SOME ARITHMETICAL PROPERTIES OF THE CONVERGENTS OF A CONTINUED FRACTION

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In this note, we consider the greatest prime factor $G(B_n)$ of the denominator of the n-th convergent A_n/B_n of an infinite continued fraction

$$\zeta = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \dots,$$

where the $a_1, a_2, \dots$ are positive integers.

We show in § 1 that, for "almost all" ζ, $G(B_n)$ increases rapidly with n (Theorem 1). In § 2, we prove that ζ is a Liouville number (*i.e.* $B_n < B_{n+1}^{\epsilon}$ for arbitrary $\epsilon > 0$ and an infinity of n) if $G(B_n)$ is bounded for all n (Theorem 2); and, in fact, there are Liouville numbers with bounded $G(B_n)$. If the denominators a_{n+1} are bounded or increase slowly, then we can prove sharper results (B and C); but we omit the proofs, since they are similar to that of Theorem 2.

Corresponding results hold for the numerators A_n of the convergents A_n/B_n of ζ.

Notation. In the following, ζ is a positive irrational number,

$$\zeta = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \dots$$

is its regular continued fraction, and

$$\frac{A_{-1}}{B_{-1}} = \frac{1}{0}, \quad \frac{A_0}{B_0} = \frac{a_0}{1}, \quad \frac{A_1}{B_1} = \frac{a_0 a_1 + 1}{a_1}, \quad \dots$$

is the sequence of its convergents. If

$$\{P\} = \{P_1, P_2, \dots, P_t\}$$

is an arbitrary finite set of different prime numbers, then $M(\{P\})$ denotes the set of all indices n for which all prime factors of B_n belong to $\{P\}$, and

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$N(x; \{P\})$ denotes the number of elements $n \leqslant x$ of this set. Finally, $G(k)$ denotes the greatest prime factor of $k \neq 0$.

I.

1. In this first paragraph, we prove that for "almost all" ζ the function $G(B_n)$ increases rapidly with n.

LEMMA 1. *Let S be the set of all positive integers k for which*

$$k \geqslant \xi, \quad G(k) \leqslant \exp\left(\frac{\log k}{20 \log\log k}\right);$$

then, for large $\xi > 0$,

$$\sum_{k \text{ in } S} \frac{1}{k} = O\big((\log \xi)^{-3}\big).$$

Proof. We divide the set of all positive integers k for which

$$(1) \qquad k \leqslant x, \quad G(k) \leqslant \exp\left(\frac{\log k}{20 \log\log k}\right)$$

into three classes A, B, and C, such that A consists of those elements which are divisible by a square greater than or equal to $(\log x)^{10}$, and the remaining elements k belong to B or C, according as $k \geqslant \sqrt{x}$ or $k < \sqrt{x}$. Then A has at most

$$\sum_{r \geqslant (\log x)^5} \frac{x}{r^2} = O\left(\frac{x}{(\log x)^5}\right)$$

elements. Next, let k be an element of B, and let

$$k = P_1^{h_1} P_2^{h_2} \ldots P_t^{h_t}$$

be its representation as a product of powers of different primes. Then, if an exponent $h \geqslant 2$, either P^{h-1} or P^h is a square factor of k, and therefore

$$P^{h-1} < (\log x)^{10}.$$

Since $\sqrt{x} \leqslant k \leqslant x$, for large x, we have

$$P^h \leqslant (\log x)^{10h/(h-1)} \leqslant (\log x)^{20} \leqslant \exp\left(\frac{\log x}{40 \log\log x}\right) \leqslant \exp\left(\frac{\log k}{20 \log\log k}\right).$$

Since this inequality holds also for $h = 1$, we have

$$k \leqslant \exp\left(\frac{t \log k}{20 \log \log k}\right),$$

and k is divisible by at least $20 \log \log k \geqslant 10 \log \log x$ different prime numbers, when x is sufficiently large. Therefore the number of divisors of k

$$d(k) \geqslant 2^{10 \log \log x} \geqslant (\log x)^5.$$

Now

$$\sum_{k \leqslant x} d(k) = O(x \log x),$$

so that B has at most

$$(\log x)^{-5} O(x \log x) = O\left(\frac{x}{(\log x)^4}\right)$$

elements. Since C has less than $\sqrt{x}$ elements, there are therefore only

$$O\left(\frac{x}{(\log x)^5}\right) + O\left(\frac{x}{(\log x)^4}\right) + \sqrt{x} = O\left(\frac{x}{(\log x)^4}\right)$$

integers k satisfying (1).

Suppose now that

$$k_1, \quad k_2, \quad k_3, \quad \ldots \quad (1 \leqslant k_1 < k_2 < k_3 < \ldots)$$

is the sequence of all positive integers k for which

$$G(k) \leqslant \exp\left(\frac{\log k}{20 \log \log k}\right).$$

Then, by the last result,

$$\frac{1}{k_\nu} = O\left(\frac{1}{\nu (\log \nu)^4}\right),$$

and the lemma follows immediately, since

$$\sum_{\nu \geqslant n} \frac{1}{\nu (\log \nu)^4} = O\Big((\log n)^{-3}\Big).$$

Lemma 2. *The measure of the set of all ζ in $0 \leqslant \zeta \leqslant 1$, such that the denominator B_n of one of the convergents A_n/B_n of ζ is equal to a given integer $k \geqslant 1$, is not greater than $1/k$.*

This is trivial, since

$$\left|\zeta - \frac{A_n}{B_n}\right| \leqslant \frac{1}{2B_n^2}.$$

THEOREM 1. *The set of all* ζ *in* $0 \leqslant \zeta \leqslant 1$, *for which an infinity of indices* n *exist satisfying*

$$G(B_n) \leqslant \exp\left(\frac{\log B_n}{20 \log\log B_n}\right), \tag{2}$$

is of measure zero.

Proof. Obviously $B_n \geqslant B_n'$, where A_n'/B_n' is the convergent of order n of the special continued fraction

$$\frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \dots.$$

Now
$$B_n' = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\},$$

and therefore

$$B_n' \geqslant \tfrac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n \quad (n = 1, 2, 3, \dots). \tag{3}$$

Let n be an arbitrary index. Then, by Lemmas 1 and 2, the measure of all ζ in $0 \leqslant \zeta \leqslant 1$, for which (2) holds, is not greater than

$$\Sigma' \frac{1}{k} = O(n^{-3}),$$

where the summation extends over all integers k for which

$$k \geqslant \tfrac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n, \quad G(k) \leqslant \exp\left(\frac{\log k}{20 \log\log k}\right).$$

Therefore the measure of all ζ in $0 \leqslant \zeta \leqslant 1$, for which (2) is satisfied for an infinity of indices $n \geqslant N$, is not greater than

$$O\left(\sum_{n \geqslant N} n^{-3}\right) = O(N^{-2}) = o(1),$$

and hence the theorem follows immediately.

In particular, from Theorem 1 and (3), for "almost all" ζ in $0 \leqslant \zeta \leqslant 1$ and all sufficiently large n, we have

$$G(B_n) \geqslant \exp\left(\frac{n}{50 \log n}\right).$$

II.

2. In this second paragraph, we give some properties of the set $M(\{P\})$ and the arithmetical function $N(x;\ \{P\})$ for special classes of irrational numbers ζ.

LEMMA 3. *For every* $\epsilon > 0$ *and every finite system* $\{P\}$ *of given prime numbers* $P_1, \ldots, P_t$, *there is at most a finite number of systems of three integers*

$$X_1 \neq 0, \quad X_2 \neq 0, \quad X_3 = \xi X_3^* \neq 0,$$

such that

$$X_1 - X_2 = X_3, \quad (X_1, X_2) = 1, \quad |X_3^*| \geqslant \max(|X_1|, |X_2|)^\epsilon,$$

where ξ *and* X_3^* *are integers, and all prime factors of* $X_1 X_2 X_3^*$ *belong to* $\{P\}$.

Proof. Take a prime number n, for which

$$n \geqslant 5, \quad \frac{1+2\sqrt{(n-1)}}{n} < \epsilon, \quad \frac{2}{\sqrt{n}} < \epsilon.$$

By hypothesis, X_1 and X_2 can be written in the form

$$X_1 = \eta_1 P_1^{h_1} \ldots P_t^{h_t}, \quad X_2 = \eta_2 P_1^{k_1} \ldots P_t^{k_t} \quad (\eta_1 = \pm 1, \ \eta_2 = \pm 1),$$

with non-negative exponents. Dividing them by n, we get, say,

$$h_\tau = n h_\tau' + h_\tau'', \quad k_\tau = n k_\tau' + k_\tau'' \quad (\tau = 1, 2, \ldots, t),$$

where $h_\tau', h_\tau'', k_\tau', k_\tau''$ are integers, and

$$0 \leqslant h_\tau'' \leqslant n-1, \quad 0 \leqslant k_\tau'' \leqslant n-1 \quad (\tau = 1, 2, \ldots, t).$$

Put

$$x = P_1^{h_1'} \ldots P_t^{h_t'}, \quad y = P_1^{k_1'} \ldots P_t^{k_t'}, \quad a = P_1^{h_1''} \ldots P_t^{h_t''} \eta_1, \quad b = P_1^{k_1''} \ldots P_t^{k_t''} \eta_2.$$

Then there are only $4n^{2t}$ possible sets (a, b). Also

$$X_1 = ax^n, \quad X_2 = by^n;$$

hence $(x, y) = 1$ and

$$ax^n - by^n = \xi X_3^*.$$

The binary form on the left-hand side is either irreducible, or is the product of an irreducible form of degree $n-1 \geqslant 3$ and a linear factor. Also there are only a finite number of possible forms. On the right, all prime factors of X_3^* belong to $\{P\}$. Hence, by the p-adic generalization of the Thue-Siegel theorem†, we must have

$$X^* = O\Big(\max(|x|, |y|)^{\max\{2\sqrt{n},\, 1+2\sqrt{(n-1)}\}}\Big)$$
$$= O\Big(\max(|X_1|, |X_2|)^{2/\sqrt{n},\, 1+2\sqrt{(n-1)/n}}\Big),$$

except for at most a finite number of solutions. Hence the lemma follows immediately.

THEOREM 2. *Suppose that for an infinity of different indices*

$$n = n_1,\ \ n_2,\ \ n_3,\ \ \ldots \tag{4}$$

the denominators B_{n-1}, B_n, B_{n+1} *of three consecutive convergents of* ζ *are divisible by only a finite system of prime numbers* $\{P\}$. *Then* ζ *is a Liouville number, and is therefore transcendental.*

Proof. Obviously

$$B_{n+1} - B_{n-1} = a_{n+1} B_n. \tag{5}$$

Put $d = (B_{n-1}, B_{n+1})$. Then $(d, B_n) = 1$, since any two consecutive B's are relatively prime; and so, by (5), d is a divisor of a_{n+1}. Write

$$B_{n+1} = dB^*_{n+1},\quad B_{n-1} = dB^*_{n-1},\quad a_{n+1} = da^*_{n+1}.$$

Then $(B^*_{n-1}, B^*_{n+1}) = 1$, and all prime factors of B^*_{n-1}, B_n, B^*_{n+1} belong to $\{P\}$ if n is an element of the sequence (4). From (5) and Lemma 3,

$$B^*_{n+1} - B^*_{n-1} = a^*_{n+1} B_n;$$

hence
$$B_n \leqslant \max(|B^*_{n-1}|, |B^*_{n+1}|)^{\epsilon} \leqslant B^{\epsilon}_{n+1}$$

for sufficiently large n; and this is the defining property of a Liouville number.

3. By the last proof, for all sufficiently large n at least one of any three consecutive indices $n-1$, n, $n+1$ does not belong to $M(\{P\})$, if ζ is not a Liouville number; hence we have the inequality

(A) $$\limsup_{x\to\infty} N(x;\ \{P\})/x \leqslant \tfrac{2}{3},\quad \text{if}\quad \log a_{n+1} = O(\log B_n).$$

† K. Mahler, *Math. Annalen*, 107 (1933), 691–730, Satz 2, 722.

In a similar way, by considering a sufficiently large number of consecutive indices and applying a lemma similar to Lemma 3, we can prove the following two results:

(B) $$\lim_{x\to\infty} N(x;\ \{P\})/x = 0, \quad \text{if} \quad \log a_{n+1} = o(\log B_n),$$

(C) $$N(x;\ \{P\}) = O(\log x), \quad \text{if} \quad \text{all } a_{n+1} \text{ are bounded.}$$

We mention as examples for these theorems:

(A) All real irrational algebraic numbers, the number π, the logarithms of all real rational numbers, the powers e^{a} with real irrational algebraic exponents.

(B) The powers e^{a} with rational exponents $a \neq 0$.

(C) All real quadratic irrational numbers.

(For these quadratic irrationals, it is even possible to show that $M(\{P\})$ has only a finite number of elements.)

Since any two consecutive B_n are relatively prime, all B_n cannot be powers of one single prime number. It is, however, easy to construct a Liouville number for which all B_n are only divisible by two arbitrary given prime numbers. On the other hand there are Liouville numbers for which all B_n are prime numbers. Assuming Riemann's hypothesis to be true, it is easy to show that there exist also non-Liouville numbers with this property.

Added 28 *October*, 1938. With respect to Theorem 2, it may be remarked that there are *transcendental non-Liouville* numbers, for which the greatest prime divisor of q_n is bounded for an infinity of indices n; *e.g.*,

$$\zeta = 3^{-1} + 3^{-3} + 3^{-9} + 3^{-27} + 3^{-81} + \dots.$$

We can show further that there exist real numbers ζ, for which the greatest prime factor of both p_n and q_n is bounded for an infinity of n. These numbers are necessarily transcendental, and probably they are Liouville numbers; but this we have not yet proved.

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