(x_\mu)^2 (x-x_\mu)^2}\right]^{-1} \sum_{\mu=1}^{n} \frac{l_\mu(x)^2}{k_\mu} \\ &\geqq |x - x_d|^2 \frac{m(b-a)^{2n-1}}{(2n-1)\binom{2n-2}{n-1}^2} \sum_{\nu=1}^{n} \frac{l_\nu(x)^2}{k_\nu}.\end{aligned} \tag{39}$$

But since $k_\nu^{(n)} < \int_{-1}^{1} p(t)\, dt$ (we made use of this at the upper estimate, too) then if e.g. $x_{d+1}^{(n)} \leqq x \leqq x_d^{(n)}$, we may write

$$\omega_n(x)^2 > \frac{m}{\int_{-1}^{1} p(t)\, dt} \frac{(b-a)^{2n-1}}{(2n-1)\binom{2n-2}{n-1}^2} [l_d(x)^2 + l_{d+1}(x)^2]\, |x - x_d^{(n)}|^2. \tag{40}$$

Now by lemma IV for any x in the interval in question

$$l_d(x)^2 + l_{d+1}(x)^2 \geqq \tfrac{1}{2}[l_d(x) + l_{d+1}(x)]^2 \geqq \tfrac{1}{2}$$

i.e. by

$$(2n-1)\binom{2n-2}{n-1}^2 \sim c_{35}4^n$$

we obtain that

$$|\omega_n(x)| > c_{36}\left\{\frac{m}{b-a}\left[\int_{-1}^{1} p(t)\,dt\right]^{-1}\right\}^{\frac{1}{2}}\left(\frac{b-a}{4}\right)^n |x - x_d^{(n)}|,$$

which proves the theorem.

THEOREM IV. *Let us add to the hypotheses of theorem III that, throughout a subinterval* $[c, d]$ *of* $[a, b]$, $m \leqq p(x) \leqq \dfrac{M}{\sqrt{(1-x^2)}}$. *Then, if* $x_d^{(n)}$ *denotes again the root of* $\omega_n(x)$ *nearest to* x, *we have in* $[c+\epsilon, d-\epsilon]$ *for* $n > n_0(\epsilon, c, d, p)$

$$|\omega_n(x)| > \frac{c_{37}}{\sqrt{(b-a)}}\left[\frac{m}{M + \int_{-1}^{1} p(t)\,dt}\right]^{\frac{1}{2}} |x - x_d^{(n)}| \left(\frac{b-a}{4}\right)^n \sqrt{n}.$$

For the proof we require

LEMMA V. *In* $[-1, +1]$ *suppose* $p(x) \geqq 0$ *and L-integrable; suppose furthermore, throughout a subinterval* $[u, v]$ $p(x) \leqq \dfrac{M}{\sqrt{(1-x^2)}}$; *then for the Christoffel-numbers* $k_\nu^{(n)}$ *belonging to the* $x_\nu^{(n)}$*'s lying in* $[u+\eta, v-\eta]$ $(\eta > 0)$, *we have*

$$k_\nu^{(n)} < \frac{c_{37}}{n}\left[M + \frac{c_{38}}{\eta^2 n}\int_{-1}^{1} p(t)\,dt\right].$$

PROOF. According to the first corollary due to Shohat of Lemma II we may write

$$\begin{aligned} k_\nu^{(n)} &= \min_{f(x_\nu^{(n)})=1,\, f=a_0x^{n-1}+\cdots} \int_{-1}^{1} |f(t)|^2 p(t)\,dt \\ &= \min_{F(\vartheta_\nu^{(n)})=1,\, F(\vartheta)=b_0+\cdots+b_{n-1}\cos(n-1)\vartheta} \int_0^\pi |F(\vartheta)|^2 p(\cos\vartheta)\sin\vartheta\,d\vartheta \\ &\leqq \int_0^\pi \left[\left(\frac{\sin n\dfrac{\vartheta+\vartheta_\nu^{(n)}}{2}}{n\sin\dfrac{\vartheta+\vartheta_\nu^{(n)}}{2}}\right)^2 + \left(\frac{\sin n\dfrac{\vartheta-\vartheta_\nu^{(n)}}{2}}{n\sin\dfrac{\vartheta-\vartheta_\nu^{(n)}}{2}}\right)^2\right] p(\cos\vartheta)\sin\vartheta\,d\vartheta \\ &\equiv \int_{\alpha_\nu}^{\beta} + \int_C \equiv I_1 + I_2, \end{aligned}$$

where $u = \cos\beta$, $v = \cos\alpha$ and C stands for the intervals complementary to $[\alpha, \beta]$ in $[0, \pi]$. Evidently

$$I_1 < M \frac{2}{n^2} \int_0^{2\pi} \left(\frac{\sin n \frac{\vartheta - \vartheta_\nu^{(n)}}{2}}{\sin \frac{\vartheta - \vartheta_\nu^{(n)}}{2}} \right)^2 d\vartheta = \frac{c_{38} M}{n}$$

and

$$I_2 < \frac{c_{39}}{\eta^2 n^2} \int_{-1}^{1} p(t)\, dt,$$

which proves the lemma.

PROOF OF THEOREM IV. From (39)

$$\omega_n(x)^2 \geqq m \frac{(b-a)^{2n-1}(x - x_d^{(n)})^2}{(2n-1)\binom{2n-2}{n-1}^2} \left[\frac{l_d(x)^2}{k_d} + \frac{l_{d+1}(x)^2}{k_{d+1}} \right].$$

If $c + \epsilon \leqq x \leqq d - \epsilon$ and $n > n_1(c, d, \epsilon, p)$ then, according to footnote,[7] the interval $[x_{d+1}^{(n)}, x_d^{(n)}]$ containing x lies in $[c + \frac{1}{2}\epsilon, d - \frac{1}{2}\epsilon]$, and hence, by applying lemma IV, and with the substitution $u = c$, $v = d$, $\eta = \frac{1}{2}\epsilon$, Lemma V we obtain for $n > \max(n_1, 4/\epsilon^2) = n_0$

$$\omega(x)^2 > c_{40} \frac{mn}{M + \int_{-1}^{1} p(t)\, dt} \left(\frac{b-a}{4} \right)^{2n-1} | x - x_d^{(n)} |^2,$$

which establishes the theorem.

REMARK I. In the special case $m \leqq p(x) \leqq \dfrac{M}{\sqrt{(1-x^2)}}$ throughout $[-1, +1]$ we have by the aforesaid in $[-1 + \epsilon, 1 - \epsilon]$

$$c_{41}(p, \epsilon) \frac{\sqrt{n}}{2^n} | x - x_d^{(n)} | \leqq | \omega_n(x) | \leqq c_{42}(p, \epsilon) \frac{\sqrt{n}}{2^n}.$$

REMARK II. In lemma V we required the upper estimate of the Christoffel-numbers. Although we shall not use it, we mention, that if in $[-1, +1]$ $p(x) \geqq m > 0$, then

$$k_\nu^{(n)} \geqq \frac{2m}{(1 - x_\nu^2) P_n'(x_\nu)^2 + n^2 P_n(x_\nu)^2},$$

equality only for $p(x) \equiv m$; here $P_n(x)$ means the n^{th} Legendre-polynomial with normalization $P_n(1) = 1$. By this

$$k_\nu^{(n)} \geqq \frac{m}{n^2}, \qquad \nu = 1, 2, \cdots n, \qquad n = 1, 2, \cdots.$$

Further it is easy to obtain that if in $[-1, +1]$ $p(x) \geqq m > 0$, then

$$k_\nu^{(n)} \geqq \frac{c_{43}(\epsilon)m}{n}, \qquad -1 + \epsilon \leqq x_\nu^{(n)} \leqq 1 - \epsilon,$$

and, if in $[-1, +1]$ $p(x) \geqq \dfrac{m}{\sqrt{(1 - x^2)}}$ holds, then

$$k_\nu^{(n)} \geqq \frac{c_{44}m}{n}, \qquad \nu = 1, 2, \cdots n.$$

REMARK III. If in Lemma V $[u, v] \equiv [-1, +1]$, then we have the sharper result

$$k_\nu^{(n)} \leqq \frac{c_{45}M}{n}, \qquad \nu = 1, 2, \cdots n.$$

REMARK IV. We obtain from the proof of Lemma III

$$\omega_{n-1}(x) = \frac{\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)} l_\nu(x)}{\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2}} = \frac{\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2} \cdot \frac{\omega_n(x)}{x - x_\nu}}{\sum_{\nu=1}^{n} \frac{1}{k_\nu \omega_n'(x_\nu)^2}},$$

where $l_\nu(x)$ are the fundamental functions of the Lagrange interpolation formed upon the roots of $\omega_n(x)$. As $k_\nu > 0$ we evidently have for sufficiently small ϵ (see (2))

$$\text{sg}\, \omega_{n-1}(x_\nu^{(n)} - \epsilon) = -\text{sg}\, \omega_n(x_\nu^{(n)} - \epsilon),$$
$$\text{sg}\, \omega_{n-1}(x_{\nu+1}^{(n)} + \epsilon) = \text{sg}\, \omega_n(x_{\nu+1}^{(n)} + \epsilon),$$

i.e. we obtained the well known fact that there is a root of $\omega_{n-1}(x)$ between each pair of roots of $\omega_n(x)$.

2.

THEOREM V. *If for the fundamental functions belonging to the matrix* $\mathfrak{M}$

(41) $$[| l_k(x) |]^{1/n} \leqq 1 + \epsilon, \qquad -1 \leqq x \leqq +1, k = 1, 2, \cdots n, n > n_2(\epsilon),$$

holds for any sufficiently small ϵ, *then, at any fixed point* z *of the complex plane cut up along* $[-1, +1]$, *we have*

$$\lim_{n \to \infty} [\omega_n(z)]^{1/n} = \frac{z + \sqrt{(z^2 - 1)}}{2}.$$

Here we must take those values of the roots which are positive on the positive real axis for $z > 1$.

For the proof we require

LEMMA VI. *From* (41) *it follows that, for any small positive* η, *if* $n > n_3(\eta)$,

$$| \omega_n'(x_\nu) | > (\tfrac{1}{2} - \eta)^n, \qquad \nu = 1, 2, \cdots n.$$

PROOF. Suppose that the lemma is not true. In this case there is a positive absolute constant δ and a sequence of positive integers $n_1 < n_2 < \cdots$ such, that to any n_k we could give an integer ν_k with $1 \leqq \nu_k \leqq n_k$ and

$$|\omega'_{n_k}(x_{\nu_k}^{(n_k)})| < (\tfrac{1}{2} - \delta)^{n_k}.$$

But, as, according to a classical theorem of Tchebycheff, there is in $[-1, +1]$ a ξ_k for which the value of the polynomial $\dfrac{\omega_{n_k}(x)}{x - x_{\nu_k}^{(n_k)}}$ is not less than $\dfrac{1}{2^{n_k - 2}}$, at the same $x = \xi_k$

$$|l_{\nu_k, n_k}(\xi_k)| > \frac{1}{2^{n_k-2}} \cdot \frac{1}{(\frac{1}{2} - \delta)^{n_k}}$$

which, for $k \to \infty$ contradicts (41) and thus proves the lemma.

REMARK. It follows from lemma VI and lemma I that (41) implies

$$\lim_{n \to \infty} \left[\sum_{\nu=1}^{n} \frac{1}{|\omega'_n(x_\nu)|} \right]^{1/n} = 2.$$

We shall make no use of this statement in this paper.

PROOF OF THEOREM V. Let

$$\eta_\nu = \cos \frac{2\nu - 1}{2n + 2} \pi, \qquad \nu = 1, 2, \cdots (n + 1),$$

the roots of the $(n + 1)^{\text{th}}$ Tchebycheff-polynomial $T_{n+1}(x)$ $(T_n(\cos \vartheta) = \cos n\vartheta)$, and represent $\omega_n(x)$ by the Lagrange-interpolatoric-polynomial taken at these η_ν's. If $L_\nu(z)$ $(\nu = 1, 2 \cdots (n + 1))$ denote *these* fundamental functions, then

$$\omega_n(z) = \sum_{\nu=1}^{n+1} \omega_n(\eta_\nu) L_\nu(z) \tag{42}$$

(z any number). But by (41) and lemma I, in $-1 \leqq x \leqq +1$, for $n > n_2(\epsilon)$ we have

$$|\omega_n(x)| 2^{n-2} \leqq 2 \sum_{\nu=1}^{n} \left| \frac{\omega_n(x)}{\omega'_n(x_\nu)(x - x_\nu)} \right| \leqq 2(n + 1)(1 + \epsilon)^n$$

i.e. by $|T'_{n+1}(\eta_\nu)| = \dfrac{n + 1}{\sqrt{(1 - \eta_\nu^2)}}$ from (42) for $n > n_3(\epsilon)$

$$|\omega_n(z)| < \sum_{\nu=1}^{n+1} \frac{8(n + 1)(1 + \epsilon)^n}{2^n} \frac{|z + \sqrt{(z^2 - 1)}|^{n+1}}{(n + 1)|z - \eta_\nu|}$$

$$< 17 \left| \frac{1 + \epsilon}{2} (z + \sqrt{(z^2 - 1)}) \right|^{n+1} \max_{\nu=1,2,\cdots(n+1)} \frac{1}{|z - \eta_\nu|};$$

hence, as z is not in $[-1, +1]$, for $n > n_4(\epsilon, z)$

$$[|\omega_n(z)|]^{1/n} < \frac{1 + 2\epsilon}{2} |z + \sqrt{(z^2 - 1)}|.$$

Now let us consider the lower estimate and represent the $(n-1)^{\text{th}}$ Tchebycheff-polynomial $T_{n-1}(z)$ by the Lagrange-interpolatoric-polynomial taken at the roots of $\omega_n(x)$. We obtain

$$T_{n-1}(z) \equiv \sum_{\nu=1}^{n} T_{n-1}(x_\nu^{(n)}) l_\nu(x).$$

As $| T_{n-1}(x_\nu) | \leqq 1$, there is an integer ν_0 with $1 \leqq \nu_0 \leqq n$ and

$$| l_{\nu_0}(z) | > \frac{1}{n} | T_{n-1}(z) |,$$

thus for $n > n_5(\epsilon)$

$$(43) \qquad | \omega_n(z) | > \frac{1}{2n} (1 - \epsilon) | z + \sqrt{(z^2 - 1)} |^{n-1} | \omega_n'(x_{\nu_0}^{(n)}) | | z - x_{\nu_0}^{(n)} |.$$

From (43), by lemma VI, and as z is not at $[-1, +1]$, we obtain for $n > n_6(\epsilon)$

$$(44) \qquad | \omega_n(z) | \geqq \frac{1}{2n} \left| \frac{1 - 2\epsilon}{2} (z + \sqrt{(z^2 - 1)}) \right|^{n-1} | z - x_{\nu_0}^{(n)} |.$$

From (44) and $n > n_7(\epsilon, B)$ it is evident that in any closed bounded set B of the complex plane cut up along $[-1, +1]$ we have uniformly

$$(45) \qquad [| \omega_n(z) |]^{1/n} > (1 - 3\epsilon) \left| \frac{z + \sqrt{(z^2 - 1)}}{2} \right|,$$

where we are to take that value of the square root, for which the right side $\sim z$ for $z \to \infty$. This proves the theorem, since on the positive real axis, for $z \to \infty$, the two sides of (45) are equal without the sign of the absolute value too, and upon the cut plane both sides are one-valued and regular functions of z.

We already mentioned in the introduction that in the case of strongly normal polynomials, the asymptotic formula for $[\omega_n(z)]^{1/n}$ upon the cut plane is a consequence of the above Fejérian-theorem. In order to state an analogous theorem for a class of orthogonal polynomials more general than that of Szegö's we require a further lemma.

Lemma VII. *If the L-integrable weight-function $p(x)$ is non-negative in $[-1, +1]$ and if its roots form an aggregate of measure 0, then for the fundamental functions connected with the matrix p and $n > n_8(\epsilon)$ we have*

$$| l_{\nu,n}(x) | \leqq (1 + \epsilon)^n, \qquad -1 \leqq x \leqq +1, \qquad \nu = 1, 2, \cdots, n.$$

Proof. We employ following theorem of E. Remes.[20] Let in $[-1, +1]$ be given a finite set of disjoint intervals of total length ϑ. If throughout this

[20] E. Remes: *Sur une propriété extrémale des polynomes de Tchebichef*. Communications de l'Institut de Sciences etc., Kharkow, 1936, série 4, XIII, fasc. 1, pp. 93–95.

aggregate the absolute value of the polynomial $f(x)$ of degree n is not greater than M then in $[-1, +1]$

$$|f(x)| \leqq M \left| T_n\left(\frac{4}{\vartheta} - 1\right)\right|,$$

where $T_n(\cos \vartheta) = \cos n\vartheta$.

Now suppose lemma VII to be untrue. Then we have an infinite sequence of positive integers $n_1 < n_2 < \cdots$ and a c_{46} such, that to every n_k there is a positive integer ν_k with $1 \leqq \nu_k \leqq n_k$ and a ξ_{ν_k} with $-1 \leqq \xi_{\nu_k} \leqq +1$ to satisfy

$$(46) \qquad |l_{\nu_k,n_k}(\xi_k)| > (1 + c_{46})^{n_k}.$$

Apply now for $l_{\nu_k,n_k}(x)$ Remes's theorem taking for the aggregate of intervals of length ϑ those intervals throughout which

$$|l_{\nu_k,n_k}(x)| \leqq \left(1 + \frac{c_{46}}{2}\right)^{n_k}.$$

Thus we obtain

$$\left(1 + \frac{c_{46}}{2}\right)^{n_k} \left| T_{n_k}\left(\frac{4}{\vartheta} - 1\right)\right| \geqq (1 + c_{46})^{n_k}$$

from which $0 < \vartheta < 2 - c_{47}$ where c_{47} depends only upon c_{46} and is independent of n_k. Hence throughout intervals $\mathfrak{A}$ of total length greater than c_{47}, $|l_{\nu_k,n_k}(x)| \geqq (1 + \frac{1}{2}c_{46})^{n_k}$. But then by the assumption made for the roots of $p(x)$ we may omit from $\mathfrak{A}$ intervals of smaller total length than $\frac{1}{2}c_{47}$ such that throughout the remaining $\mathfrak{A}^+$ $p(x) \geqq c_{48}$, where c_{48} depends only upon c_{47}. Hence we had

$$\int_{-1}^{1} l_{\nu_k,n_k}(x)^2 p(x)\,dx \geqq \int_{\mathfrak{A}^+} l_{\nu_k,n_k}(x)^2 p(x)\,dx$$

$$\geqq c_{48} \int_{\mathfrak{A}^+} l_{\nu_k,n_k}(x)^2\,dx > c_{48} \cdot \frac{c_{47}}{2}\left(1 + \frac{c_{46}}{2}\right)^{n_k}$$

which contradicts the Shohat minimum property of the $l_\nu(x)$'s (Lemma II. Coroll. I.). Hence lemma VII is established.

By theorem V and lemma VII we state

THEOREM VI. *Let the weight-function $p(x)$ be non-negative and L-integrable in $[-1, +1]$ and assume the aggregate of its roots to be of measure 0. Then, taking the suitable values of the roots we have upon the plane cut up along $[-1, +1]$*

$$\lim_{n\to\infty} [\omega_n(z)]^{1/n} = \frac{z + \sqrt{(z^2 - 1)}}{2}$$

uniformly in each interior domain.

3.

In this section we consider c) problems. We prove following Fejérian theorem:

THEOREM VII. *Assume the matrix* $\mathfrak{M}$ *to be such that* $[-1, +1]$ *possesses a subinterval* $[b, a] \equiv [\cos \beta, \cos \alpha]$ *with*

$$| l_k(x) | \leqq c_{49}, \qquad k = \nu, \nu + 1, \cdots \mu$$

and for the other fundamental functions

$$| l_k(x) | < c_{50} n^{c_{51}}$$

throughout $[b, a]$, *if*

$$\vartheta_{\nu-1}^{(n)} < \alpha \leqq \vartheta_{\nu}^{(n)} < \vartheta_{\nu+1}^{(n)} < \cdots < \vartheta_{\mu}^{(n)} \leqq \beta < \vartheta_{\mu+1}^{(n)}.$$

Then

$$\frac{[\epsilon(b-a)]^{\frac{1}{2}}}{c_{49}} \cdot \frac{1}{n} \leqq \vartheta_{k+1}^{(n)} - \vartheta_k^{(n)} \leqq \frac{c_{49} \cdot c_{52}(\epsilon, a, b, c_{50}, c_{51})}{n},$$

if $\vartheta_k^{(n)}$ *and* $\vartheta_{k+1}^{(n)}$ *are in* $[\alpha + \epsilon, \beta - \epsilon]$, ϵ *denoting any small positive number.*

PROOF. First we prove the lower estimate

$$\frac{1}{| \vartheta_{k+1}^{(n)} - \vartheta_k^{(n)} |} = \left| \frac{l_k(\cos \vartheta_k^{(n)}) - l_k(\cos \vartheta_{k+1}^{(n)})}{\vartheta_k^{(n)} - \vartheta_{k+1}^{(n)}} \right| = \left| \frac{dl_k(\cos \vartheta)}{d\vartheta} \right|_{\vartheta = \vartheta^+},$$

where $\vartheta_k^{(n)} \leqq \vartheta^+ \leqq \vartheta_{k+1}^{(n)}$. But then, $l_k(\cos \vartheta)$ being a trigonometric polynomial of order $(n-1)$, by the Bernstein-Fejér-theorem we have

$$\frac{1}{| \vartheta_{k+1}^{(n)} - \vartheta_k^{(n)} |} \leqq \frac{c_{49}(n-1)}{[\epsilon(b-a)]^{\frac{1}{2}}},$$

which proves the lower estimate.

Let us now consider the upper estimate. Suppose

$$\max (\vartheta_{l+1}^{(n)} - \vartheta_l^{(n)}) = \frac{2A(n)}{n} = \vartheta_{k+1}^{(n)} - \vartheta_k^{(n)}, \qquad \alpha + \epsilon \leqq \vartheta_l^{(n)} < \vartheta_{l+1}^{(n)} \leqq \beta - \epsilon$$

and we have to prove that $A(n)$ remains smaller than a number independent of n. We can suppose $A(n) > 10$. Let r be the smallest positive even integer greater than $[c_{51}] + 4$, μ the largest integer with $\frac{\mu - 1}{2} r \leqq n - 1$, $\frac{\vartheta_k^{(n)} + \vartheta_{k+1}^{(n)}}{2} = \delta_0$ and

$$\varphi(\vartheta) = \frac{1}{\mu^r} \left[\left(\frac{\sin \mu \frac{\vartheta + \delta_0}{2}}{\sin \frac{\vartheta + \delta_0}{2}} \right)^r + \left(\frac{\sin \mu \frac{\vartheta - \delta_0}{2}}{\sin \frac{\vartheta - \delta_0}{2}} \right)^r \right]. \tag{47}$$

Then $\varphi(\vartheta)$ is a non-negative, pure cosine polynomial of order $(n-1)$ at the utmost, for which

$$\varphi(\delta_0) \geqq 1, \tag{48a}$$

and if, without any loss of generality, we assume $0 \leqq \delta_0 \leqq \frac{\pi}{2}$,

$$| \varphi(\vartheta) | \leqq \frac{1}{\mu^r}\left[\frac{1}{\sin^r \frac{\vartheta+\delta_0}{2}} + \frac{1}{\sin^r \frac{\vartheta-\delta_0}{2}}\right] \leqq \left(\frac{9\pi^2}{2\mu}\right)^r \frac{2}{(\vartheta-\delta_0)^r}. \tag{48b}$$

Let us now represent $\varphi(\vartheta)$ by the n^{th} Lagrange-interpolatory polynomial taken upon $\mathfrak{M}$. Then by (48a) and (48b) we have

$$\begin{aligned} 1 \leqq | \varphi(\delta_0) | &= \left| \sum_{\nu=1}^{n} \varphi(\vartheta_\nu^{(n)}) l_\nu(\cos \delta_0) \right| < \sum_{\nu=1}^{n} \left(\frac{9\pi^2}{2\mu}\right)^r \frac{| l_\nu(\cos \delta_0) |}{(\vartheta_\nu^{(n)} - \delta_0)^r} \\ &< c_{49} \sum_{\alpha+\frac{\epsilon}{2} \leqq \vartheta_\nu^{(n)} \leqq \beta-\frac{\epsilon}{2}} \frac{1}{(\vartheta_\nu^{(n)} - \delta_0)^r} \cdot \left(\frac{9\pi^2}{2\mu}\right)^r + \left(\frac{9\pi^2}{2\mu}\right)^r c_{50} n^{c_{51}} \cdot \left(\frac{2}{\epsilon}\right)^r n. \end{aligned} \tag{49}$$

Let us divide the sum upon the right-hand side into two parts with $\nu \leqq k$ or $\nu \geqq k+1$ respectively. As, according to the already proved lower estimate in the first sum we have

$$\delta_0 - \vartheta_\nu^{(n)} > \frac{A(n)}{n} + \frac{(k-\nu)[\epsilon(b-a)]^{\frac{1}{2}}}{\sqrt{2}\, c_{49} n} \qquad \text{for} \quad \vartheta_\nu^{(n)} \geqq \alpha + \frac{\epsilon}{2}$$

and in the second one

$$\vartheta_\nu^{(n)} - \delta_0 > \frac{A(n)}{n} + \frac{(k+1-\nu)[\epsilon(b-a)]^{\frac{1}{2}}}{\sqrt{2}\, c_{49} n} \qquad \text{for} \quad \vartheta_\nu^{(n)} \leqq \beta - \frac{\epsilon}{2},$$

we obtain that the sum on the right hand side of (49)

$$\begin{aligned} &< 2n^r \sum_{\nu=0}^{\infty} \frac{1}{\left(A(n) + \nu \frac{[\epsilon(b-a)]^{\frac{1}{2}}}{c_{49}\sqrt{2}}\right)^r} \\ &\qquad < \frac{2n^r}{r-1} \frac{c_{49}\sqrt{2}}{[\epsilon(b-a)]^{\frac{1}{2}}} \frac{1}{\left(A - \frac{[\epsilon(b-a)]^{\frac{1}{2}}}{c_{49}\sqrt{2}}\right)^{r-1}} < \frac{c_{53}(\epsilon, a, b) c_{49} n^r}{\left(\frac{A}{2}\right)^{r-1}}, \end{aligned}$$

as $r \geqq 4$ and $\frac{A}{2} > 5 > \frac{[\epsilon(b-a)]^{\frac{1}{2}}}{c_{49}\sqrt{2}}$ $(c_{49} \geqq 1)$. By substituting this into (49) we obtain

$$1 < \frac{c_{54}(c_{51}, \epsilon, c_{50})}{n^2} + \frac{c_{49}^2 c_{55}(\epsilon, a, b, c_{51})}{A^2},$$

which establishes the upper estimate.

The consequences of this theorem for sequences of strongly normal polynomials, we already mentioned in the introduction. For orthogonal polynomials we have

THEOREM VIII. *If the weight-function $p(x)$ is non-negative and L-integrable throughout $[-1, +1]$, and if, throughout the subinterval $[b, a] \equiv [\cos \beta, \cos \alpha]$ $0 < m \leqq p(x) \leqq \dfrac{M}{\sqrt{(1-x^2)}}$, then for any $\epsilon > 0$*

$$\frac{c_{56}(a, b, p, \epsilon)}{n} \leqq \vartheta_{k+1}^{(n)} - \vartheta_k^{(n)} \leqq \frac{c_{57}(a, b, p, \epsilon)}{n}$$

if $\alpha + \epsilon \leqq \vartheta_k^{(n)} < \vartheta_{k+1}^{(n)} \leqq \beta - \epsilon$.

PROOF. By (34a) we have for $b \leqq x \leqq a$

$$\sum_{\nu=1}^{n} \frac{l_\nu(x)^2}{k_\nu} < \frac{2}{m(a-b)} n^2.$$

As $k_\nu < \int_{-1}^{1} p(t)\, dt$, for $b \leqq x \leqq a$,

$$(50) \quad |l_\nu(x)| < \left[\frac{2}{m(a-b)} \int_{-1}^{1} p(t)\, dt\right]^{\frac{1}{2}} \cdot n, \qquad \nu = 1, 2, \cdots n, \qquad n = 1, 2, \cdots.$$

Further *a fortiori* from (34b)

$$\frac{l_\mu(x)^2}{k_\mu} \leqq \frac{c_{33}\left(\frac{\epsilon}{2}\right)}{m(a-b)} n, \qquad \mu = 1, 2, \cdots n, \qquad b + \frac{\epsilon}{2} \leqq x \leqq a - \frac{\epsilon}{2},$$

i.e. by applying lemma V with $u = b$, $v = a$, $\eta = \frac{1}{2}\epsilon$ for those $l_\mu(x)$, for which $b + \frac{1}{2}\epsilon \leqq x_\mu^{(n)} \leqq a - \frac{1}{2}\epsilon$ we obtain

$$|l_\mu(x)|^2 < \frac{c_{33}\left(\frac{\epsilon}{2}\right)}{m(a-b)} n \cdot \frac{c_{58}(p, \epsilon)}{n} = \frac{c_{59}(p, \epsilon)}{a-b},$$

$$b + \frac{\epsilon}{2} \leqq x \leqq a - \frac{\epsilon}{2}.$$

Thus the premises of theorem VII are satisfied for $[b + \frac{1}{2}\epsilon, a - \frac{1}{2}\epsilon]$ with $c_{49} = \left[\dfrac{c_{59}(p, \epsilon)}{a-b}\right]^{\frac{1}{2}}$, $c_{50} = \left[\dfrac{2}{m(a-b)} \displaystyle\int_{-1}^{1} p(t)\, dt\right]^{\frac{1}{2}}$, $c_{51} = 1$; hence theorem VIII is proved.

REMARK I. From corollary III of lemma II and from remark III of theorem IV it follows that if $\dfrac{m}{\sqrt{(1-x^2)}} \leqq p(x) \leqq \dfrac{M}{\sqrt{(1-x^2)}}$ in $[-1, +1]$, then

$$\sum_{\nu=1}^{n} l_\nu(x)^2 \leqq c_{60} \frac{M}{m}.$$

REMARK II. By theorems IV and VIII we obtain that if in $[-1, +1]$ the non-negative $p(x)$ is L-integrable and in a subinterval $[b, a]$ $0 < m \leqq p(x) \leqq$

$\frac{M}{\sqrt{(1-x^2)}}$, then $|\omega_n(x)|$ takes between any two roots lying in $[b+\epsilon, a-\epsilon]$ a value greater than $\frac{c_{61}(a, b, p, \epsilon)}{\sqrt{n}}\left(\frac{b-a}{4}\right)^n$.

We already mentioned in the introduction that if in the subinterval the weight-function is supposed to be continuous there are asymptotic theorems to be obtained. For sake of simplicity let us ascribe to $p(x)$ besides the properties of being non-negative and L-integrable in $[-1, +1]$ also continuity and positiveness of $p(x)\sqrt{(1-x^2)}$ throughout $[-1, +1]$. The results are new for this case, too, but the argument is very much clearer.

In (34c) for $x = \xi_0 = \cos\varphi_0$ we had

$$(51)\qquad \min_{\substack{f(t)=c_0+\cdots+c_{n-1}t^{n-1}\\ f(\xi_0)=1}} \int_{-1}^{1} \frac{f(t)^2}{\sqrt{(1-t^2)}}\,dt = \frac{\pi}{n-\frac{1}{2}+\frac{1}{2}\frac{\sin(2n-1)\varphi_0}{\sin\varphi_0}}.$$

Consider the polynomial

$$(52)\quad \phi_{n-1}(x) = \frac{T_{n-1}(\xi_0)T_n(x) - T_n(\xi_0)T_{n-1}(x)}{x-\xi_0}\cdot\frac{1}{n-\frac{1}{2}+\frac{1}{2}\frac{\sin(2n-1)\varphi_0}{\sin\varphi_0}}$$

with $T_0(x) = 1/\sqrt{2}$, $T_\nu(\cos\vartheta) = \cos\nu\vartheta$, $\nu \geqq 1$. It is evident that for $n > 1$ $\phi_{n-1}(\xi_0) = 1$ and that by the formula of Christoffel-Darboux (which holds for $n > 1$)

$$(53)\qquad \frac{T_{n-1}(\xi_0)T_n(x) - T_n(\xi_0)T_{n-1}(x)}{x-\xi_0} = 2\sum_{\nu=0}^{n-1} T_\nu(\xi_0)T_\nu(x)$$

we may write for $n > 1$

$$(54)\qquad \phi_{n-1}(x) \equiv \frac{\sum_{\nu=0}^{n-1} T_\nu(\xi_0)T_\nu(x)}{\sum_{\nu=0}^{n-1} T_\nu(\xi_0)^2} \equiv 2\,\frac{\sum_{\nu=0}^{n-1} T_\nu(\xi_0)T_\nu(x)}{n-\frac{1}{2}+\frac{1}{2}\frac{\sin(2n-1)\varphi_0}{\sin\varphi_0}},$$

which means that $\phi_{n-1}(x)$ is that polynomial which, amongst the $f(x)$ polynomials of degree $(n-1)$ with $f(\xi_0) = 1$ minimizes the integral $\int_{-1}^{1} f(t)^2 \frac{dt}{\sqrt{(1-t^2)}}$. By (54) we have for any $-1 \leqq \xi_0 \leqq 1$, $-1 \leqq x \leqq +1$

$$(55)\qquad |\phi_{n-1}(x)| \leqq c_{61}.$$

For $\phi_{n-1}(x)$ we require following two lemmas.

LEMMA VIII. *If for $n \to \infty$ $n\varphi_0 \to +\infty$ and $n(\pi - \varphi_0) \to \infty$, then the distance between φ_0 and the root of $\phi_{n-1}(\cos\vartheta) = 0$ nearest to $\vartheta = \varphi_0$ is $\sim\pi/n$.*

PROOF. According to (52) it will suffice to consider the roots differing from $\vartheta = \varphi_0$ of the equation

$$(56)\qquad \cos(n-1)\varphi_0\cos n\vartheta - \cos n\varphi_0\cdot\cos(n-1)\vartheta = 0.$$

As each of the $(n-1)$ intervals given by the n different real roots of $T_n(x) = 0$ contains just one root of the equation $T_{n-1}(x) = 0$ it is clear that any equation of the form $\lambda T_n(x) + \mu T_{n-1}(x)$ hence also $\phi_{n-1}(x) = 0$, have a root in the interval $\left[\cos \frac{2l+1}{2n}\pi, \cos \frac{2l-1}{2n}\pi\right]$ $(l = 1, \cdots (n-1))$. Thus equation (56) has a root in each interval of length $2\pi/n$ and consequently for the rightwards root φ_0' next to φ_0 we surely have $|\varphi_0' - \varphi_0| \leqq 2\pi/n$. Then by simple transformation we obtain from (56)

$$\sin \frac{\vartheta - \varphi_0}{2} \sin (n - \tfrac{1}{2})(\vartheta + \varphi_0) - \sin \frac{\vartheta + \varphi_0}{2} \cdot \sin (n - \tfrac{1}{2})(\varphi_0 - \vartheta) = 0$$

which for $n\varphi_0 \to \infty$, $n(\pi - \varphi_0) \to \infty$ immediately leads to $A \to \pi$ if $\varphi_0' = \varphi_0 + A/n$.

LEMMA IX. *If for the weight-function $p(x)$ in $[-1, +1]$, $q(x) \equiv p(x)\sqrt{(1 - x^2)} \geqq m > 0$ and $q(x)$ is continuous throughout the same interval, then for $0 \leqq \varphi_0 \leqq \pi$, $\xi_0 = \cos \varphi_0$*

$$\lim_{n\to\infty} n \int_{-1}^{1} \phi_{n-1}(t)^2 p(t)\, dt = \pi p(\xi_0)\sqrt{(1 - \xi_0^2)} \lim_{n\to\infty} \frac{1}{1 + \frac{1}{2n} \frac{\sin (2n-1)\varphi_0}{\sin \varphi_0}}$$

i.e. if $n\varphi_0 \to \infty$, $n(\pi - \varphi_0) \to \infty$

$$\lim_{n\to\infty} n \int_{-1}^{1} \phi_{n-1}(t)^2 p(t)\, dt = \pi p(\xi_0)\sqrt{(1 - \xi_0^2)}.$$

PROOF. For $|x - \xi_0| \geqq 1/n^{\frac{1}{4}}$ it is easy to obtain from (52)

$$|\phi_{n-1}(x)| \leqq 20 n^{-\frac{3}{4}}$$

i.e.

$$\int\limits_{\substack{-1 \leqq t \leqq +1 \\ |t - \xi_0| \geqq n^{-1/4}}} \phi_{n-1}(t)^2 p(t)\, dt < \frac{400}{n^{3/2}} \int_{-1}^{1} p(t)\, dt. \tag{57}$$

If $|p(x')\sqrt{(1 - x'^2)} - p(\xi_0)\sqrt{(1 - \xi_0^2)}| \leqq \delta$ for any $|x' - \xi_0| \leqq n^{-\frac{1}{4}}$, we have

$$\left| \int\limits_{\substack{-1 \leqq t \leqq +1 \\ |t - \xi_0| \leqq n^{-1/4}}} \phi_{n-1}(t)^2 p(t)\, dt - p(\xi_0)\sqrt{(1 - \xi_0^2)} \int_{-1}^{1} \frac{\phi_{n-1}(t)^2}{\sqrt{(1 - t^2)}}\, dt \right| \tag{58}$$

$$< \frac{400}{n^{3/2}}\pi \max_{-1 \leqq x \leqq +1} p(x)\sqrt{(1 - x^2)} + \delta \int_{-1}^{1} \frac{\phi_{n-1}(t)^2}{\sqrt{(1 - t^2)}}\, dt.$$

From (57) and (58) evidently

$$\left| \int_{-1}^{1} \phi_{n-1}(t)^2 p(t)\, dt - \frac{\pi p(\xi_0)\sqrt{(1-\xi_0^2)}}{n - \frac{1}{2} + \frac{1}{2}\frac{\sin(2n-1)\varphi_0}{\sin\varphi_0}} \right|$$

$$= \left| \int_{-1}^{1} \phi_{n-1}(t)^2 \left[p(t) - \frac{p(\xi_0)\sqrt{(1-\xi_0^2)}}{\sqrt{(1-t^2)}} \right] dt \right|$$

$$< \frac{400\pi}{n^{3/2}} \left[\int_{-1}^{1} p(t)\, dt + \max_{-1 \leqq x \leqq +1} p(x)\sqrt{(1-x^2)} \right] + \frac{10\pi\delta}{n}. \quad \text{Q.e.d.}$$

REMARK. If $p(x)\sqrt{(1-x^2)} = q(x)$ has at $x = \xi_0$ a discontinuity of first order, then $q(\xi_0)$ at the right-hand side in Lemma IX is replaced by $\dfrac{q(\xi_0+0) + q(\xi_0-0)}{2}$.

LEMMA X. *If for the polynomial $f(x)$ of degree k $f(\xi_0) = 1$, $\xi_0 = \cos\varphi_0$ and any positive ϵ and $\delta < 1$, $k \geqq 16/\delta$ and $\sin\varphi_0 > 8/k\delta$ the inequality*

$$\int_{\substack{|t-\xi_0| \leqq \epsilon \\ -1 \leqq t \leqq +1}} \frac{f(t)^2}{\sqrt{(1-t^2)}}\, dt \leqq (1-\delta) \frac{\pi}{k + \frac{1}{2} + \frac{1}{2}\frac{\sin(2k+1)\varphi_0}{\sin\varphi_0}}$$

holds, then

$$\int_{-1}^{1} \frac{f(t)^2}{\sqrt{(1-t^2)}}\, dt > \frac{\delta\left(1 + \frac{\epsilon^2}{4}\right)^{\frac{1}{4}k\delta}}{400k}.$$

REMARK. This lemma means that if the quadratic integral of the polynomial normalized for 1 at $x = \xi_0$ is "too small" in the interval $[\xi_0 - \epsilon, \xi_0 + \epsilon]$, it must be very large in some other parts of $[-1, +1]$.

PROOF. Without any loss of generality we may suppose $\xi_0 \geqq 0$. Then construct with the above $f(x)$

$$F(x) = f(x)\left[1 - \left(\frac{x - \xi_0}{1 + \xi_0}\right)^2\right]^{[\frac{1}{4}k\delta]} \tag{59}$$

a polynomial, whose degree is less than $k(1 + \frac{1}{2}\delta)$. In $[-1, +1]$ we evidently have $\left|\dfrac{x-\xi_0}{1+\xi_0}\right| \leqq 1$. From (34c)

$$\int_{-1}^{1} \frac{F(t)^2}{\sqrt{(1-t^2)}}\, dt \geqq \frac{\pi}{k + 2\left[\frac{k\delta}{4}\right] + \frac{1}{2} + \frac{1}{2}\frac{\sin\left(2k + 4\left[\frac{k\delta}{4}\right] + 1\right)\varphi_0}{\sin\varphi_0}}; \tag{60}$$

on the other hand

$$\int_{-1}^{1} \frac{F(t)^2}{\sqrt{(1-t^2)}}\, dt < \left(1 - \frac{\epsilon^2}{4}\right)^{[\frac{1}{4}k\delta]} \int_{|t-\xi_0| \geqq \epsilon} \frac{f(t)^2}{\sqrt{(1-t^2)}}\, dt + \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} \frac{f(t)^2}{\sqrt{(1-t^2)}}\, dt, \tag{61}$$

as in $[-1, +1]$ the second factor of $F(x)$ is non-negative but $\leqq 1$. Making use of the hypothesis after the arrangement we obtain from (60) and (61)

$$\int_{-1}^{1} \frac{f(t)^2}{\sqrt{(1-t^2)}}\,dt > \int_{|t-\xi_0|\geqq\epsilon} \frac{f(t)^2}{\sqrt{(1-t^2)}}\,dt$$

$$> \frac{\pi}{\left(1-\frac{\epsilon^2}{4}\right)^{\frac{1}{2}k\delta}} \left[\frac{1}{k + \frac{k\delta}{2} + \frac{1}{2} + \frac{1}{2}\frac{\sin\left(2k + 4\left[\frac{k\delta}{4}\right] + 1\right)\varphi_0}{\sin\varphi_0}} \right.$$

$$\left. - \frac{1-\delta}{k + \frac{1}{2} + \frac{1}{2}\frac{\sin(2k+1)\varphi_0}{\sin\varphi_0}} \right] > \frac{\left(1+\frac{\epsilon^2}{4}\right)^{\frac{1}{2}k\delta}}{100k^2} \left[\frac{k\delta}{2} - \frac{2}{\sin\varphi_0}\right] > \frac{\delta\left(1+\frac{\epsilon^2}{4}\right)^{\frac{1}{2}k\delta}}{400k}.$$

Q.e.d.

According to what has been said before we may deduce asymptotic formulas for the Christoffel numbers belonging to $p(x)$.

THEOREM IX. *Let $p(x)\sqrt{(1-x^2)}$ be continuous and $p(x)\sqrt{(1-x^2)} \geqq m > 0$ in $[-1, +1]$; then, if $n \to \infty$, we have for any $x_\nu^{(n)}$ lying in* $-\left[1 - \frac{\log n}{n^2}\right]^{\frac{1}{2}} \leqq x_\nu^{(n)} \leqq \left[1 - \frac{\log n}{n^2}\right]^{\frac{1}{2}}$

$$k_\nu^{(n)} = \int_{-1}^{1} l_\nu(t)^2 p(t)\,dt \sim \frac{\pi p(x_\nu^{(n)})\sqrt{(1-x_\nu^{(n)2})}}{n}.$$

PROOF. First we show that we have for any ϵ and δ independent of n if $n > n_8(\delta, \epsilon)$ and $|x_\nu^{(n)}| \leqq \left[1 - \frac{64}{n^2\delta^2}\right]^{\frac{1}{2}}$

$$(62) \qquad \int\limits_{\substack{-1\leqq t\leqq +1\\ |t-x_\nu^{(n)}|\leqq\epsilon}} \frac{l_\nu(t)^2}{\sqrt{(1-t^2)}}\,dt > (1-\delta)\frac{\pi}{n - \frac{1}{2} + \frac{1}{2}\frac{\sin(2n-1)\vartheta_\nu^{(n)}}{\sin\vartheta_\nu^{(n)}}}.$$

Suppose the contrary, then by lemma X we had

$$\int_{-1}^{1} \frac{l_\nu(t)^2}{\sqrt{(1-t^2)}}\,dt > \frac{\delta\left(1+\frac{\epsilon^2}{4}\right)^{\frac{1}{2}\delta n}}{500n},$$

i.e. *a fortiori*

$$\int_{-1}^{1} l_\nu(t)^2 p(t)\,dt \geqq m\,\frac{\delta\left(1+\frac{\epsilon^2}{4}\right)^{\frac{1}{2}\delta n}}{500n},$$

which contradicts the minimum-property of Shohat if n is sufficiently large. Thus (62) is proved. But then by (62) for $n > n_9(\delta, \epsilon)$

$$\int_{-1}^{1} l_\nu(t)^2 p(t)\,dt > \min_{|x-x_\nu^{(n)}|\leq\epsilon} p(x)\sqrt{(1-x^2)} \int_{\substack{|x-x_\nu^{(n)}|\leq\epsilon \\ -1\leq x\leq+1}} \frac{l_\nu(t)^2}{\sqrt{(1-t^2)}}\,dt$$

(63)

$$> (1-\delta)\,\frac{\pi p(x_\nu^{(n)})\sqrt{(1-x_\nu^{(n)2})}}{n-\frac{1}{2}+\frac{1}{2}\frac{\sin(2n-1)\vartheta_\nu^{(n)}}{\sin\vartheta_\nu^{(n)}}}.$$

On the other hand by the minimum-property

$$\text{(64)}\qquad \int_{-1}^{1} l_\nu(t)^2 p(t)\,dt \leqq \int_{-1}^{1} \phi_{n-1}(t)^2 p(t)\,dt \sim \frac{\pi p(x_\nu^{(n)})\sqrt{(1-x_\nu^{(n)2})}}{n-\frac{1}{2}+\frac{1}{2}\frac{\sin(2n-1)\vartheta_\nu^{(n)}}{\sin\vartheta_\nu^{(n)}}}$$

by lemma IX, if we replace ξ_0 by $x_\nu^{(n)}$; (63) and (64) lead to theorem IX.

And now we may go over to the asymptotic representation of the fundamental functions.

THEOREM X. *In* $[-1, +1]$ *let* $p(x)\sqrt{(1-x^2)}$ *be continuous and such that* $p(x)\sqrt{(1-x^2)} \geqq m > 0$. *Then, if* $\epsilon > 0$, $n > n_0(\epsilon)$ $|x_\nu^{(n)}| \leqq \left[1-\frac{\log n}{n^2}\right]^{\frac{1}{2}}$, *for* $-1 \leqq x \leqq +1$ *we have uniformly* ($T_r(\cos\vartheta) = \cos r\vartheta$, $r > 1$)

$$|l_{\nu,n}(x) - \phi_{n-1}(x)| \equiv \left| l_\nu(x) - \frac{T_{n-1}(x_\nu^{(n)})T_n(x) - T_n(x_\nu^{(n)})T_{n-1}(x)}{\left(n-\frac{1}{2}+\frac{1}{2}\frac{\sin(2n-1)\vartheta_\nu^{(n)}}{\sin\vartheta_\nu^{(n)}}\right)(x-x_\nu^{(n)})}\right| < \epsilon.$$

PROOF. Let us consider, with the above $\phi_{n-1}(x)$, the integral

$$\text{(65)}\qquad I_\nu \equiv \int_{-1}^{1} [l_\nu(t) - \phi_{n-1}(t)]^2 p(t)\,dt.$$

As $\phi_{n-1}(x) \equiv \sum_{k=1}^{n} \phi(x_k^{(n)}) l_k(x)$ and $\phi_{n-1}(x_\nu^{(n)}) = 1$, we obtain

$$I_\nu = k_\nu^{(n)} - 2\sum_{d=1}^{n} \phi_{n-1}(x_d^{(n)}) \int_{-1}^{1} l_d(t) l_\nu(t) p(t)\,dt$$

(66)

$$+ \int_{-1}^{1} \phi_{n-1}(t)^2 p(t)\,dt = -k_\nu^{(n)} + \int_{-1}^{1} \phi_{n-1}(t)^2 p(t)\,dt.$$

But then by theorem IX and lemma IX, for $n \to \infty$, $|x_\nu^{(n)}| \leqq \left[1-\frac{\log n}{n^2}\right]^{\frac{1}{2}}$ uniformly in ν

$$\text{(67)}\qquad \lim_{n\to\infty} nI_\nu = 0.$$

From the premise and the remark to theorem VIII it follows for $[-1, +1]$ and $\nu = 1, 2, \cdots n$

$$| l_\nu(x) | \leqq c_{60} \frac{M}{m}.$$

By this and (55) for $l_\nu (\cos \vartheta) - \phi_{n-1} (\cos \vartheta) = \psi(\vartheta)$ we obtain

$$| \psi(\vartheta) | \leqq c_{62}(p).$$

But then by the theorem of Bernstein-Fejér, in $[0, \pi]$ we have

$$| \psi'(\vartheta) | \leqq c_{62}(p)n. \tag{68}$$

Let $\psi(\vartheta)$ assume its absolute maximum in $[0, \pi]$ at $\vartheta = \gamma_0$ and let this maximum value be D_ν. Then in the subinterval $i \equiv \left[\gamma_0 - \frac{D_\nu}{2c_{62}n}, \gamma_0 + \frac{D_\nu}{2c_{62}n}\right]$ of $[0, \pi]$, by (68) we have

$$|\psi(\vartheta)| > D_\nu - \frac{D_\nu}{2} = \frac{D_\nu}{2}$$

i.e.

$$I_\nu \equiv \int_{-1}^{1} [l_\nu(t) - \phi_{n-1}(t)]^2 p(t)\, dt = \int_0^\pi \psi(\vartheta)^2 p\, (\cos \vartheta) \sin \vartheta\, d\vartheta$$

$$> m \int_{(i)} \psi(\vartheta)^2\, d\vartheta > m \frac{D_\nu}{c_{62} n} \cdot \frac{D_\nu^2}{4}.$$

Thus by (67), for $n \to \infty$ *a fortiori*

$$\frac{mD_\nu^3}{4c_{62}} \to 0$$

i.e. $D_\nu \to 0$, which establishes the theorem.

From theorem IX and lemma VIII we easily deduce

THEOREM XI. *Let* $p(x)\sqrt{(1 - x^2)}$ *be continuous in* $[-1, +1]$ *with* $p(x)\sqrt{(1 - x^2)} \geqq m > 0$, *further let* $C(n)$, *for* $n \to \infty$ *arbitrarily slowly, tend to infinity; then for those roots* $\cos \vartheta_\nu^{(n)}$ *of the* n^{th} *polynomial orthogonal to* $p(x)$, *which satisfy*

$$\frac{C(n)}{n} \leqq \vartheta_k^{(n)} < \vartheta_{k+1}^{(n)} \leqq \pi - \frac{C(n)}{n},$$

we have uniformly in k

$$\lim_{n \to \infty} n(\vartheta_{k+1}^{(n)} - \vartheta_k^{(n)}) = \pi.$$

REMARK I. If we assume not $p(x)\sqrt{(1 - x^2)}$, but $p(x)$ to be $\geqq m$ and continuous and not in $[-1, +1]$ but in the subinterval $[a, b]$, then $\phi_{n-1}(x)$ is to be replaced by a polynomial of similar form but the Tchebycheff polynomials

$T_n(x)$ are to be replaced by $P_n(x)$ Legendre-polynomials. Theorem IX and X remain true for those k_ν and $l_\nu(x)$, for which $a + \epsilon \leqq x_\nu^{(n)} \leqq b - \epsilon$ and $a + \epsilon \leqq x \leqq b - \epsilon$.

REMARK II. If we attribute again continuity and positiveness in $[-1, +1]$ to $p(x)\sqrt{(1 - x^2)}$, it is probable, that theorem X holds for the fundamental functions belonging to *every* $x_\nu^{(n)}$. Theorem XI does not hold for *every* $x_\nu^{(n)}$ in the original form; the difference $\vartheta_{k+1}^{(n)} - \vartheta_k^{(n)}$ will be asymptotically equal to the distance between $\vartheta_k^{(n)}$ and that root of

$$\cos (n - 1)\vartheta_k^{(n)} \cos n\vartheta - \cos n\vartheta_k^{(n)} \cos (n - 1)\vartheta = 0,$$

which is nearest, on the right, to $\vartheta_k^{(n)}$.

4.

In this section we consider the number of roots of polynomials in a given interval. We already mentioned in the introduction that if

$$\overline{\lim_{n\to\infty}} \, [| \, l_\nu(x) \, |]^{1/n} \leqq 1 \tag{69}$$

uniformly for $-1 \leqq x \leqq +1$ and $\nu = 1, 2, \cdots, n$, then the fundamental points of $\mathfrak{M}$ are uniformly distributed. We present an elementary proof for this Fejérian-theorem. Here and also later a theorem of M. Riesz[21] plays a most important part, so—because of its shortness—we reproduce it as

LEMMA XI. *If a trigonometric polynomial of order n, $f(\vartheta)$, takes its absolute maximum in $[0, 2\pi]$ at $\vartheta = \vartheta_0$, then there is no root of $f(\vartheta)$ in $\left[\vartheta_0 - \frac{\pi}{2n}, \vartheta_0 + \frac{\pi}{2n}\right]$.*

Suppose the theorem to be untrue. Without loss of generality let $\vartheta_0 = 0$, $f(0) = 1$ and suppose, that the nearest root, the distance of which is less than $\pi/2n$, lies to the right. But in this case the curves $y = f(\vartheta)$ and $y = \cos n\vartheta$ would have at $\vartheta = 0$ at least a double point of intersection and by the premise a third one in $[0, \pi/n]$. In any of the intervals $[\pi/n, 2\pi/n]$, $[2\pi/n, 3\pi/n]$, $\cdots$, $[(2n - 2)\pi/n, (2n - 1)\pi/n]$ they would also have at least one intersection; hence the trigonometric polynomial $f(\vartheta) - \cos n\vartheta$ of order n had in $[0, 2\pi]$ $(2n + 1)$ roots, which is impossible.

COROLLARY. *If a trigonometric polynomial of order n takes its absolute maximum between two real roots, the distance of these roots cannot be less than π/n. The statement holds for all n.*

THEOREM XII. *If upon the matrix $\mathfrak{M}$*

$$[| \, l_k(x) \, |]^{1/n} \leqq 1 + \epsilon, \qquad k = 1, 2, \cdots n, \qquad -1 \leqq x \leqq +1,$$

for $n > n_{10}(\epsilon)$, then we have for any $0 \leqq \alpha < \beta \leqq \pi$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{\nu \\ \alpha \leqq \vartheta_\nu^{(n)} \leqq \beta}} 1 = \frac{\beta - \alpha}{\pi}.$$

[21] M. Riesz: *Eine trigonometrische Interpolationsformel usw.*, Jahresbericht der deutschen Mathematischerver, 1915, pp. 354–368.

PROOF. If the theorem would be untrue, we had in $[0, \pi]$ a subinterval $[\alpha, \beta]$ and a c_{63} such, that there would be an infinity of integers $n_1 < n_2 < \cdots$ for which the number of the n_k^{th} fundamental points lying in $[\alpha, \beta]$ is less than $\frac{1}{\pi}(\beta - \alpha - c_{63})n_k$. We may assume $c_{63} < \frac{\beta - \alpha}{6}$ and write instead of n_k simply n. Let

$$\varphi_k = \alpha + k\frac{\pi}{n+1} \tag{70a}$$

where k runs over the integers (positive, negative and 0), for which φ_k lies in $\left[0, \alpha + \frac{c_{63}}{4}\right]$; for $n \to \infty$ the number of these φ_k asymptotically equals $\frac{1}{\pi}\left(\alpha + \frac{c_{63}}{4}\right)n$. Let further

$$\psi_l = \beta + l\frac{\pi}{n+1}, \tag{70b}$$

where l runs over the integers (positive, negative and 0) for which ψ_l lies in $\left[\beta - \frac{c_{63}}{4}, \pi\right]$; for $n \to \infty$ the number of these asymptotically equals $\frac{1}{\pi}\left(\pi - \beta + \frac{c_{63}}{4}\right)n$. Let further

$$G(x) \equiv \prod_{\nu}{}' (x - \cos\vartheta_\nu^{(n)}) \prod_k (x - \cos\varphi_k) \prod_l (x - \cos\psi_l), \tag{71}$$

where $\prod'$ is to be extended over the $\vartheta_k^{(n)}$ lying in $[\alpha, \beta]$. The degree of $G(x)$ is by the premise and the definition of φ_k and ψ_l, for sufficiently large n,

$$< \frac{1}{\pi}(\beta - \alpha - c_{63})n + \frac{1}{\pi}\left(\alpha + \frac{c_{63}}{3}\right)n + \frac{1}{\pi}\left(\pi - \beta + \frac{c_{63}}{3}\right)n = \left(1 - \frac{c_{63}}{3\pi}\right)n. \tag{72}$$

As the order of the trigonometric polynomial $G(\cos\vartheta)$ is less than n and the distance of its consecutive roots in $\left[0, \alpha + \frac{c_{63}}{4}\right]$ and $\left[\beta - \frac{c_{63}}{4}, \pi\right]$ is less than π/n, $G(\cos\vartheta)$ takes, by lemma XI, its absolute maximum in $\left[\alpha + \frac{c_{63}}{4}, \beta - \frac{c_{63}}{4}\right]$, at a place $\vartheta = \gamma$ say. Let finally

$$F(x) = G(x)\left\{1 - \frac{(x - \cos\gamma)^2}{4}\right\}^{[c_{63}n/8\pi]}, \tag{73}$$

where the brackets in the exponent denote the greatest integer contained. Then, by (72) the degree of $F(x)$ is

$$< \left(1 - \frac{c_{63}}{3\pi}\right)n + \frac{c_{63}}{4\pi}n = \left(1 - \frac{c_{63}}{12\pi}\right)n \tag{74}$$

and, like $G(x)$, $F(x)$ takes its absolute maximum at $\vartheta = \gamma$, too. Then

$$F(x) = \sum_{\nu=1}^{n} F(x_\nu^{(n)}) l_\nu(x)$$

i.e. for $x = \cos\gamma$

$$|G(\cos\gamma)| = |F(\cos\gamma)| = \left|\sum_{\nu=1}^{n} F(x_\nu^{(n)}) l_\nu(\cos\gamma)\right| = \Big|\sum_\nu{}'' F(x_\nu^{(n)}) l_\nu(\cos\gamma)\Big|,$$

where, by definition of $F(x)$, $\sum''$ refers only to those $\vartheta_\nu^{(n)}$, which are not in $[\alpha, \beta]$. But then, by the hypothesis, for $n > n_{10}(\epsilon)$

$$|G(\cos\gamma)| < (1+\epsilon)^n \sum_\nu{}'' |F(x_\nu^{(n)})|$$

$$< (1+\epsilon)^n \sum_\nu{}'' |G(x_\nu^{(n)})| \left(1 - \frac{(x_\nu^{(n)} - \cos\gamma)^2}{4}\right)^{c_{63}n/8\pi}.$$

As by definition of γ

$$|G(\cos\gamma)| \geqq |G(x_\nu^{(n)})|, \qquad \nu = 1, 2, \cdots n,$$

we have *a fortiori*

$$1 \leqq (1+\epsilon)^n \sum_\nu{}'' \left(1 - \frac{(x_\nu^{(n)} - \cos\gamma)^2}{4}\right)^{c_{63}n/8\pi}$$

$$< n(1+\epsilon)^n \max\left[\left(1 - \frac{\left(\cos\alpha - \cos\left(\alpha + \frac{c_{63}}{4}\right)\right)^2}{4}\right)^{c_{63}n/8\pi}, \left(1 - \frac{\left(\cos\left(\beta - \frac{c_{63}}{4}\right) - \cos\beta\right)^2}{4}\right)^{c_{63}n/8\pi}\right],$$

which is, with sufficiently small ϵ, untrue for $n > n_{11}(\epsilon)$ and thus the theorem is proved.

By theorem XII and lemma VII we obtain

THEOREM XIII. *Let $p(x)$ be in $[-1, +1]$ non-negative, and L-integrable further suppose that its roots form an aggregate of measure 0; then for the roots of the n^{th} polynomial $\cos\vartheta_\nu^{(n)}$ belonging to $p(x)$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{\nu \\ \alpha \leqq \vartheta_\nu^{(n)} \leqq \beta}} 1 = \frac{\beta - \alpha}{\pi},$$

where $[\alpha, \beta]$ denotes any fixed subinterval of $[0, \pi]$.

If we want to secure uniform distribution with error-term, we require the following Fejérian

THEOREM XIV. *If for a matrix*

$$|l_\nu(x)| \leqq D, \qquad -1 \leqq x \leqq +1, \qquad \nu = 1, 2, \cdots n, \qquad n = 1, 2, \cdots,$$

then for the elements of the n^{th} row, $\cos \vartheta_\nu^{(n)}$ ($\nu = 1, 2, \cdots, n$) and for any subinterval $[\alpha, \beta]$ of $[0, \pi]$ satisfying $(\beta - \alpha)n \geqq c_{69}(D, \epsilon)$ we have

$$\left| \sum_{\alpha \leqq \vartheta_\nu^{(n)} \leqq \beta} 1 - \frac{\beta - \alpha}{\pi} n \right| < c_{70}(D, \epsilon)\{(\beta - \alpha)n\}^{\frac{1}{2}+\epsilon},$$

where we emphasize that $c_{70}(D, \epsilon)$ is independent of α and β, too.

PROOF. Consider first the upper estimate. Let $[\alpha, \beta]$ be a fixed subinterval—without loss of generality we may suppose $\beta - \alpha \leqq \frac{\pi}{4}$ —$\left[\frac{\beta - \alpha}{\pi} n\right] = k$ and assume

$$\sum_{\alpha \leqq \vartheta_\nu^{(n)} \leqq \beta} 1 = k + l. \tag{75}$$

From α rightwards let us cut off k-times the distance π/n and leftwards $[\frac{1}{4}l]$-times until we reach A, further from β rightwards also $[\frac{1}{4}l]$-times until we reach B; if $\alpha - \pi/n\,[\frac{1}{4}l] \leqq 0$ or $\beta + \pi/n\,[\frac{1}{4}l] \geqq \pi$, set $A = 0$, correspondingly $B = \pi$. Let the points of division be φ_ν. Let further

$$G_1(x) = \prod_\nu (x - \cos \varphi_\nu) \prod_\mu{}' (x - \cos \vartheta_\mu^{(n)}), \tag{76}$$

where $\prod'$ in the second product runs over the $\vartheta_\nu^{(n)}$ lying outside of $[\alpha, \beta]$. In this case $G_1(\cos \vartheta)$ is a pure cosine-polynomial, whose order $\leqq n - k - l + k + 2[\frac{1}{4}l] < n - \frac{1}{2}l$. Then the distance of two consecutive roots of $G_1(\cos \vartheta)$ in $[A, B]$ is not greater than π/n, i.e., by lemma XI $G_1(\cos \vartheta)$ takes its absolute maximum outside of this interval, at a point $\vartheta = \lambda$, say. Let

$$F_1(x) = G_1(x) T_{[\frac{1}{4}l]}\left(-1 + 2\,\frac{x - \cos \beta}{\cos \alpha - \cos \beta}\right), \tag{77}$$

where $T_{[\frac{1}{4}l]}(\cos \varphi) = \cos\,[\frac{1}{4}l]\varphi$. As the degree of $F_1(x)$ is not greater, than $n - \frac{1}{4}l$, we represent $F_1(x)$ by the Lagrange-interpolatory polynomial formed upon the n^{th} row of the matrix $\mathfrak{M}$ and obtain

$$F_1(x) \equiv \sum_{\nu=1}^{n} F_1(x_\nu^{(n)}) l_\nu(x)$$

i.e.

$$| F_1(\cos \lambda) | = \left| \sum_{\nu=1}^{n} F_1(x_\nu^{(n)}) l_\nu(\cos \lambda) \right| = | \sum_\nu{}'' F_1(x_\nu^{(n)}) l_\nu(\cos \lambda) |, \tag{78}$$

where $\sum''$ runs over integers, for which $\vartheta_\nu^{(n)}$ lies in $[\alpha, \beta]$. As for any $1 \leqq \nu \leqq n$

$$| G_1(\cos \lambda) | \geqq | G_1(x_\nu^{(n)}) |,$$

by (78) and the hypothesis we have

$$\left| T_{[\frac{1}{4}l]}\left(-1 + 2\,\frac{\cos\lambda - \cos\beta}{\cos\alpha - \cos\beta}\right)\right| \tag{79}$$

$$\leqq D \sum_{\alpha \leqq \vartheta_\nu^{(n)} \leqq \beta} \left| T_{[\frac{1}{4}l]}\left(-1 + 2\,\frac{\cos\vartheta_\nu^{(n)} - \cos\beta}{\cos\alpha - \cos\beta}\right)\right|.$$

Each term of the right-hand-side-sum is not greater than 1 and the number of terms is less[8] than $c_{71}(D)k$, i.e. by (79)

$$\left| T_{[\frac{1}{4}l]}\left(-1 + 2\,\frac{\cos\lambda - \cos\beta}{\cos\alpha - \cos\beta}\right)\right| \leqq c_{72}(D)\cdot k. \tag{80}$$

If A or B fall upon one of the borders of $[0, \pi]$ λ cannot be there, thus from λ it may be assumed

$$\min\left(|\lambda - \alpha|, |\lambda - \beta|\right) \geqq \frac{\pi}{n}\left[\frac{l}{4}\right].$$

Without loss of generality we may suppose $\lambda \geqq \beta + [\frac{1}{4}l]\,\pi/n$. Then we have

$$\left|-1 + 2\,\frac{\cos\lambda - \cos\beta}{\cos\alpha - \cos\beta}\right| > 1 + 2\,\frac{\cos\beta - \cos\left(\beta + \dfrac{\pi}{n}\left[\dfrac{l}{4}\right]\right)}{\cos\alpha - \cos\beta}. \tag{81}$$

Suppose $0 \leqq \alpha_1 < \alpha_2 < \alpha_3 \leqq \pi$, $\alpha_3 - \alpha_2 = \delta_1$, $\alpha_2 - \alpha_1 = \delta_2$; then we obviously have

$$\frac{\cos\alpha_2 - \cos\alpha_3}{\cos\alpha_1 - \cos\alpha_2} = \frac{\sin\dfrac{\delta_1}{2}\sin\left(\alpha_2 + \dfrac{\delta_1}{2}\right)}{\sin\dfrac{\delta_2}{2}\sin\left(\alpha_2 - \dfrac{\delta_2}{2}\right)} \tag{82}$$

$$\geqq \frac{\sin^2\dfrac{\delta_1}{2}}{\sin\dfrac{\delta_2}{2}\sin\left(\delta_1 + \dfrac{\delta_2}{2}\right)} > c_{73}\,\frac{\delta_1^2}{\delta_2(\delta_1 + \delta_2)}.$$

Case I. $l \geqq k \geqq 20$. We apply (82) with $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \beta + [\frac{1}{4}l]\pi/n$; then we have

$$\delta_1 \geqq \frac{\pi}{n}\left(\frac{l}{4} - 1\right) \geqq \frac{\pi}{n}\left(\frac{k}{4} - 1\right) \geqq \frac{k\pi}{5n} > \frac{1}{6}\,\delta_2$$

and

$$\frac{\cos\beta - \cos\left(\beta + \left[\dfrac{l}{4}\right]\dfrac{\pi}{n}\right)}{\cos\alpha - \cos\beta} > \frac{c_{73}}{7}\,\frac{\delta_1}{\delta_2} > c_{75}\,\frac{l}{k} > c_{75}. \tag{83}$$

As $T_\nu(x)$ increases monotonely for $x \geqq 1$ and satisfies for $\rho \geqq 0$ the inequality

$$T_\nu(1 + \rho) \geqq \tfrac{1}{2}\left(1 + \sqrt{(2\rho)}\right)^\nu, \tag{84}$$

we have from (80), (81), (83) and (84)

$$c_{76}^{\frac{1}{4}l} \leqq c_{72}(D)\ k$$

$$l < c_{77}(D) \log k$$

which together with $k \leqq l$ gives $k \leqq k_0(D)$.

CASE II. $l < k, k \geqq k_0(D)$. In this case $\delta_1 \leqq \frac{l\pi}{4n} < \frac{k\pi}{4n} < \frac{\delta_2}{4}$ and

$$\frac{\cos\beta - \cos\left(\beta + \frac{\pi}{n}\left[\frac{l}{4}\right]\right)}{\cos\alpha - \cos_{\bullet}\beta} \geqq c_{78}\left(\frac{\delta_1}{\delta_2}\right)^2 > c_{79}\left(\frac{l}{k}\right)^2.$$

From (90), (81) and (84) we have

$$\frac{1}{2}\left(1 + 2\sqrt{c_{79}}\frac{l}{k}\right)^{[\frac{1}{4}l]} \leqq T_{[\frac{1}{4}l]}\left(1 + 2c_{79}\left(\frac{l}{k}\right)^2\right) < c_{72}(D)k$$

$$l < c_{80}(D)\ (k \log k)^{\frac{1}{2}}. \quad \text{Q.e.d.}$$

Let us now consider the lower estimate; when again $\left[\frac{\beta - \alpha}{\pi} n\right] = k$, let

(85) $$\sum_{\alpha \leqq \vartheta_\nu^{(n)} \leqq \beta} 1 = k - l.$$

Thus we have to estimate l from above. Now cut off from α to 0 leftward distances of the length $\frac{\pi}{n}\left(1 + \frac{1}{10k^{\frac{1}{2}-\epsilon}}\right)$, as many times as possible and rightward $[\frac{1}{4}l]$-times until A'; furthermore, from β rightward to π as many times, as possible and leftwards $[\frac{1}{4}l]$-times until B'. As

$$2\left[\frac{l}{4}\right]\frac{\pi}{n} \leqq \frac{l\pi}{2n} < \frac{k\pi}{2n} \leqq \frac{\beta - \alpha}{2},$$

we have $A' < B'$. We denote the points of division by φ_μ'. Let

(86) $$G_2(x) = \prod_\mu (x - \cos \varphi_\mu') \cdot \prod_\kappa (x - \cos \vartheta_\kappa^{(n)}),$$

where the second product refers to the $\vartheta_\kappa^{(n)}$'s lying inside of $[\alpha, \beta]$. In this case $G_2(\cos \vartheta)$ is a pure cosine-polynomial of order

$$k - l + 2\left[\frac{l}{4}\right] + \left[\frac{n\alpha}{\pi\left(1 + \frac{1}{10k^{\frac{1}{2}-\epsilon}}\right)}\right] + \left[\frac{n(\pi - \beta)}{\pi\left(1 + \frac{1}{10k^{\frac{1}{2}-\epsilon}}\right)}\right]$$

$$< k - \frac{l}{2} + \frac{n}{\pi\left(1 + \frac{1}{10k^{\frac{1}{2}-\epsilon}}\right)}(\pi - (\beta - \alpha))$$

$$< k - \frac{l}{2} + \left(\frac{10nk^{\frac{1}{2}-\epsilon}}{\pi(10k^{\frac{1}{2}-\epsilon} + 1)}\right)\left(\pi - \frac{k\pi}{n}\right)$$

$$= n + \frac{n}{10k^{\frac{1}{2}-\epsilon}}\cdot\frac{10k^{\frac{1}{2}-\epsilon}}{1 + 10k^{\frac{1}{2}-\epsilon}} + \frac{k^{\frac{1}{2}+\epsilon}}{10 + k^{\epsilon-\frac{1}{2}}} - \frac{l}{2} < n - \frac{n}{10k^{\frac{1}{2}-\epsilon} + 1} + \frac{k^{\frac{1}{2}+\epsilon}}{10} - \frac{l}{2}$$

Suppose that $l > \frac{1}{5}k^{\frac{1}{2}+\epsilon}$ is true; then the order of $G_2(\cos\vartheta)$ would be

(87) $$< n - \frac{n}{10k^{\frac{1}{2}-\epsilon}+1}$$

and after lemma XI the place $\vartheta = \lambda$, on which $G_2(\cos\vartheta)$ takes its absolute maximum could be only in a root interval, whose length is

$$\geqq \frac{\pi}{\text{degree of } G_2(\cos\vartheta)} > \frac{\pi}{n}\left(1 + \frac{1}{10k^{\frac{1}{2}-\epsilon}}\right).$$

But on the outside of $[A', B']$ the distance of the consecutive roots of $G_2(\cos\vartheta)$ is $\leqq \frac{\pi}{n}\left(1 + \frac{1}{10k^{\frac{1}{2}-\epsilon}}\right)$; so λ could be only in $[A', B']$. Without loss of generality let $0 \leqq \lambda \leqq \frac{\pi}{2}$; then we have for $0 \leqq \vartheta \leqq \pi$

$$\left|\frac{1}{\sin\frac{\vartheta+\lambda}{2}}\right| < \frac{c_{81}}{|\vartheta-\lambda|}, \qquad \left|\frac{1}{\sin\frac{\vartheta-\lambda}{2}}\right| < \frac{c_{81}}{|\vartheta-\lambda|}.$$

With this c_{81} let M be the greatest odd number not exceeding $400\, c_{81} \frac{n}{10k^{\frac{1}{2}+\epsilon}+1} + 1$, $N = 2\left[\frac{k^{2\epsilon}}{800c_{81}}\right]$

(88a) $$\psi(\cos\vartheta) = \frac{1}{M^N}\left\{\left(\frac{\sin M\frac{\vartheta+\lambda}{2}}{\sin\frac{\vartheta+\lambda}{2}}\right)^N + \left(\frac{\sin M\frac{\vartheta-\lambda}{2}}{\sin\frac{\vartheta-\lambda}{2}}\right)^N\right\}$$

and finally

(88b) $$F_2(x) = G_2(x)\psi(x).$$

The degree of the polynomial $\psi(x)$ is $\frac{M-1}{2}N < \frac{200c_{81}n}{10k^{\frac{1}{2}+\epsilon}}\frac{k^{2\epsilon}}{400c_{81}} = \frac{n}{20k^{\frac{1}{2}-\epsilon}} < \frac{n}{10k^{\frac{1}{2}-\epsilon}+1}$ and according to (87) and (886) the degree of $F_2(x)$ is $\leqq n-1$, i.e.

(89) $$|F_2(\cos\lambda)| = \left|\sum_{\nu=1}^{n} F_2(x_\nu^{(n)})l_\nu(\cos\lambda)\right| = \left|\sum_\nu{}' F_2(x_\nu^{(n)})l_\nu(\cos\lambda)\right| < D\sum_\nu{}' |F_2(x_\nu^{(n)})|,$$

the last two summations refer to the $\vartheta_\nu^{(n)}$'s lying outside of $[\alpha, \beta]$. As $|G_2(\cos\lambda)| \geqq |G_2(x_\nu^{(n)})|$ $(\nu = 1, 2, \cdots, n)$, we should have from (89)

(90) $$|\psi(\cos\lambda)| < D\sum_\nu{}' |\psi(\cos\vartheta_\nu^{(n)})|.$$

It is easy to see that

$$\psi(\cos\lambda) \geqq 1, \qquad |\psi(\cos\vartheta)| < 2\left(\frac{c_{81}}{M(\vartheta-\lambda)}\right)^N,$$

and comparing this with (90) and with our theorem, we have

$$\begin{aligned} 1 &< 2D\left(\frac{c_{81}}{M}\right)^N {\sum_\nu}' \frac{1}{(\vartheta_\nu^{(n)}-\lambda)^N} \\ &< 2D\left(\frac{c_{81}}{M}\right)^N\Bigg[\frac{1}{(\alpha-\lambda)^N}+\frac{1}{\left(\alpha-\frac{c_{82}\pi}{5n}-\lambda\right)^N}+\frac{1}{\left(\alpha-2\frac{c_{82}\pi}{5n}-\lambda\right)^N}+\cdots \\ &\quad +\frac{1}{(\beta-\lambda)^N}+\frac{1}{\left(\beta+\frac{c_{82}\pi}{5n}-\lambda\right)^N}+\frac{1}{\left(\beta+2\frac{c_{82}\pi}{5n}-\lambda\right)^N}+\cdots\Bigg]. \end{aligned} \tag{91}$$

Bur for $l > c_{83}$ we have

$$|\lambda-\alpha| > \frac{l\pi}{5n}, \qquad |\lambda-\beta| > \frac{l\pi}{5n}$$

and (91) gives $k > c_{84}$

$$1 < 4D\left(\frac{5c_{81}n}{\pi M}\right)^N\left[\frac{1}{l^N}+\frac{1}{(l+c_{82})^N}+\frac{1}{(l+2c_{82})^N}+\cdots\right] < \frac{4D}{c_{82}}\left(\frac{10c_{81}n}{\pi Ml}\right)^N k \tag{92}$$

further for $l > c_{85}$ we have

$$M > \frac{40c_{81}n}{k^{\frac{1}{2}+\epsilon}}, \qquad N > \frac{k^{2\epsilon}}{800c_{81}},$$

which gives from (92)

$$1 < \frac{4D}{c_{82}}k\left(\frac{k^{\frac{1}{2}+\epsilon}}{4\pi l}\right)^N < \frac{4D}{c_{82}}k\left(\frac{5}{4\pi}\right)^{k^{2\epsilon}/800c_{81}}.$$

This means a contradiction for $l > c_{86}(D, \epsilon)$. Q.c.d.

THEOREM XV. *If upon the matrix* $\mathfrak{M}$

$$|l_k(x)| \leqq c_{87}n^{c_{88}}, \qquad -1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots, n, \qquad n = 1, 2, \cdots,$$

then for any subinterval $[\alpha, \beta]$ *of* $[0, \pi]$

$$\left|\sum_{\substack{\nu \\ \alpha\leqq\vartheta_\nu^{(n)}\leqq\beta}} 1 - \frac{\beta-\alpha}{\pi}n\right| < c_{89}(c_{87}, c_{88}, \epsilon)n^{\frac{1}{2}+\epsilon}.$$

PROOF. It will be sufficient to prove that for any subinterval the upper estimate holds, as in this case the respective application for the intervals $[0, \alpha]$ and $[\beta, \pi]$ leads to

$$\left|\sum_{\substack{\nu \\ 0\leqq\vartheta_\nu^{(n)}\leqq\alpha}} 1 - \frac{\alpha}{\pi}n\right| < c_{89}n^{\frac{1}{2}+\epsilon}$$

and respectively to

$$\left| \sum_{\beta \leq \vartheta_\nu^{(n)} \leq \pi} 1 - \frac{\pi - \beta}{\pi} n \right| < c_{89} n^{\frac{1}{2}+\epsilon}$$

i.e.

$$\sum_{\alpha \leq \vartheta_\nu^{(n)} \leq \beta} 1 = n - \sum_{0 \leq \vartheta_\nu^{(n)} \leq \alpha} 1 - \sum_{\beta \leq \vartheta_\nu^{(n)} \leq \pi} 1 > \frac{\beta - \alpha}{\pi} n - 2c_{89} n^{\frac{1}{2}+\epsilon},$$

which establishes the lower estimate. The proof of the upper estimate is completely analogous to that applied in theorem XIV.

For a sequence of strongly normal polynomials, theorem XIV immediately presents the uniform distribution of roots in $[0, \pi]$ with the error-terms mentioned, but we do not state this in a separate theorem. For orthogonal polynomials according to (50) and to the first remark appended to theorem VIII we may state that if throughout $[-1, +1]$ we have $p(x) \geqq m > 0$ and L-integrable, then

$$| l_\nu(x) | \leqq \left[\frac{1}{m} \int_{-1}^{1} p(t)\, dt \right]^{\frac{1}{2}} \cdot n, \quad \nu = 1, 2, \cdots n, \quad n = 1, 2, \cdots, \quad -1 \leqq x \leqq +1,$$

or respectively, if in $[-1, +1]$ is $p(x)$ L-integrable and $m \leqq p(x) \sqrt{(1 - x^2)} \leqq M$ then

$$| l_\nu(x) | \leqq \left[c_{66} \frac{M}{m} \right]^{\frac{1}{2}}, \quad -1 \leqq x \leqq +1, \quad \nu = 1, 2, \cdots n, \quad n = 1, 2, \cdots.$$

Hence theorem XV and XIV are applicable and we obtain following two theorems:

THEOREM XVI. *If the weight function is L-integrable and satisfies in* $[-1, +1]$ $p(x) \geqq m > 0$ *and the roots of the* n^{th} *orthogonal polynomial are* $\cos \vartheta_\nu^{(n)}$, *then for any* $[\alpha, \beta]$ *of* $[0, \pi]$ *we have*

$$\left| \sum_{\alpha \leq \vartheta_\nu^{(n)} \leq \beta} 1 - \frac{\beta - \alpha}{\pi} n \right| < c_{90}(p, \epsilon) n^{\frac{1}{2}+\epsilon}.$$

THEOREM XVII. *If the weight function $p(x)$ is L-integrable and satisfies in* $[-1, +1]$ $0 < m \leqq p(x) \sqrt{(1 - x^2)} \leqq M$, *then for the roots of the* n^{th} *orthogonal polynomial* $\cos \vartheta_\nu^{(n)}$ *and for any subinterval* $[\alpha, \beta]$ *of* $[0, \pi]$ *we have*

$$\left| \sum_{\alpha \leq \vartheta_\nu^{(n)} \leq \beta} 1 - \frac{\beta - \alpha}{\pi} n \right| < c_{91}(p, \epsilon) \{ (\beta - \alpha) n \}^{\frac{1}{2}+\epsilon}$$

if $n(\beta - \alpha) > c_{92}(p, \epsilon)$.

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