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ON EXTREMAL PROPERTIES OF THE DERIVATIVES OF POLYNOMIALS

BY P. ERDÖS

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Throughout this paper let $f(x)$ be a polynomial of degree n satisfying the inequality $|f(x)| \leqq 1$ for $-1 \leqq x \leqq 1$. A. Markoff[1] showed that for $-1 \leqq x \leqq 1$, $|f'(x)| \leqq n^2$. Equality is obtained only for the Tchebicheff polynomial $T_n(x)$ for $x = \pm 1$. In the present paper we shall prove the following analogous

THEOREM.[2] *Suppose $f(x)$ has only real roots and no roots in $-1, +1$; then for $-1 \leqq x \leqq 1$, $|f'(x)| < \frac{1}{2}en$. This is the best possible result.*

PROOF. We distinguish two cases. First we assume that $f'(x)$ has a root x_0 such that $-1 \leqq x_0 \leqq 1$. A simple linear transformation enables us to put $\max f(x) = 1$ for $-1 \leqq x \leqq 1$ and $f(-1) = f(+1) = 0$. We prove the following

LEMMA. *Suppose $f(x)$ has only real roots none of which lie in the interval $-1, +1$ and let $f(-1) = f(+1) = 0$, $\max\limits_{-1 \leqq x \leqq 1} f(x) = 1$, then*

$$f(x) < e\,\frac{1+x}{1+x_0}.$$

PROOF. Put $x_0 - x = d$. Without loss of generality we may assume $-1 < x < x_0$. Then if x_i $(i = 1, 2, \cdots, n)$, $x_1 = -1$, $x_2 = +1$ denote the roots of $f(x)$, we have

$$(1) \quad 1 = f(x_0) = c \prod_{i \leqq n} (x_0 - x_i), \quad \text{and} \quad \sum_{i \leqq n} \frac{1}{x_0 - x_i} = 0 (f(x) = cx^n + \cdots).$$

Now

$$f(x) = c \prod_{i \leqq n} (x - x_i) = c(1 + x) \prod_{i=2}^{n} (-d + x_0 - x_i).$$

Thus

$$f(x) = \frac{f(x)}{f(x_0)} = \frac{1+x}{1+x_0} \prod_{i=2}^{n} \left(1 - \frac{d}{x_0 - x_i}\right).$$

But by (1)

$$\sum_{i=2}^{n} \frac{1}{x_0 - x_i} = -\frac{1}{1+x_0}.$$

Hence from $\prod (1 + a_i) < \exp \sum a_i$, we have

$$f(x) < \frac{1+x}{1+x_0} \exp \frac{d}{1+x_0} < e\,\frac{1+x}{1+x_0}. \quad \text{q.e.d.}$$

[1] A. Markoff, Abh. Akad. Wiss. St. Petersburg, 1889, vol. 62, pp. 1–24.

[2] The same result was obtained by Mr. Eröd by a different method.

By a slightly longer calculation we could show that

$$(2) \qquad f(x) \leqq \left(1 - \frac{1}{n}\right)^{-n+1} \frac{1 + x}{1 + x_0},$$

equality occurring only for

$$f(x) = \frac{n}{2^n \left(1 - \frac{1}{n}\right)^{n-1}} (x + 1)(x - 1)^{n-1}$$

and

$$f(x) = \frac{n}{2^n \left(1 - \frac{1}{n}\right)^{n-1}} (x + 1)^{n-1}(x - 1).$$

In these cases

$$f'(1) = \frac{n}{2\left(1 - \frac{1}{n}\right)^{n-1}} \rightarrow \frac{e}{2} n.$$

Thus it can be shown by an easy calculation that the constant e of our Lemma cannot be improved. We have

$$(3) \qquad \sum_{x_i > x_0} \frac{1}{x_i - x_0} = \sum_{x_i < x_0} \frac{1}{x_0 - x_i} \leqq \min\left(\frac{k}{1 - x_0}, \frac{n - k}{1 + x_0}\right) \leqq \frac{n}{2},$$

where k denotes the number of roots $> x_0$ of $f(x)$. Now by our Lemma we obtain

$$f(x) = f(x) \sum_{i=1}^{n} \frac{1}{x - x_i} < e \frac{1 + x}{1 + x_0} \sum_{x_i < x_0} \frac{1}{x - x_i} \leqq e \frac{1 + x}{1 + x_0} \frac{1 + x_0}{1 + x} \sum_{x_i < x_0} \frac{1}{x_0 - x_i}$$

$$< \frac{e}{2} n,$$

by (3) which proves our Theorem for the first case. From (2) we can deduce that

$$|f'(x)| \leqq \frac{n}{2\left(1 - \frac{1}{n}\right)^{n-1}} \quad \text{for} \quad -1 \leqq x \leqq 1.$$

Suppose now that $f'(x) \neq 0$ for $-1 \leqq x \leqq 1$. A linear transformation enables us to put $f(-1) = 0$, $f(+1) = 1$. The roots of $f(x)$ are $x_1 = -1$, x_2, x_3, $\cdots$, x_n. Now

$$f'(1) \leqq \sum_{x_i < 1} \frac{1}{1 - x_i} \leqq \frac{n}{2}.$$

We have as in our Lemma

$$f(x) = \frac{f(x)}{f(1)} = \frac{1+x}{2}\prod_{i=2}^{n}\left(\frac{x-x_i}{1-x_i}\right) = \frac{1+x}{2}\prod_{i=2}^{n}\left(1 - \frac{1-x}{1-x_i}\right)$$
$$< \frac{1+x}{2}\exp\frac{1-x}{2} < e\,\frac{1+x}{2}.$$

In the above we used the fact that

$$\frac{1}{2} + \sum_{i=2}^{n}\frac{1}{1-x_i} \geqq 0,$$

along with the inequality $\prod (1 + a_j) < \exp \sum a_j$. Thus

$$f'(x) = f(x)\sum_{i=1}^{n}\frac{1}{x-x_i} < e\,\frac{1+x}{2}\sum_{x_i<x}\frac{1}{x-x_i} \leqq e\,\frac{1+x}{2}\,\frac{2}{1+x}\sum_{x_i<1}\frac{1}{1-x_i} \leqq \frac{e}{2}n.$$

This completes the proof of our Theorem. A slightly longer calculation would show that in the second case $f'(x) \leqq \frac{1}{2}n$, where equality holds only for $f(x) = (1 \pm x)^n/2^n$. A theorem of S. Bernstein[3] states that for $-1 \leqq x \leqq 1$, $|f'(x)| \leqq n/(1-x^2)^{\frac{1}{2}}$. For every subinterval of -1, $+1$ this result is very much better than the theorem of Markoff. By analogy we prove

THEOREM.[4] *Let $f(x)$ be a (real valued) polynomial having no root in the interior of the unit circle; then for $-1 + c < x < 1 + c$, $|f'(x)| < \frac{4}{c^2}n^{\frac{1}{2}}$ for $n > n_0$.*

PROOF. Suppose that for a certain x_0 in $-1 + c$, $1 - c$ $|f'(x_0)| \geqq \frac{4}{c^2}$. Put $|x - x_0| < n^{-\frac{1}{2}}$ and denote by $x_1, x_2, \cdots, x_n$ the roots of $f(x)$. Then since $f(x)$ has no root in the unit circle

$$\left|\frac{1}{x-x_i} - \frac{1}{x_0-x_i}\right| \leqq \frac{1}{c} - 1/c - n^{-\frac{1}{2}} = n^{-\frac{1}{2}}/c(c - n^{-\frac{1}{2}}) < 2/c^2n^{\frac{1}{2}} \quad \text{for} \quad n > n_0.$$

Without loss of generality we may assume that $f'(x_0) > 0$. Then

$$f'(x) = f(x)\sum_{i=1}^{n}\frac{1}{x-x_i} > f(x)\sum_{i=1}^{n}\frac{1}{x_0-x_i} - \frac{2n^{\frac{1}{2}}}{c^2} > 0. \tag{4}$$

since

$$\sum_{i=1}^{n}\frac{1}{x_0-x_i} = \frac{f'(x_0)}{f(x_0)} \geqq \frac{4}{c^2}n^{\frac{1}{2}}. \tag{5}$$

Thus $f(x)$ increases for $x_0 < x < x_0 + n^{-\frac{1}{2}}$. Hence by (4) and (5) $f'(x) > \frac{f'(x_0)}{2} > \frac{2n^{\frac{1}{2}}}{c^2}$. But

$$1 > \int_{x_0}^{x_0+n^{-}} f'(x)\,dx > \frac{2}{c^2}$$

[3] S. Bernstein, Belg. Mém. 1912, p. 19.

[4] This problem was suggested to me by Professor D. R. Curtiss.

which leads to a contradiction. Put

$$f(x) = \frac{1}{e}(x^2 - 1)^n (1 + x)^{[n^{\frac{1}{2}}]}$$ [5]

Writing $x = an^{-\frac{1}{2}}$ we have,

$$|f(x)| = \frac{1}{e}\left(1 - \frac{a^2}{n}\right)^n (1 + an^{-\frac{1}{2}})^{[n^{\frac{1}{2}}]} < \frac{1}{e} e^{-a^2+a} < 1.$$

But $|f'(0)| = \frac{[n^{\frac{1}{2}}]}{e}$ which shows that in Theorem 2. $n^{\frac{1}{2}}$ cannot be replaced by any function tending to infinity more slowly.

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[5] $[n^{\frac{1}{2}}]$ denotes the greatest integer not exceeding $n^{\frac{1}{2}}$.